An excursion set model of hierarchical clustering: ellipsoidal collapse and the moving barrier

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Acceptance: 2001 August 27; Received: 2001 August 20; in original form: 2001 May 21

\textbf{ABSTRACT}

The excursion set approach allows one to estimate the abundance and spatial distribution of virialized dark matter haloes efficiently and accurately. The predictions of this approach depend on how the non-linear processes of collapse and virialization are modelled. We present simple analytic approximations that allow us to compare the excursion set predictions associated with spherical and ellipsoidal collapse. In particular, we present formulae for the universal unconditional mass function of bound objects and the conditional mass function which describes the mass function of the progenitors of haloes in a given mass range today. We show that the ellipsoidal collapse based moving barrier model provides a better description of what we measure in the numerical simulations than the spherical collapse based constant barrier model, although the agreement between model and simulations is better at large lookback times. Our results for the conditional mass function can be used to compute accurate approximations to the local-density mass function, which quantifies the tendency for massive haloes to populate denser regions than less massive haloes. This happens because low-density regions can be thought of as being collapsed haloes viewed at large lookback times, whereas high-density regions are collapsed haloes viewed at small lookback times. Although we have applied our analytic formulae only to two simple barrier shapes, we show that they are, in fact, accurate for a wide variety of moving barriers. We suggest how they can be used to study the case in which the initial dark matter distribution is not completely cold.

\textbf{Key words:} galaxies: clusters: general – cosmology: theory – dark matter.

\section{INTRODUCTION}

Epstein (1983) described an approach which allowed him to use the statistical properties of an initially Poisson distribution of particles to derive an estimate of the number density of collapsed dark matter haloes at later times. Peacock & Heavens (1990) and Bond et al. (1991) applied essentially the same approach, but now for an initially Gaussian random density fluctuation field, to derive an estimate of the same quantity: the so-called universal, unconditional mass function (Press & Schechter 1974). Bond et al. also showed how this model could be extended to estimate the conditional mass function of subhaloes within parent haloes. (Essentially the same estimate was also provided by Bower 1991, using a different argument.) Lacey & Cole (1993) used this to estimate the rate at which small objects merge with each other to produce larger objects. They also provided formulae for the distribution of halo formation times; Nusser & Sheth (1999) provided formulae for the distribution of the halo mass at formation. Sheth (1996) and Sheth & Pitman (1997) showed how various higher order statistical properties of the forest of merger history trees associated with the formation of these objects could also be estimated within this approach. Mo & White (1996) and Jing & White (1997) provided predictions for the higher order moments of the spatial distribution of the haloes, and Sheth (1998) showed how to use the approach to estimate the probability that a randomly placed cell contains a certain amount of mass. Clearly, the approach has been very useful.

The approach combines the simple physics of the spherical collapse model with the assumption that the initial fluctuations were Gaussian and small. The problem of estimating any one of the quantities listed above is reduced to solving a problem associated with the crossing of an appropriately chosen barrier by particles undergoing Brownian motion; the Brownian nature of the motion...
comes from the assumption of Gaussian initial conditions, and the barrier shape is specified by the spherical collapse model (e.g. Sheth 1998). Hence, this is often called the excursion set approach.

Alternative derivations of the subclump distribution (Bower 1991), and the clustering of haloes (Catelan, Matarrese & Porciani 1998) which yield very similar, if not identical, results are present in the literature. Since these do not explicitly use the properties of random walks, we did not include them in the list above. Quite different approaches to estimating the mass function (Manrique & Salvador-Sole 1995, 1996; Lee & Shandarin 1998) and the merger rate (Manrique et al. 1998; Hanami 2001) have also been presented in the literature, but we will not describe them further here.

Early work (Selb & Bertchinger 1994; Lacey & Cole 1994) had already shown that the excursion set estimates of the unconditional mass functions provided a good but not perfect fit to the number density of subclumps in their simulations. Some had argued that the spherical collapse model, on which the analytic estimates were based, is an inadequate description of the collapse (e.g. Barrow & Silk 1991; Bond & Myers 1996; Monaco 1997a,b). Then Tormen (1998) reported that the spherical collapse based excursion set predictions also did not fit the conditional mass function of subclumps in his simulations well, and Jing (1998) showed that the clustering of the haloes was also different than predicted. Motivated by these various discrepancies, Sheth et al. (2001) discussed a simple way of modifying the excursion set approach to incorporate Bond & Myers’ description of ellipsoidal collapse into the predictions of the excursion set approach.

On average, initially denser regions collapse before less dense ones. This means that, at any given epoch, there is a critical density which must be exceeded if collapse is to occur. In the spherical collapse model, this critical density does not depend on the mass of the collapsed object. However, in their parametrization of ellipsoidal collapse, Sheth et al. (2001) showed that, of the set of objects which collapse at the same time, the less massive ones must initially have been denser than the more massive ones, since the less massive ones would have had to hold themselves together against stronger tidal forces. They argued that this could be incorporated into the excursion set approach, simply and effectively, if not rigorously, by changing the barrier shape. In essence, whereas the barrier associated with spherical collapse is one whose height does not depend on distance from the origin of the walk, the one associated with ellipsoidal dynamics increases with distance. They showed that the excursion set approach with a moving barrier was able to provide a good fit to the universal halo mass function.

This paper is devoted to a more detailed discussion of moving barrier models. In general, moving barrier models have a richer structure than the constant barrier model. For example, the approach with spherical dynamics predicts that, at any given time, all the mass in the Universe is bound up in collapsed objects, whereas a small fraction of the mass remains unbound in the case of ellipsoidal dynamics. In addition, whereas clustering is strictly hierarchical in the case of spherical dynamics, incorporating ellipsoidal collapse into the excursion set approach results in a model in which fragmentation as well as mergers may occur – the approach predicts that some small haloes fragment before they are subsumed into larger ones. Although we do not use any of these features here, Appendix A describes these in more detail; some of these features may (or may not!) provide better approximations to the physics of gravitational instability than does the constant barrier model, in more detail. Appendix A also describes how the results of this paper can be used to model the halo mass function in warm dark matter scenarios such as that revisited by Bode, Ostriker & Turok (2001).

Section 2 shows how moving barrier models can be used to make simple analytic estimates of a number of statistical quantities which are routinely measured in numerical cosmological simulations. The primary results of Section 2 are equations (4) and (7), which are accurate for a large class of moving barrier shapes. To illustrate how these formulae work, we show the result of inserting the ellipsoidal collapse moving barrier of Sheth et al. (2001) into these formulae. Section 2.1 presents the number density of bound objects as a function of mass (the unconditional mass function), and Section 2.2 describes a simple, efficient algorithm for generating it. Section 2.3 presents the average number of progenitor subhaloes as a function of subhalo mass, for a wide range of specified parent halo masses (the conditional mass function), and Section 2.4 shows how the halo mass function depends on the surrounding density field (the local-density mass function). Comparison with simulations shows that the approach, with the ellipsoidal collapse based moving barrier shape, is quite accurate. Section 2.5 shows that, at small lookback times, neither the constant nor the moving barrier predictions describe the conditional mass functions in the simulations particularly well, although the agreement at large lookback times is quite good if the spherical collapse constant barrier is used, and even better if the ellipsoidal collapse moving barrier is used.

