Interfacial instabilities driven by self-gravity: first numerical results

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ABSTRACT

The evolution of the interstellar medium (ISM) is driven by a variety of phenomena, including turbulence, shearing flows, magnetic fields and the thermal properties of the gas. Among the most important forces at work is self-gravity, which ultimately drives protostellar collapse. As part of an ongoing study of instabilities in the ISM, Hunter, Whitaker & Lovelace have discovered another process driven by self-gravity: the instability of an interface of discontinuous density. Theory predicts that this self-gravity driven interfacial instability persists in the static limit and in the absence of a constant background acceleration. Disturbances to a density interface are found to grow on a time-scale of the order of the free-fall time, even when the perturbation wavelength is much less than the Jeans length. Here we present the first numerical simulations of this instability. The theoretical growth rate is confirmed and the non-linear morphology displayed. The self-gravity interfacial instability is shown to be fundamentally different from the Rayleigh–Taylor instability, although both exhibit similar morphologies under the condition of a high density contrast, such as is commonly found in the ISM. Such instabilities are a possible mechanism by which observed features, such as the pillars of gas seen near the boundaries of interstellar clouds, are formed.

Key words: hydrodynamics – instabilities – ISM: evolution – ISM: structure.

1 INTRODUCTION

The interstellar medium (ISM) is a dynamic, ever-changing medium. Hydrodynamic instabilities play a large role in determining the morphology of the ISM. Well-known examples include global types, such as Jeans and thermal instabilities, and interfacial types, such as Kelvin–Helmholtz and Rayleigh–Taylor (RT) instabilities. Hunter, Whitaker & Lovelace (1997) have identified a new interfacial instability that acts upon an interface of discontinuous density and is driven by self-gravity. Their analytic results show that the self-gravity interfacial instability (SGI) persists in the static limit. They determined the theoretical growth rate for a planar interface, which in the limit of non-stratified, incompressible media is independent from the perturbation wavelength. The instability therefore exists for perturbation wavelengths much less than the Jeans length. The SGI may play a role in the formation of pillar-like structures in the ISM. In this paper, we present the first numerical models of the SGI.

In Section 2, we summarize the theory behind the SGI in the incompressible limit. In Section 3, we describe the numerical models. Section 4 contains the results in which we contrast SGI models with RT models. Finally, in Section 5 we present conclusions and further considerations.

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growth of the RT instability depends upon the density contrast and the perturbation wavelength, but not the background density. For a fixed value of \( g \), the SGI grows more rapidly than the RT instability in relatively dense regions with relatively long wavelength perturbations, such that \( \lambda = 2\pi/k > g/G|p_2 - p_1| \).

The physical origin of the SGI can be explained as follows (Hunter et al. 1997). For specificity, let the density below the interface \( (p_1) \) be greater than the density above the interface \( (p_2) \). The first term in equation (1) can be traced to the pressure difference across the interface. An upward distortion of the interface, \( \varepsilon_w(x) \), results in a mass excess proportional to \( (p_1 - p_2)\varepsilon_w(x) \). This in turn produces a positive perturbation of the gravitational potential \( \Phi \); thus, \( \Phi' > 0 \) in the nearby space above and below the interface. An increase in the fluid pressure results owing to Bernoulli’s equation, which in the static limit reduces to \( p' = \rho \Phi' \). The potential is continuous across the interface, so the pressure below the interface, \( p_1' = p_1 \Phi' \), is larger than that above, \( p_2' = p_2 \Phi' \). The pressure difference across the interface causes the distortion to grow.

In the incompressible limit, the growth rate for the SGI is independent of the perturbation wavelength. In the fully compressible case, the growth rate depends only weakly upon the Jeans criterion (Hunter et al. 1997). At first thought it seems counter-intuitive that self-gravity can drive an instability for an underlying reason is that, if \( p_1 \neq p_2 \), self-gravity gives rise to a strain at a crenulated interface that has nothing to do with the Jeans length for either medium taken separately. The energy principle makes this clear; a planar density interface is unstable because the configuration is not one of minimum energy.