Section 3 shows the result of considering more complicated moving barrier excursion set models. In particular, it shows the result of considering the full six-dimensional random walk associated with Gaussian random fields, rather than the one-dimensional simplification proposed by Sheth et al. (2001). It shows that their simplification is actually quite accurate. Details associated with the calculations in this section are presented in Appendix B.

Section 4 discusses some simple implications of our findings. Although, in this paper, we concentrate on the moving barrier derived by Sheth et al. (2001), we think it worth stressing that our analytic formulae are more general: they are accurate for a wide variety of moving barrier shapes.

In the paper we show results for two choices of the initial fluctuation spectrum belonging to the cold dark matter family: SCDM and ΛCDM, for which (Ω0,h,σ8) are (1.0,5.0,6) and (0.3,0.7,0.9) respectively, and Λ = 1 – Ω. Here Ω0 is the density in units of the critical density today, the Hubble constant today is H0 = 100h km s−1 Mpc−1, and σ8 describes the amplitude of the initial fluctuation spectrum. The simulations to which we compare all our results have been kindly made available to the public by the Virgo consortium (Jenkins et al. 1998). We actually use the subset called the GIF simulations; these were performed with 2563 particles each, in a box of size L = 85 Mpc/h for the SCDM run, and L = 141 Mpc/h for the OCDM and ΛCDM runs.

2 THE MOVING BARRIER MODEL

As discussed in the introduction, we will mainly be interested in the first-crossing distributions of uncorrelated Brownian motion random walks. Following Bond et al. (1991) and Lacey & Cole (1993), these first-crossing distributions can be used to provide useful approximations to what have come to be called the conditional and unconditional mass functions of the dark halo distribution. The results of this section should be thought of as generalizations of the results in Lacey & Cole (1993). Whereas they restricted their attention to a barrier of fixed height, this
section presents analytic formulae which approximate the barrier crossing distribution for a wide class of moving barriers.

Before we begin, we think a word on notation is helpful. We have chosen to present our formulae for the first-crossing distributions using the same notation as Lacey & Cole (1993). This means that we use the symbol $S$ to represent the variance in the density fluctuation field when smoothed on some scale, which is usually denoted $\sigma^2$. However, some of our formulae can be written in terms of the scaled variable $(\sigma / \sigma_S)^2 = S / S_\alpha$ for some suitably defined $S_\alpha$. When this is possible, we also write our formulae in terms of $\nu = S / S$.

### 2.1 The unconditional mass function

To illustrate how our formulae work, rather than use the spherical collapse barrier of constant height, we will use the moving barrier shape defined above, with the density fluctuation field when smoothed on some scale, which is usually denoted $\sigma^2$. However, some of our formulae can be written in terms of the scaled variable $(\sigma / \sigma_S)^2 = S / S_\alpha$, for some suitably defined $S_\alpha$. When this is possible, we also write our formulae in terms of $\nu = S / S$.

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constructing an algorithm for generating it is not so straightforward. We have found that first generating a Gaussian variate \( x \), and then setting \( \nu = |x|^{3/6}/(1 + |x|^{1/6}) \) is accurate to within 1 per cent or so over the range \( 0.01 \leq \nu \leq 100 \). The speed of this algorithm compensates for the fact that it does not exactly produce variates drawn from the ellipsoidal collapse mass function.

### 2.3 The conditional mass function

As stated above, we do not know of an analytic expression for the first-crossing distribution associated with barriers which have the form given in equation (1). However, we do have two reasonably accurate fitting formulae—equations (2) and (4)—to this distribution. One might have thought that we could use them to make an estimate of the conditional mass function as follows.

Bond et al. (1991) and Lacey & Cole (1993) argued that conditional mass functions could be estimated by considering the successive crossings of boundaries associated with different redshifts. The first crossing of two constant barriers of different heights has an analytic solution, so they were able to provide analytic estimates for the conditional mass function associated with the spherical collapse model. Such a formula is very useful because, once the conditional mass function is known, the forest of merger history trees can be constructed using the algorithm described by Sheth & Lemson (1999b), from which the non-linear stochastic biasing associated with this mass function can be derived using the logic of Mo & White (1996) and Sheth & Lemson (1999a).

In the constant barrier case, the conditional mass function is computed by considering walks which start from \( \sigma^2 = S, \delta_c(z_0) \) rather than from the origin, and then intersect the constant barrier \( \delta_c(z_1) \) at, say, \( z \). This is easily computed because, despite the shift in the origin, the second barrier is still one of constant height. Since the first-crossing distribution of a barrier of constant height is known, (recall it was just such a distribution which was associated with the universal, unconditional mass function), the conditional mass function can also be written analytically. Essentially, \( f_1(S) \) has the same form as the unconditional mass function \( f(S) \), but with the change of variables: \( \delta_c(z_2) \rightarrow \delta_c(z_1) - \delta_c(z_0) \) and \( S \rightarrow S - S \).

One might have wondered if the same change of variables in equation (2) provides a good description of the conditional mass function associated with the ellipsoidal collapse moving barrier. Unfortunately, because the barrier shape is not linear in \( S \), changing the origin of the coordinate system does not yield a barrier of exactly the same functional form. Specifically, the shape of

\[
B(s, z_1) - B(s, z_0) = \sqrt{a}\delta_1[1 + \beta S^q/(a\delta_1^q)]^q - \sqrt{a}\delta_0[1 + \beta S^q/(a\delta_0^q)]^q,
\]

where \( \delta_0 = \delta_c(z_0) \), etc. can be written as a constant plus a term which scales as \( (s - S) \) only when \( \alpha \) equals zero or one. This means that, formally, the solution to the two-barrier problem associated with ellipsoidal dynamics is not given by a simple rescaling of the unconditional ellipsoidal collapse mass function. Therefore we cannot simply rescale the fitting function of equation (2) to get a reliable estimate of the conditional mass function: the two-barrier problem associated with moving barriers must, in general, be solved numerically.

This is discouraging because it means that, in principle, we must find a different fitting function for each choice of condition, because each condition corresponds to a different origin, say, \( (B_0, S_0) \), and so to a slightly different barrier shape. Of course, the result

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**Figure 1.** First crossing distributions and the universal unconditional halo mass function. The histogram shows the distribution obtained by simulating random walks that are absorbed on the ellipsoidal collapse moving barrier; the solid curve shows our analytic approximation to this distribution (equation 4). The dashed curve shows the distribution that fits the halo mass function. The histogram shows the distribution obtained by simulating random walks that are absorbed on the ellipsoidal collapse moving barrier; the solid curve shows our analytic approximation to this distribution (equation 4). The dashed curve shows the distribution that fits the halo mass function.
can be generated relatively quickly in at least two ways. The first is
to solve the integral equation associated with this barrier
numerically (Monaco 1997b; Sheth 1998). The second is to
simply simulate the random walk trajectories and so construct the
first-crossing distribution directly.