### 3 MODELS

Numerical simulations are performed using CFDLIB (Computational Fluid Dynamics Library), which was developed at the Los Alamos National Laboratory (Kashiwa et al. 1994). CFDLIB is a finite-volume code well suited for problems of all flow speeds. The self-gravitational potential is solved for in two-dimensions using the MUDPACK multigrid code developed at the National Center for Atmospheric Research (Adams 1989, 1991). The models shown here utilize a two-dimensional Cartesian \((x,y)\) grid with 257 cells in both directions. In all models, the denser fluid (fluid 1) fills the bottom half of the grid, and the more tenuous fluid (fluid 2) fills the upper half of the grid. Parameters for the models considered are listed in Table 1. Models sghot and sgcold are self-gravitating models in which \( g = 0 \). Models rhot and rtcold are models in which the self-gravity is excluded and \( g > 0 \) (\( g \) acts in the upward direction). Reflective boundary conditions are used to confine the normal velocity components of the gas along all four boundaries. The potential is confined by periodic boundary conditions along the \( x \) boundaries, and by specified gradient conditions along the \( y \) boundaries. In order to isolate the SGI, models are begun from a state of hydrostatic equilibrium. Although the assumption of hydrostatic equilibrium is not realistic for the ISM, this represents a necessary step in understanding the SGI. The coupling of the SGI with other physical processes in the ISM is left for future work. Before perturbed models were considered, unperturbed models were run for very long times to test the equilibrium. The procedure used for determining the initial equilibrium configurations for SGI and RT models are detailed next. In all models, the gas is assigned a molecular weight \( \mu = 2 \) and a ratio of specific heats \( \gamma = 1.4 \).

#### 3.1 SGI

In setting up the equilibrium for a self-gravitating gas, the two-dimensional planar problem reduces to a one-dimensional problem. The equilibrium configuration in each region is solved for independently from the other as a function of \( \gamma \), where \( \gamma \geq 0 \) is the distance from the interface. The initial density and temperature distributions for the SGI models are determined by the simultaneous solution to the equation of hydrostatic equilibrium and the one-dimensional Poisson equation,

\[
\frac{1}{\rho} \frac{d\rho}{dy} - \frac{d\Phi}{dy} = -\frac{4\pi Gp}{\rho}.
\]

We assume a polytropic equation of state, \( p = \rho \gamma \), where the constant \( \kappa \) is determined by the values of \( p \) and \( \rho \) at the interface, \( \kappa = p_G/p_0 \). The polytropic relation allows the replacement of the pressure term in the hydrostatic equation with a simple function of the density. The combined result of these equations is a differential equation for the density distribution,

\[
\frac{d^3\rho}{dy^2} + \frac{(\gamma - 2)}{\rho} \left( \frac{d\rho}{dy} \right)^2 + \frac{4\pi Gp^{(3-\gamma)}}{\gamma \kappa} = 0.
\]

Analytic solutions exist for the isothermal case \( \gamma = 1 \), \( \gamma = 2 \), the incompressible case \( \gamma = \infty \), and \( \gamma = 2/3 \) (the analogue of \( \gamma = 6/5 \) for spherical polytropes). For general values of \( \gamma \) a numerical solver is needed. The problem is simplified by recasting equation (3) in terms of the Emden variable \( \theta \), such that

\[
T = T_0 \theta, \quad p = p_0 \theta^\gamma, \quad \rho = p_0 \theta^{(\gamma+1)}.
\]

As before, the subscripts denote values at the interface. The polytropic index \( n \) is related to \( \gamma \) by \( n = 1/(\gamma - 1) \). Rewriting equation (3) as a function of \( \theta \) results in a simpler equation,

\[
a^2 \frac{d^2 \theta}{dy^2} = -\theta^\gamma.
\]

The scaleheight \( a \) is determined from the density and temperature at the interface,

\[
a^2 = \frac{(\kappa + 1)}{4\pi G} \left( \frac{p_1}{\rho_0} \right)^{(1-n)/n} = (n + 1) \frac{R T_0}{4\pi G \mu p_0}.
\]

where \( R \) is the gas constant. The problem is simplified further by the definition of another dimensionless variable \( \xi \), such that \( \gamma = a \xi \). When \( \xi \) is substituted into equation (5), the result is a Cartesian analogue to the Lane–Emden equation,

\[
a^2 \frac{d^2 \theta}{d\xi^2} = -\theta^\gamma.
\]

The boundary values for each region are such that at \( y = 0 \) the density is a maximum, \( \rho = \rho_0 \), and the gradient of the density goes

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**Table 1. Model parameters.**