Fig. 2 shows an example of what is involved in solving the two
barrier problem numerically in this way. The smooth dotted and
solid curves show the ellipsoidal collapse barrier $B(S, z)$ of
equation (1) scaled to $z = 0$ and $z = 2$, respectively. Jagged curves
show a few representative random walk trajectories: they start at
the barrier position $B(S, z = 0)$, where $S(M)$ is given by the GIF
SCDM power spectrum, and $M/M_\text{SCDM}^\star = 2$. These random walks are
followed until they first cross the barrier $B(s, z = 2)$. The value of $S$
at which this happens is stored and used to make plots like those
shown below. The random walks were generated by the same
Monte Carlo code that was used to generate Fig. A1, except that
there the barrier shape was given by equation (A1). Fig. A1 shows
that this Monte Carlo code works correctly. One possible approach
to the excursion set conditional mass function, then, is simply to
generate it numerically, as the need arises.

Before providing a detailed comparison of the conditional mass
functions generated using this Monte Carlo model and those in
numerical simulations, it is useful to study a simple limiting case.
Fig. 2 shows that for $S/S_\text{SCDM} < 0.5$, the height of the barrier is
approximately constant. At small $S$, the only difference between
the barrier at two redshifts, and the spherical collapse constant
barriers, arises from the factor of $a = 0.707$. This has the following
consequence. At small lookback times (small redshift differences),
most random walk trajectories will intersect the barrier before they
have travelled very far along the $S$-axis. For these trajectories, the
barrier is effectively one of constant height. This means that the
conditional mass function for massive haloes at small lookback
times will have the same shape as that predicted by the constant
barrier, with one small difference. The factor of $a = 0.707$ has the
effect of slightly reducing (by a factor of $\sqrt{a}$) the effective redshift
difference relative to the original constant barrier model. As a
result, the GIF barrier suggests that massive haloes at small
lookback times will be slightly more massive than the original
constant barrier model predicts. Since $\sqrt{0.707}$ is close to unity, this
effect will be small. In practice, we expect the barrier predictions to
differ significantly from those of the constant barrier only for small
haloes, or at large lookback times (high redshift). This is
encouraging, because these are precisely the regimes in which
simulations suggest that the constant barrier model is inaccurate
(Tormen 1998).

In addition to Monte Carlo the conditional mass functions,
we can use the results of the previous subsection to derive a simple
analytic expression for their shape. We can do this because our
barrier crossing formula is reasonably accurate for a rather wide
range of barrier shapes. A glance at Fig. 2 shows that the barrier
shapes associated with the conditional mass functions are not likely
to be too different from those associated with the unconditional
function, so we should be able to use equation (4) to approximate
most of the conditional mass functions we will be interested in.
In practice, this can be done by simply making the appropriate
replacements $B \rightarrow B(s) = B(S)$ and $S \rightarrow s \sim S$ in equation (4). At
the risk of being repetitive, our approximation for the conditional
mass function is

$$N(m, \delta| M, \delta_0) dm = \frac{M}{m^2} f(m, \delta| M, \delta_0) dm,$$

where $f(m|M)$ dm = $f(s|M) ds$ with

$$f(s|M) ds = \frac{T(s|M)}{\sqrt{2\pi(s - S)}} \exp\left\{ - \frac{[B(s) - B(S)]^2}{2(s - S)} \right\} \frac{ds}{s - S}$$

and

$$T(s|M) = \sum_{n=0}^{5} \frac{(S - s)^n \delta^n |B(s) - B(S)|}{\delta^n s^n}.$$

Figs 3 and 4 compare this approximation with the numerical Monte
Carlo distribution, and compare both with the actual distribution
measured in the cosmological simulations. The figures show the
conditional mass functions measured in the simulations (symbols
with error bars), and histograms show the conditional mass
functions generated using our Monte Carlo code, for parent haloes
in the mass range $1 \leq M/M_\text{SCDM} \leq 2$ (upper curves) and $16 \leq M/M_\text{SCDM} \leq 32$ (lower curves); the upper curves have been shifted
upwards by a factor of 10 for clarity. Smooth curves show the
various analytic approximations discussed so far. These are
equation (7) (solid), the distribution associated with rescaling the
unconditional mass function of equation (2) (dashed), and the
conditional distribution associated with the constant barrier,
spherical collapse model of Bond et al. (1991) and Lacey & Cole
(1993) (dotted). (The finite mass resolution of the simulations
means that the distributions are artificially truncated at low
masses.)

The spherical collapse based dotted curves are often quite
different from the $N$-body simulation symbols. This discrepancy
is similar to that first noticed by Tormen (1998); haloes in the
simulations seem to hold themselves together at higher redshift
than the spherical collapse model predicts. Whereas the dashed
curves obtained by rescaling the unconditional halo mass function
are better fits to the cosmological simulations, the solid curves and
histograms, in which the relation between the excursion set model
and the conditional mass function are accounted for more carefully,
are almost always even more accurate. The agreement between the
solid curves and the histograms suggests that our formula
(equation 7) is quite accurate. Therefore, in the comparisons to
follow, we will sometimes show only the analytic curves. Recall
that the analytic formula should be most accurate for the high-
redshift progenitors of massive parents, and least accurate when the

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Figure 2. Examples of random walks used to construct the conditional mass functions associated with the ellipsoidal collapse moving barriers at $z = 0$ (dotted curve) and $z = 2$ (solid curve).

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2.4 Dependence on local density

Following Mo & White (1996), knowledge of the conditional mass function allows one to estimate how the distribution of dark matter haloes today depends on the average density in which the haloes are. In essence, they argued that a dense region should be thought of as an object which will collapse and form a virialized halo in the near future. This means that the haloes in it today can be thought of as ‘progenitor subhaloes’ viewed at ‘low redshift’. In contrast, it will be a much longer time before an underdense region collapses (if it collapses at all), so the haloes within it today are like the progenitor haloes viewed at high redshift. In hierarchical models, massive haloes form later, and less massive haloes form earlier.

Therefore one expects haloes in dense regions to be more massive, on average, than in underdense regions.

The precise dependence of halo mass on local density depends on the precise relation between the local density today and the effective ‘redshift’. Mo & White (1996) used the spherical collapse model to provide this relation. They provided a simple fitting formula to this relation in an Einstein–de Sitter universe. We have checked that the following simple modification to their fitting formula is reasonably accurate for all cosmologies of interest:

\[
\delta_0(\delta, z_0) = \delta_{sc}(z_0) \times \left[ \frac{1.68647 - 1.35}{(1 + \delta)^2} \right] - \frac{1.12431 + 0.78785}{(1 + \delta)^{3.58661}},
\]

where \(M/\rho V = (1 + \delta)\) is the non-linear density of a region containing mass \(M\) within the volume \(V\) at \(z_0\), and \(\delta_{sc}(z_0)\) denotes the critical density for spherical collapse at \(z_0\).

Mo & White (1996) argued that the number density of \(m\)-haloes in regions of non-linear density \(\delta\) (which contain mass \(M\)) is obtained by inserting this expression for \(\delta_0\) in the formula for the conditional mass function \(N(m, \delta, z_1)\) given in equation (6). Lemson & Kauffmann (1999) and Sheth & Lemson (1999a) showed that this provided a reasonably good description of how, in their simulations, the density of haloes depended on local density. The agreement between the model and simulations is not perfect: Lemson & Kauffmann noted that the spherical collapse based formulae overpredict the number densities by a factor of 1.5.