<table>
<thead>
<tr>
<th>Model</th>
<th>( \rho_1 )</th>
<th>( T_1 )</th>
<th>( \xi_0 \rho_0 )</th>
<th>( \lambda )</th>
<th>( \lambda_{11} )</th>
<th>( u_0/c_1 )</th>
<th>( \Delta x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sghot</td>
<td>2.0</td>
<td>1000</td>
<td>2</td>
<td>6.425</td>
<td>370.1</td>
<td>0.005</td>
<td>1.0</td>
</tr>
<tr>
<td>rhot</td>
<td>2.0</td>
<td>1000</td>
<td>2</td>
<td>6.425</td>
<td>370.1</td>
<td>0.005</td>
<td>1.0</td>
</tr>
<tr>
<td>sgcold</td>
<td>10.0</td>
<td>10</td>
<td>5</td>
<td>6.425</td>
<td>16.55</td>
<td>0.05</td>
<td>0.5</td>
</tr>
<tr>
<td>rtcold</td>
<td>10.0</td>
<td>10</td>
<td>5</td>
<td>6.425</td>
<td>16.55</td>
<td>0.05</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Units for Table 1 are as follows: \( \rho_1 = 10^{-21} \text{g cm}^{-3} \), \( T_1 = K \), \( \lambda = 10^7 \text{ cm} \), \( \lambda_{11} = 10^7 \text{ cm} \), \( \Delta x = 10^{16} \text{ cm} \).
to zero. A fourth-order Runge–Kutta numerical integration scheme (Press et al. 1992) is employed with these conditions to solve for \(\xi_t\), which is then used to determine the initial density and temperature distributions according to equations (4). The numerical and analytic solutions show excellent agreement for both the \(\gamma = 1\) and \(\gamma = 2\) cases. The density and temperature values at the interface (separate \(\rho_0\) and \(T_0\) for each region) are chosen so that the pressure is continuous across the interface.

The rate at which the temperature and density diminish with increasing \(y\) is dependent upon the scaleheight \(\hat{y}\). If the scaleheight is very large compared with the grid size, the density and temperature decrease very little. In the opposite extreme, a relatively small scaleheight results in a rapid fall-off in density and temperature. As an example, the initial density and temperature distributions for the SGI model sgcold, which has a relatively small scaleheight, are shown in Fig. 1. The scaleheight for fluid 2 is five times larger than the scaleheight for fluid 1, so fluid 1 is far more stratified. The size of the scaleheight places a limitation upon the allowable grid size. If the spatial extent of region 1 were any larger than the scaleheight for fluid 1, so fluid 1 is far more stratified. The size of the scaleheight places a limitation upon the allowable grid size. If the spatial extent of region 1 were any greater, the density values would become negative. The robustness of the equilibria varies amongst the models, with generally good results. The best equilibrium is achieved when the scaleheight is very large, resulting in a nearly uniform distribution. For such models, the maximum velocities observed after 10 e-folding times \((1\tau_e = \omega^{-1})\) are of order 1 cm s\(^{-1}\), a tiny fraction of the sound speed. For models with significant stratification, such as model sgcold, the maximum velocities after 10 e-folding times are an order of magnitude larger, but still only a small fraction of the sound speed. Deviations from equilibrium do not give rise to velocities large enough to interfere with the evolution of a perturbed model.

### 3.2 Rayleigh–Taylor

Although an in-depth consideration of the RT instability is not the goal of this study, it is of interest to compare the morphological similarities and differences between the RT and SGI instabilities. To provide an even basis for comparison, RT models are also begun from a state of hydrostatic equilibrium. The underlying method is the same as that used for self-gravitating models, but the equations are easier to solve. For a gas that is not self-gravitating, but feels a constant acceleration \(g\), the equation of hydrostatic equilibrium is

\[
\hat{p} = \frac{\hat{y}}{\gamma \hat{y}} \frac{\partial \hat{y}}{\partial y}.
\]

Integrating and setting \(\hat{p} = \hat{p}_0\) at the interface gives

\[
\hat{p} = \hat{p}_0 \left( \frac{(y - 1)\mu \hat{g}}{\gamma \hat{R}T_0} \right)^{1/(\gamma - 1)}. \tag{11}
\]

The temperature distribution is

\[
T = \frac{(y - 1)\mu \hat{g}}{\gamma \hat{R}} y + T_0. \tag{12}
\]

The density and temperature are both increasing functions of \(y\) everywhere except across the interface. For the case of a fluid with no self-gravity, the analytic solution applies to all values of \(\gamma\), so no Runge–Kutta solver is needed. The initial density and temperature distributions for the RT model rtcold are shown in Fig. 2. As in the SGI models, the departure from equilibrium is very small and does not affect the perturbed models.