The previous section showed that, in fact, the conditional mass functions are better fitted by ellipsoidal collapse based curves. Therefore one might reasonably expect that the same will be true for \(n(m|\delta)\). To emphasize how similar \(n(m|\delta)\) is to the conditional mass function, we have chosen to do the following. We divided the simulation volume up into cubes, each 10 Mpc/h on a side. We then divided the cubes into three classes: the densest, and least dense 10 per cent of the cells, and the 10 per cent around the median density. The average overdensity in these classes is \((1 + \delta) = 0.32, 0.99\) and \(3.2\) for the SCDM simulation, and \(0.18, 0.67\) and \(4.5\) for ΛCDM.

Fig. 5 shows \(m^2 n(m|\delta)\) for haloes of mass \(m\) plotted versus the ratio of \(m\) to the mass of the cell they inhabit. Symbols with error bars show the measurements in the SCDM simulations, histograms show the associated random walk distribution, solid curves show our analytic approximation to the random walks, dashed curves show the result one gets by rescaling the unconditional mass function, and dotted curves are for the spherical collapse, constant barrier model. The curves for the three types of cells have been offset from each other for clarity; the lowest density cells are the lowest curves, and they have not been shifted, average density cells.
have been shifted upwards by a factor of 10, and the densest cells have been shifted upwards by a factor of 100.

Notice how qualitatively similar the measurements in the simulations are to those shown in Fig. 3. Most haloes in the densest cells are a significant fraction of the total mass in the cell, so the mass function in dense cells looks very similar to the low-redshift conditional mass functions. In contrast, the mass function in underdense cells looks much more like the high-redshift conditional mass functions. This is the qualitative behaviour predicted by the model.

A more quantitative comparison shows that the spherical collapse model (dotted lines) overpredicts the number density of haloes by about the same amount as reported by Lemson & Kauffmann (1999). The result of rescaling the unconditional mass function (dashed lines) fares better, although it appears to be more accurate in the denser cells. In all but the densest cells, the moving barrier model (solid curves) provides the best fit to the simulation data.

Fig. 6 shows what this trend means for the number density of haloes in dense and less dense regions in the ΛCDM simulation. The various symbols and curves are the same as in the previous figure; the only difference is that now the x-axis has been multiplied by the total mass in the cell to show physical, rather than relative, mass. Clearly, less dense cells have essentially no massive haloes. In addition, the ratio of massive to less massive haloes is higher in denser cells, and the density of less massive haloes in dense regions is actually smaller than the density of less massive haloes in underdense regions.

Note again that the spherical collapse based formula (dashed lines) overpredicts the counts by a factor of 2 or so, that the result of rescaling the unconditional mass function works reasonably well in the dense cells but is not as accurate at low density (dashed lines), and that the best fit is provided by the moving barrier model (solid lines). This is not as trivial as it might at first seem. Although we argued that the conditional and local-density mass functions are very similar, this is only exactly true for the constant barrier model. For the moving barrier, there is a subtle difference. The moving barrier for the local-density mass function always has the same shape – that associated with $z_0$. The mass function depends on the local density because the starting point of the walk depends on $\delta_c(z_0)$, and on the mass $M$ in the cell. In contrast, the moving barrier associated with the conditional mass function changes as the redshift $z_1$ at which the mass function is evaluated changes (see, e.g., Fig. 2). In general, the local-density barrier increases more strongly with distance along the walk; so, for the same ‘effective redshift’, one expects it to predict fewer massive haloes compared to the conditional mass function.

A comparison of Figs 3 and 5 shows that this is indeed the case. At large $m/M$, the curves in the conditional mass function plots are ordered solid, dashed dotted (e.g., top left panel of Fig. 3), whereas the ordering is dashed, dotted, solid in Fig. 5. This subtle difference is actually very obvious in low-density cells of the ΛCDM simulation (middle and bottom curves in Fig. 6). Since the low-density cells occupy a substantial fraction of the total volume, it is encouraging that our moving barrier predictions work so well there.

In summary, we argued that the conditional and the local-density mass functions provide different and complimentary tests of the moving barrier shape, and that our moving barrier based formulae provide a more accurate fit to the simulations, in both cases, than does the spherical collapse based constant barrier model. One might have thought that dense regions simply have more haloes on average, say $n(m|\delta) = (1 + \delta)n(m)$. Our results show that this is wrong, but that the ellipsoidal collapse moving barrier model allows one to compute a good estimate of $n(m|\delta)$ analytically.

### 2.5 Rescaling the conditional mass function

In the excursion set model with a constant barrier height, the unconditional mass function, when expressed as a function of $n = \delta_c^2/\sigma^2(m)$, is expected to be a universal function which is independent of redshift, cosmology or initial power spectrum. In addition, if the conditional mass function is expressed in units of $(\delta_{c1} - \delta_0)^2/(s - 3)$, then it is expected to have the same shape as the unconditional mass function.

The previous section we noted that, although the unconditional mass function is a universal function of $n$, this function is not the one predicted by the constant barrier model. We argued that if we
interpret the unconditional mass function as coming from a moving barrier, then we no longer expect the conditional mass function to be a universal function of \( \nu \). Figs 7 and 8 show this explicitly; they show the result of applying this rescaling to the conditional mass functions in the SCDM and the \( \Lambda \)CDM simulations we presented earlier.

The four panels in each figure show the conditional mass function at each of the four redshifts we have been studying so far. The symbols show the result of rescaling the conditional mass functions in the simulations for parent haloes with mass in the range 1–2\( M_\odot \) (filled circles) and 8–32\( M_\odot \) (open triangles) at \( z = 0 \). Notice that the symbols in each panel do not overlap exactly – at fixed \( z \), the conditional mass functions for different parent haloes do not rescale exactly. In addition, the band traced out by the symbols at \( z = 4 \) is quite different from the band traced out at \( z = 0.5 \); the mass functions at different output times do not rescale either. Both these findings illustrate our main point: the conditional mass function is not a universal function of the scaling variable \( \nu \).

The dotted curves, which are the same in all the panels, show the predictions of the constant barrier model; they do not provide a good fit at any time for any mass range. The dashed curves, which are also the same in all the panels, show the result of assuming that, upon rescaling, the conditional mass function will have the same shape as the unconditional mass function; although they provide a good fit at large \( z \), they are increasingly in error at small \( z \). This is true for both the SCDM and the \( \Lambda \)CDM models.

The solid lines in the various panels show the predictions of our moving barrier model. In this case, the predictions depend both on the parent mass range, and on the redshift at which the progenitors are identified: we have chosen to show the predictions for the 1–2\( M_\odot \) haloes only. Whereas the model is in reasonable agreement with the simulations at large \( z \), it has the wrong shape at small \( z \). This is not very surprising, because our formula for the first-crossing distribution of the moving barrier model was supposed to work only at large \( z \), but it is unfortunate that the disagreement at low \( z \) is so bad! One might have thought that the actual first-crossing distribution may be in good agreement with the GIF simulations, and that it is only the analytic approximation which is in error at small \( z \). Unfortunately, this is not so. The histograms in Fig. 8 show the result of simulating an ensemble of random walks to construct the conditional mass functions; although they are in slightly better agreement with the simulations, they are still quite different.

The discrepancy between the simulations and our moving barrier model predictions are most pronounced when the subclump mass is \( m/\bar{m} \approx 1/2 \). This suggests that our model is unable to describe the histories of clumps at small lookback times. At small lookback times, one might worry that the spherical overdensity group-finder we use to identify the subhaloes in the simulations might find a different set of objects than a friends-of-friends algorithm. Plots of the rescaled conditional mass function constructed using a friends-of-friends algorithm look very similar to the spherical overdensity results presented above – the discrepancy between model and simulations is independent of the choice of groupfinder.

In addition, because the theory assumes that mass is conserved – all the mass of a subclump becomes part of the final halo – whereas this is not true in the simulations: some of the particles which make up the final object may have come from a subclump which merged along with most of its particles into a different object. This means that there is some freedom associated with how we decide whether a clump at an early time should be counted as a subclump of a halo at the final time. We have tried two schemes for identifying subclumps: a progenitor is a clump which donates at least half its mass to the final object, or which donates at least one particle to the final object. Once we have made this decision, we must also decide what we wish to count as the mass of the parent object: two natural choices are the mass at the final time (which, by definition, is fixed for all earlier redshifts), or the mass which is obtained by summing up the masses of all the progenitor subclumps (which may depend on redshift). The figures above are for the case in which a subclump is any clump which donates at least one particle to the final object, and the parent mass is defined as the mass at the final time (so it is independent of redshift). Although the actual conditional mass

\[
\nu = \frac{\langle \delta_1 - \delta_0 \rangle^2}{(s-S)}
\]

Figure 7. Rescaled conditional mass functions in the SCDM model. The panels show the different redshift bins we studied earlier. Symbols in each panel show the different mass ranges we considered. Dotted lines show the constant barrier prediction (in these variables it is the same as the unconditional mass function), dashed lines show the result of rescaling the actual unconditional mass function, and solid curves show the result of rescaling the moving barrier predictions.

\[
\nu = \frac{\langle \delta_1 - \delta_0 \rangle^2}{(s-S)}
\]

Figure 8. Same as Fig. 7, but for \( \Lambda \)CDM.
functions depend slightly on which combination of the above choices we make, the generic results shown above are independent of this choice.

Before moving on, we think it worth noting that the discrepancies between the SCDM simulations and the dotted or dashed curves are qualitatively similar to the discrepancies in the ΛCDM case. This suggests that one should be able to find a model which can account for these discrepancies in a way which is independent of power spectrum, redshift or cosmology. Our moving barrier model is just not up to the task. The next section studies why.

3 AN EXTENSION

In the ellipsoidal collapse model envisaged by Bond & Myers (1996) and implemented by Sheth et al. (2001), the collapse of a patch is determined by the surrounding shear field. In a Gaussian random field, the field around a patch may differ from patch to patch. Appendix B provides a simple prescription for choosing a set of patches which have the correct ensemble averaged properties – in essence, this requires studying the first-crossing distribution of six-dimensional random walks.

Because a six-dimensional walk is computationally expensive, rather than choosing the distribution of initial patches from this exact statistical distribution, Sheth et al. (2001) suggested it should be a good approximation to use an appropriately chosen mean value, and neglect the scatter around this value. This allowed them to reduce what is a six-dimensional random walk to a one-dimensional walk. It is this one-dimensional walk which we have considered so far. One might worry, however, that the discrepancy between model predictions and simulation results we found in the previous section may actually be due to our neglect of the scatter around the average value. The main purpose of this section is to study this possibility.

Fig. 9 shows the result of generating an ensemble of 4000 six-dimensional random walks associated with Gaussian random fields as described in Appendix B. The walks are stopped when they cross the barrier associated with the ellipsoidal collapse model of Bond & Myers (1996). The crosses show the values of \(\delta\) and \(\sigma\) at which the six-dimensional walks crossed the ellipsoidal collapse barrier \(\delta_{\mathrm{rel}}(\nu, p)\): we actually used the simple fit, equation (3) in Sheth et al. (2001), to the critical density required for collapse \(\delta_{\mathrm{crit}}(\nu, p)\). The solid curve shows the approximation used by Sheth et al.; it provides a rather good description of how \(\delta_{\mathrm{rel}}\) increases with increasing \(\nu\). Appendix B describes the reason for this in more detail. For now, we simply note that because the solid curve provides a reasonably good description of the crosses, the first-crossing distributions of the six-dimensional walks considered here are unlikely to be very different from the first-crossing distributions associated with the (considerably simpler) one-dimensional walks studied in the previous sections of this paper.

The two jagged curves in Fig. 10 show this explicitly. They show the first-crossing distributions associated with the six-dimensional walks which cross the \(z = 0\) and \(z = 0.5\) six-dimensional ellipsoidal collapse barriers. They have been rescaled similarly to how the unconditional mass functions in simulations rescale: \(\nu = (\delta_{\nu}/\sigma)^2\). After rescaling, the two curves appear similar, as they should; note that they are reasonably like the dashed curve, which shows equation (2), and they are rather different from the dotted curve which shows the spherical collapse prediction.

Recall that the dashed curve is very similar to the mass function one gets by simulating one-dimensional random walks. Because the jagged curves are in quite good agreement with the dashed curve, the first-crossing distributions associated with the six-dimensional walks are actually rather similar to those associated with the (considerably simpler) one-dimensional walks studied in the previous sections. This agrees with what our conclusions from Fig. 9. Appendix B describes the reason for this in more detail.

Having shown that the unconditional mass functions associated...
with the six-dimensional random walks are in reasonable with numerical simulations, we now turn to the conditional mass functions – the test which our one-dimensional random walks failed. The solid symbols with error bars show the conditional mass functions associated with the six-dimensional walks, expressed in the scaled units of the previous section: \( \nu = 0.707 \delta_c^2 \sigma^{-2} (s - \hat{s}) \), where \( s \) denotes the value of \( \sigma^2 \) at which the \( z = 0.5 \) six-dimensional boundary was crossed, and \( \hat{s} \) is the scale on which the lower \( z = 0 \) six-dimensional boundary was crossed. Note the excess of points at large values of \( \nu \); these conditional mass functions are quite different from the conventional mass functions in simulations. Indeed, the discrepancy between the excursion set predictions and the simulations of hierarchical clustering has got worse!

There are at least two reasons why the excursion set approach with one-dimensional random walks may fail at small lookback times. The first is the excursion set neglect of correlations between scales; at large lookback times most subclumps are a small fraction of the mass of the parent halo, so the smoothing scale associated with the subclumps is sufficiently different from that of the parent that the neglect of correlations between the two scales is justified. At smaller lookback times the parent and subclump scales are not so well separated, so the neglect of correlations is a bad approximation. This is one possible reason for the agreement at large lookback times despite the discrepancy at low redshift. The second possibility is that the one-dimensional parametrization of ellipsoidal collapse outlined by Sheth et al. (2001) is too simple. The results of this section suggest that, in fact, it is the first possibility which is the cause of the discrepancy.