### 4 RESULTS

Perturbed SGI and RT models are initiated with a velocity perturbation of the form

\[
\tau(x, y) = \tau_0 \cos(kx) e^{-ijy}, \tag{13}
\]

where \(\tau_0\) is the maximum velocity amplitude, typically expressed as some fraction of the sound speed in the denser medium \(c_1\). In order to approximate incompressible behaviour, \(\tau_0\) must be much

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Figure 1. Plot of initial density and temperature distributions along a line of constant \(x\) for model sgcold. The denser fluid has a faster fall-off away from the interface because the scaleheight is smaller than that in the more tenuous fluid. The temperature axis has been displaced to distinguish better between density and temperature plots.

Figure 2. Plot of initial density and temperature distributions along a line of constant \(x\) for model rtcold. The temperature axis has been displaced to distinguish better between density and temperature plots.
less than the sound speed. The theoretical growth rates described by equation (1) were derived for incompressible, non-stratified media. For the numerical models, the degrees of compressibility and stratification are parametrized by the Jeans parameter $J$, defined as the ratio of the perturbation wavelength to the Jeans length, $J = \lambda / \lambda_J$. The Jeans length is defined as

$$\lambda_J = \sqrt{\frac{\pi \gamma RT_0}{GM_0}}$$  \hspace{1cm} (14)$$

Compressibility increases and the gas approaches global collapse as $J$ approaches unity from below. The value of $J$ also measures the degree of stratification, because the Jeans length and the scaleheight vary in the same way with density and temperature. A larger value of $J$ indicates a larger degree of stratification. Two sets of models are considered here: a high-temperature model with $p_1/p_2 = 2$, and a low-temperature model with $p_1/p_2 = 5$. An SGI model and an RT model are presented for each regime. We emphasize that these results are only applicable for planar interfaces. Cylindrical and spherical interfaces behave quite differently (Hunter, Whitaker & Lovelace 1998).

### 4.1 High-temperature models

The models sghot and rhot are characterized by relatively hot, tenuous gas. As a result, the scaleheight is large compared with the grid size, so the initial density and temperature distributions are only mildly stratified. For these models, the interfacial values are $p_1 = 2 \times 10^{-21}$ g m$^{-3}$, $T_1 = 1000$ K, $p_2 = 1 \times 10^{-21}$ g cm$^{-3}$ and $T_2 = 2000$ K. The theoretical e-folding time is $8.458 \times 10^{13}$ s. The perturbation wavelength is much less than the Jeans length, $J = 0.0174$, so the gas is everywhere Jeans-stable against global collapse. However, the interface is unstable to the SGI. A sample of the SGI growth is shown in Fig. 3. The corresponding RT model is shown in Fig. 4. The gravitational acceleration for model rhot is $g = 4.29 \times 10^{-14}$ cm s$^{-2}$. In the early stages of development, the SGI resembles the RT instability. The wavelength of the perturbation is preserved in both cases, and their general morphologies are similar. However, whereas the RT instability is characterized by fingers of dense gas protruding into the tenuous gas, the SGI shows the opposite; namely, thin fingers of tenuous gas protruding into the dense gas. At later times, during the non-linear regime, the SGI and RT models look very different. Model rhot exhibits the mushroom cap morphology and widely spaced dense columns common to RT instabilities. Model sghot also exhibits some mushroom cap morphology, but the dense columns are not widely spaced. In fact, the separation of the dense columns decreases as the SGI evolves. At later times, clearly defined pockets of low-density material are seen embedded in the denser material, a result very different from that for model rhot. Other SGI models with a density ratio of 2 at the interface evolve in a similar manner, but more study is needed to understand this non-linear behaviour. The structures seen near the bottom of the grid in Fig. 3(d) are a result of the reflective boundary condition.

The velocity growth for a downward moving peak in model sghot is shown in Fig. 5 as a ln–linear plot normalized to the velocity at $t_0 = 1.0 \tau_s$. Time intervals are labelled in units of the theoretical e-folding time. Multiple modes are excited by the initial perturbation, so the velocity initially falls because of the presence of small-amplitude sound waves. An experimental growth rate in perfect agreement with the theoretical prediction would result in a slope of 1.0 on the ln–linear plot once the excited SGI or RT mode dominates the motion. A least-squares fit reveals a growth rate of $1.03 \tau_s^{-1}$ for model sghot, which shows very good agreement with the theoretical prediction. Simulations performed with different perturbation wavelengths (but otherwise identical) grow at the same rate, an outcome predicted by theory. When the same sort of analysis is performed for an upward moving peak in model rhot, the numerical growth rate is found to be $0.57 \tau_s^{-1}$. We do not understand this result.