### 4 DISCUSSION

Sheth et al. (2001) argued that a simple modification to the original excursion set approach was enough to improve agreement between the predictions of the approach and numerical simulations. The modification they suggested was to the value of the linearly extrapolated critical overdensity \( \delta_c \) associated with the collapse of an object. The spherical collapse model assumes that this value is independent of the mass of the collapsed object, whereas ellipsoidal collapse makes \( \delta_c \) depend on \( m \). In the context of the excursion set approach, this corresponds to studying the first crossing statistics of a set of moving, rather than constant barriers. We also argued that a moving barrier also provides a simple way in which the excursion set approach can be extended to apply to models in which the initial dark matter distribution is not completely cold.

Because moving barrier models are so useful, we provided analytic approximations for the required first-crossing distributions (equations 4 and 7), and showed that they were reasonably accurate. Although our formulae for the conditional mass functions (solid curves in Figs 3 and 4) are slightly different from, and usually more accurate than, those one obtains by a simple rescaling of the unconditional mass function (dashed curves in the same figures), this simple rescaling of the unconditional mass function is still more accurate than what one gets if one rescales the constant barrier formulae (dotted curves). Our moving barrier model also provides a better description of how the number density of haloes depends on local density (Figs 5 and 6), although the model was more accurate at low density.

However, we showed that neither the constant nor the moving barrier models were able to describe the simulation results at small lookback times (Figs 7 and 8). This means that our results for the conditional mass functions cannot be used to generate realizations of the forest of merger history trees. We argued that this discrepancy was most likely due to the neglect by the excursion set approach of correlations between scales (Section 3). While this neglect is a bad approximation at small lookback times, it is reasonable at large lookback times. This is why the excursion set approach is able to provide a reasonably good description of clustering at high redshift, even though it is inaccurate at small redshifts. This is also why it is more accurate in lower-density regions; fortunately, the low-density regions occupy most of the volume.

Before concluding, we will consider how some of our results are related to other work in the literature. Recently, Jenkins et al. (2001) showed that, although the mass functions in their simulations scaled in accordance with the excursion set prediction, our equation (2) slightly overestimated the unconditional mass functions in their simulations. We thought it would be interesting to show the various approximations to the mass function which we presented in this paper on one plot. The dot-dashed curve in Fig. 11, with cut-offs at low and high masses, shows the fitting function proposed by Jenkins et al., which fits their simulations well, the dashed curve shows the fitting formula of equation (2) with \( a = 0.707 \) (following Sheth & Tormen 1999), the histogram shows the distribution one gets by simulating random walks with the ellipsoidal collapse moving barrier (equation 1, following Sheth et al. 2001), the solid curve shows our approximate formula for this first-crossing distribution (equation 4), and the dotted curve shows the spherical collapse, constant barrier prediction (Press & Schechter 1974; Bond et al. 1991). Simulations currently available do not probe the regime where \( \nu \approx 0.3 \) or so (the Jeans mass is at about \( \nu \approx 0.03 \)).

The upper set of curves show the residuals between our formulæ and the one provided by Jenkins et al. (2001), in the regime to the right of the low-mass cut-off (marked by an arrowhead). In addition to the previously mentioned formulæ, we have included a new short-dashed long-dashed curve which shows the result of changing \( a \) in equation (2) from 0.707 to 0.75. This simple change appears to be all that is necessary to reduce the discrepancy.
between it and the simulations substantially. Our formula differs dramatically from the one proposed by Jenkins et al. (2001) at small masses. We hope that simulations in the near future will be able to address which low-mass behaviour is correct.

To illustrate that our formulae really do work for a large class of barrier shapes, Fig. 12 shows the first-crossing distribution associated with the barrier discussed by Monaco (1997a,b):

\[
B(S) = \delta_{\text{sc}}(1.82/\delta_{\text{sc}} - 0.69 \sqrt{S/S_{\text{sc}}}),
\]

where, as throughout this paper, \( S_{\text{sc}} = \delta_{\text{sc}}^2 \). The height of this barrier decreases with \( S \), so it really is quite different from ours (Sheth et al. 2001 discuss the physical reason why). We chose not to present results for the barrier shape studied by Del Popolo & Gambera (1998) because their shape is not so different from ours, whereas Monaco’s really is quite different. The fact that this barrier decreases with \( S \) means that all walks are guaranteed to cross it (this is in contrast to barriers whose height increases sufficiently strongly with \( S \), as discussed in Appendix A). The histogram shows the numerical Monte Carlo first-crossing distribution, and the two solid curves shows our analytic approximation, computed by inserting this barrier shape into our equation (4). The curve that provides a slightly worse fit to the histogram shows the result of using the first five terms in the series (as we did for the other figures in this paper); the other curve shows what happens if we use the first 10 terms instead. Just for comparison, the dotted and dashed curves show the spherical collapse prediction, and the one which actually fits the cosmological simulations (equation 2). The figure shows that our formula describes the first-crossing distribution of this barrier shape well. This means that one could, in principle, use our formula, with Monaco’s barrier, to study the conditional mass functions associated with his parametrization of non-linear collapse – we have not pursued this further, although comparison of this first-crossing distribution with the \( z = 0.5 \) panels in Figs 7 and 8 suggests that this might be a useful exercise.

In summary, we have provided a formula which describes the first-crossing distribution of independent uncorrelated Brownian motion random walks, for a wide class of moving barriers. This formula can be used to provide simple accurate predictions for a number of statistical quantities associated with the formation and clustering of dark matter haloes, all within the same formalism.

ACKNOWLEDGMENTS

Many thanks are due to Lauro Moscardini for his realistic skepticism that this project would ever be completed! We thank the TMR European Network ‘The formation and evolution of galaxies’ under contract ERBFMRX-CT96-086 for financial support. RKS is supported by the DOE and NASA grant NAG 5-10842 at Fermilab; he thanks the Astronomy Department at the University of Padova for hospitality in 2000 May.

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Figure 12. The unconditional mass function associated with a model of collapse which was presented in Monaco (1997a,b). Whereas the height of the barrier studied in our paper increases with distance, the one proposed by Monaco decreases with distance. Despite the dramatic differences between the two barriers, inserting this shape into our equation (4) provides a good fit (solid curve) to the exact result (histogram).
APPENDIX A: THE LINEAR BARRIER

This appendix studies a barrier whose height increases more steeply than the one considered in the main text. The main reason for doing this is that it is possible to write down exact analytical solutions for the first-crossing distributions. These analytical solutions illustrate a number of novel features associated with moving barrier models which are not present in the case of a constant height barrier. These features may allow one to model a wider variety of physical phenomena.

The barrier shape was also motivated by the observation that the GIP simulations have fewer low-mass haloes relative to high-mass ones, as compared to what is predicted by the constant barrier model. This means that, at least for some range of $S$, the moving barrier must have a positive slope, since this would make it relatively easier to cross at small $S$ (large mass) than at large $S$ (small mass), as compared to a barrier of constant height.