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**Figure 3.** Time evolution of model sghot. Panels show the filled density contours at (a) $2t_s$; (b) $3t_s$; (c) $4t_s$; and (d) $5t_s$. The density scales show the density values $\times 10^{-21}$. The e-folding time is $t_s = 8.458 \times 10^{13}$ s.

**Figure 4.** Time evolution of model rhot. Panels show the filled density contours at (a) $2t_s$; (b) $3t_s$; (c) $4t_s$; and (d) $5t_s$. The density scales show the density values $\times 10^{-21}$. The e-folding time is $t_s = 8.458 \times 10^{13}$ s.
The models sgcold and rtcold contain denser, colder gas. For these models, the interfacial values are $p_1 = 1 \times 10^{-20}$ g cm$^{-3}$, $T_1 = 10$ K, $p_2 = 2 \times 10^{-21}$ g cm$^{-3}$ and $T_2 = 50$ K. The higher background density and density jump at the interface result in a faster growing SGI than that in the high-temperature models. The theoretical e-folding time is $t_e = 2.115 \times 10^{13}$ s. The low-temperature, high-density medium has a relatively small scaleheight (compared with the grid size), as indicated by the steep density gradient in Fig. 1. The perturbation wavelength is a significant fraction of the Jeans length, $J = 0.388$, so more compression is expected. The time evolution of models sgcold and rtcold is shown in Figs 6 and 7, respectively. For the model rtcold, the gravitational acceleration is $g = 3.43 \times 10^{-10}$ cm s$^{-2}$. As expected, both models undergo more compression, as indicated by the density scales, than the high-temperature models. The morphologies of models sgcold and rtcold are strikingly similar well into the non-linear regime. Both give rise to dense columns of comparable size, even though in model sgcold the fastest growth is downward into the denser gas, whereas in model rtcold the fastest growth is upwards into the tenuous gas. The total height of the columns formed by both models after 6$ t_e$ is about 170 cells, or $8.5 \times 10^{17}$ cm = 0.28 pc. The growth rates for the two models are also in agreement. The experimental growth rates for models sgcold and rtcold are $0.85 r_e^{-1}$ and $0.84 r_e^{-1}$, respectively. Given the degree of stratification in model sgcold, the experimental growth rate agrees surprisingly well with the non-stratified, incompressible theoretical prediction.

The larger degree of similarity between models sgcold and rtcold (as compared with models sghot and rthot) owes more to the increased density contrast at the interface than it does to the low temperature. An SGI model was run with the physical and grid parameters of model sghot, but with a density contrast of 5 at the interface. The results compare well with those of model sgcold in that the SGI and the RT instability exhibit similar morphologies and growth rates. In a similar experiment, a low-temperature model with a density contrast of 2 evolved in a manner that more closely resembles the high-temperature $p_1/p_2 = 2$ model than the low-temperature $p_1/p_2 = 5$ model. We speculate that interfaces in the ISM of high density contrast may evolve into dense columns of gas through either SGI or RT instabilities.

5 CONCLUSIONS
The models presented in the previous section represent the first numerical simulations of the SGI. The results show clear differences between Rayleigh–Taylor and self-gravitating interfacial instabilities. However, the simulations also demonstrate that a large density jump at the interface results in similar morphologies for both. The numerical results have validated the theoretical growth rate determined by Hunter et al. (1997) for the planar case and have confirmed that the SGI growth rate is independent from the perturbation wavelength.

Hunter et al. (1997) concluded their first study of the SGI with
the statement, ‘nature abhors an interface of discontinuous density’. Self-gravity acts to rearrange matter at planar interfaces into configurations of lower energy, resulting in crenulations of the interface that may evolve into bound structures conducive to star formation. More work is needed to understand fully the SGI and its role in the ISM. Although our cold-temperature models are representative of the typical densities and temperatures found in cold molecular clouds, they are also very idealized. Dynamic, thermal and magnetic effects must be incorporated into the models to form a more accurate picture of the ISM. How the SGI affects the morphology of the ISM when coupled to other instabilities and driving mechanisms remains to be explored. Even in the absence of other forces, the SGI can give rise to structures on reasonable time-scales. The time-scale for the growth of the SGI is of the order of the free-fall time in the denser medium. For example, model sgcold has an e-folding time of \(2.115 \times 10^{13}\) s, compared with a free-fall time of approximately \(2.101 \times 10^{13}\) s. The columns seen in Fig. 6(d) formed in slightly more than \(4 \times 10^6\) yr, not a long time by astrophysical standards. Further work is under way to investigate the importance of the SGI under more realistic conditions and in the presence of other driving mechanisms in the ISM.

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