An additional reason for considering a barrier of increasing rather than decreasing height is the following. Recent work (Bode et al. 2001) suggests that the halo mass function in which the dark matter is warm initially has even fewer low-mass halos than cold dark matter based ellipsoidal collapse models predict. In the context of the approach outlined by Sheth et al. (2001), the physical reason for this is relatively simple: low-mass haloes do not form because they are hotter initially, so a larger overdensity is required to hold them together against the random motions which prevents collapse, or against the stronger shearing from the velocity field. This suggests that the critical density required for collapse by the present time, $\delta_c(m)$, should increase even more strongly with decreasing $m$ than it does when the dark matter is cold. The warm dark matter model is not particularly well motivated, and its free parameters have not yet been fixed; it seems premature to provide a detailed $\delta_c^{WDM}(m)$ relation at the present time. For this reason, the linear barrier considered in this appendix should be thought of as an example of what happens when the barrier height increases even more steeply with decreasing mass than it does in the Sheth et al. cold dark matter models.

For all these reasons, we will suppose that the barrier shape increases linearly with increasing variance $S = \sigma^2$:

$$B(S, z) = \delta_c(z)[1 + S/S_c(z)],$$

(A1)

where $\delta_c(z) = \delta_c(1 + z)$ and $S_c(z) = S_c(1 + z)^2$ if $\Omega = 1$. These scalings with $z$ are what is required by the self-similarity of Brownian motion. In the Bond et al. (1991) formulation of the constant barrier problem, it is customary to express the $S$ axis in the units it had at some fiducial time, say $z = 0$, and to study the successive crossings of barriers having different values of $z$. In effect, the constant height Press–Schechter barrier has $S_0 = \infty$, so in that case only the scaling of the $y$-axis was apparent.

The first-crossing distribution of trajectories that do cross this linear barrier is inverse Gaussian:

$$f(S, z) \, dS = \frac{B(0, z)}{\sqrt{2\pi S}} \exp \left[ -\frac{B^2(S, z)}{2S} \right] \, dS,$$

(A2)

(Sheth 1998). Integrating this over all $S$ shows that this distribution is not normalized to unity: only a fraction $\exp(-B^2(0, z)/2)$ of all walks cross the barrier. This is the first important difference between a moving barrier and the constant barrier model. When the barrier height is constant, all random walks are guaranteed to cross the barrier because the rms height of random walks at $S$ is proportional to $\sqrt{S}$. Therefore, at sufficiently large $S$, all walks will have crossed the constant barrier, and the associated first-crossing distribution is normalized to unity.

In the excursion set approach, each random walk is associated with a volume element in the initial Lagrangian space. If all walks cross the barrier, all the mass in the Universe is expected to be bound up in collapsed objects of some mass, however small. In contrast, the linear barrier (A1) increases to arbitrarily high values at high $S$. Because the rms height of the random walk grows more slowly than the rate at which the barrier height increases, there is no guarantee that all random walk trajectories will intercept this barrier. So, in the linear barrier model, not all initial volume elements are associated with bound haloes. Since not all particles in numerical cosmological simulations are associated with bound haloes anyway (the fraction of unbound mass is typically on the order of $\sim 10$ per cent, although how much of this is a consequence of limited resolution in the simulations is uncertain), this feature of the moving barrier model may or may not be a good thing. In any case, this is one qualitative difference between the moving boundary model and a model with constant barrier height. [Readers who dislike this feature of the linear model are invited to patch a boundary model and a model with constant barrier. When the barrier case, this is one qualitative difference between the moving barrier model may or may not be a good thing. In any case, this is one qualitative difference between the moving boundary model and a model with constant barrier height. [Readers who dislike this feature of the linear model are invited to patch a boundary model and a model with constant barrier. When the barrier]

In addition to the question of normalization, barriers whose heights increase with $S$ more rapidly than $\sqrt{S}$ will have mass functions in which the low-mass end is depleted relative to the constant barrier case. Again, this is because it becomes increasingly difficult to cross at higher $S$. In the linear barrier model, for example, the mass function (got by inserting equation A2 in equation 3) has an exponential cut-off at both low and high masses. In contrast, the constant barrier cuts off exponentially only at the high mass end. Thus the mass function associated with a linear barrier has fewer small mass objects than the constant barrier predicts, in qualitative agreement with the GIP simulations.

Since a linear barrier is linear whatever the origin of the coordinate system, the solution to the two-barrier problem is also inverse Gaussian. That is, given $z_1 > z_0$ and given that the first crossing of $B(z_0)$ occurred at $S_0 = S_{01}$, the probability that the first crossing of $B(z_1)$ occurs in the range $dS_1$ about $S_1$ is given by

$$p(S_1) \, dS_1 = B(0, z_1) \, dS_1 \exp \left[ -\frac{B^2(S_1, z_1)}{2S_1} \right],$$

where $B(z_1)$ is the barrier at $z_1$. This is the first- and second-crossing probability distribution for a constant height Press–Schechter barrier of height $S_{01}$, and the first- and second-crossing distributions for a linear barrier are both inverse Gaussian.

Figure A1. The conditional mass functions associated with the linear barrier. Smooth solid curves show the analytic linear barrier prediction, dashed curves show the conditional constant barrier distribution for comparison, and histograms show the result of our numerical Monte Carlo calculation.
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equation (A2), with the substitutions \( S \rightarrow (S_1 - S_0) \), and \( B \rightarrow B_{10} \), where

\[
B_{10}(S_1 - S_0) = B(S_1, z_1) - B(S_0, z_0).
\]

As was the case for the unconditional mass function, the height of this barrier diverges as \( S_1 - S_0 \rightarrow \infty \), so not all trajectories intersect it. Again, it seems reasonable to associate the fraction that do not with the fraction of the parent halo mass that is not associated with bound subclumps. (Those readers who computed the mass functions associated with the patched constant–linear–constant barriers may disregard the previous two sentences, provided they first compute the associated two-patch-barrier problem!)

The smooth solid curves in Fig. A1 show this conditional distribution for a few representative choices of the parent halo mass \( M \). The parents were assumed to have formed at \( z_0 = 0 \), and the progenitor distributions are shown at an earlier redshift \( z_i \). The underlying power spectrum was chosen to be the same as the GIF SCDM power spectrum. It is also straightforward to solve this problem numerically: for the values of \( M \) shown in the figure panels, the histograms show the distribution generated by simulating the crossings of the higher linear barrier by \( 10^3 \) random Brownian motion trajectories that started at the initial positions \( B(S, z = 0) \), where \( S(M) \) is given by the GIF power spectrum. Fig. A1 has been included mainly to show that the Monte Carlo code we use in the main text works. We have also verified that the numerical code gives the correct conditional and unconditional mass functions when the barrier heights are constant.

This two-barrier problem illustrates a second important qualitative difference between a moving barrier model and the original constant barrier model: whereas the y-intercept \( B(0, z) \) increases as \( z \) increases, the slope decreases as \( (1 + z)^{-1} \). This means that it is possible for barriers to intersect at finite values of \( S \). For example, two linear barriers \( B(S, z_0) \) and \( B(S, z_1) \), with \( z_1 > z_0 \), will intersect at that critical value of \( S \) at which \( B(S, z_0) = B(S, z_1) : S_0/S_{01} = (1 + z_0)/(1 + z_1) \). This means that all trajectories which first cross \( B(S_0) \) at \( S > S_{01} \) must have crossed \( B(S(z_1)) \) at a smaller value of \( S \). The logic of Lacey & Cole (1993) then says that all haloes at \( z_0 \) that are less massive than the associated critical mass \( M_{01} \) were formed by fragmentation of a halo that, at \( z_1 > z_0 \), was more massive.

Again, this may or may not be a good thing. Simulations show that \( \sim 20 \) per cent of the total mass ever associated with progenitors of a halo does not find its way to the parent halo (Tormen 1998). Presumably this reflects the fact that while small haloes may fragment, the fragmented mass is not a very large fraction of the total mass. Simulations also show that some subclumps pass through the virial radius of a given parent halo several times, each time depositing some fraction of their mass, before they finally become part of the parent halo. It may be that a moving barrier model is able to incorporate and perhaps even quantify these effects.

For completeness, we also show the bias relations associated with the linear barrier model. Following Mo & White (1996) and Sheth & Tormen (1999), the mean Lagrangian bias between haloes and mass is

\[
\delta_L^S(1) = f(S_1, z_1|S_0, z_0) - 1. \tag{A3}
\]

The limit \( M_{01} \gg M_1 \) and \( |\delta_L| \ll \delta_1 \) is sometimes called the peak–background split. In this limit

\[
\delta_L^S(1) \rightarrow (\nu/\delta_1) \delta_0, \quad \text{where} \quad \nu = \delta_L^S(z_1)/S_1
\]

for the linear barrier model. Massive haloes have small values of \( S \), so they have large values of \( \nu \). Small haloes have \( \nu \sim 0 \). Thus, in this limit, less massive haloes are unbiased relative to the mass, whereas massive haloes are positively biased. For comparison, the corresponding limit for the constant barrier is

\[
\delta_C^S(1) \rightarrow (\nu/\delta_1) (v_1 - 1/\delta_1),
\]

Thus the predictions of the linear and constant barriers are similar for massive haloes, but they differ for less massive ones. In particular, less massive haloes in Lagrangian space are antibiased in the constant barrier model, whereas they are unbiased in the linear barrier model.

The halo-to-mass bias in the evolved Eulerian space for the constant barrier can be computed by expanding

\[
\delta_C^S(1) = (1 + \delta)[1 + \delta_C^S(1)] - 1
\]

to lowest order in \( \delta \). In this limit, \( \delta = \delta_0 \), so

\[
\delta_C^S(1) = (1 + v_1 - 1/\delta_1) \delta
\]

for the constant barrier. The same logic gives

\[
\delta_L^S(1) = (1 + v_1/\delta_1) \delta
\]

for the linear barrier. Again, the predictions of the constant and linear barriers agree for massive haloes, but are different for less massive ones; the haloes in the linear barrier model are more positively biased.

The rate of increase of the ellipsoidal collapse barrier studied in the main text is shallower than for the linear barrier discussed here. Since it is intermediate between the linear barrier and the constant spherical collapse barrier, we might expect the number densities, and hence the large-scale bias factors, of less massive haloes in the ellipsoidal collapse model to be intermediate between those associated with the linear and constant barrier models. Fig. 4 of Sheth et al. (2001) shows that this is, indeed, the case.

We argued at the start of this appendix that warm dark matter models can be parametrized by making the critical density for collapse depend more strongly on mass than when the dark matter is cold. In this respect, a comparison of the linear barrier formulae given here with the results for the barrier written down by Sheth et al. (2001) shows why, generically, warm dark matter models are expected to have fewer low-mass haloes and, consequently, different bias relations, particularly at the low-mass end, than cold dark matter models. This difference at the low-mass end is in qualitative agreement with the numerical simulations of Bode et al. (2000).

APPENDIX B: DISTRIBUTION OF DENSITY AND ANGULAR MOMENTUM OF A PATCH IN A GAUSSIAN RANDOM FIELD

Let \( \delta_i = \nabla_i \phi \), where \( \phi \) is the initial potential, denote the various components of the deformation tensor \( \mathbf{D} \) (here \( 1 \leq i \leq 3 \) and similarly for \( j \)). Following, e.g., Bardeen et al. (1986), these components are

\[
\begin{align*}
\delta_{11} &= (-y_1 - 3y_2/\sqrt{15} - y_3/\sqrt{5})/3 \\
\delta_{12} &= (-y_1 + 2y_2/\sqrt{5})/3 \\
\delta_{13} &= (-y_1 + 3y_2/\sqrt{15} - y_3/\sqrt{5})/3 \\
\delta_{22} &= y_2/\sqrt{15} \\
\delta_{23} &= y_3/\sqrt{15} \\
\delta_{33} &= y_2/\sqrt{15},
\end{align*}
\]

\[
\begin{align*}
\text{(B1)}
\end{align*}
\]
where the \( y_i \) are independent Gaussian variates with zero mean and variance \( \sigma^2 \). Poisson’s equation says that the trace of this matrix, \( \text{Tr}(D) \), equals the overdensity \( \delta \). Thus
\[
\delta = d_{11} + d_{22} + d_{33} = -y_1; \tag{B2}
\]
the final expression shows explicitly that \( \delta \) is a Gaussian random variate.

The eigenvalues of this matrix are the roots of the characteristic equation
\[
P(\lambda) = \det[D - \lambda I] = -\lambda^3 - \lambda^2 a_2 - \lambda a_3 - a_2 a_3, \tag{B3}
\]
where \( I \) is the identity matrix, and the \( a_k \) are various combinations of the \( d_{ij} \) obtained by expanding the expression above and ordering by powers of \( \lambda \). Since \( D \) is a \( 3 \times 3 \) real symmetric matrix, \( P(\lambda) \) is a cubic with three real roots which satisfy
\[
\lambda_1 + \lambda_2 + \lambda_3 = -a_2
\]
\[
\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = a_1
\]
\[
\lambda_1 \lambda_2 \lambda_3 = -a_0. \tag{B4}
\]
Because rotations leave the trace unchanged, the overdensity \( \delta \) is the sum of the three eigenvalues, so \(-a_2 = \delta = -y_1\). In addition, the square of the angular momentum is proportional to
\[
r^2 = \frac{(\lambda_1 - \lambda_2)^2}{2} + \frac{(\lambda_2 - \lambda_1)^2}{2} + \frac{(\lambda_3 - \lambda_1)^2}{2} = a_2^2 - 3a_1, \tag{B5}
\]
(e.g. Heavens & Peacock 1988; Catelan & Theuns 1998). This shows that to get \( r^2 \) we do not need to solve the cubic – we just need to read off the appropriate coefficients of the characteristic equation. Thus we find that
\[
r^2 = \delta^2 + 3(d_{12}^2 + d_{13}^2 + d_{23}^2 - d_{11}d_{22} - d_{11}d_{33} - d_{22}d_{33})
= (y_2^2 + y_3^2 + y_1^2 + y_2^2 + y_3^2)y_5. \tag{B6}
\]

Although the first line suggests that \( r^2 \) is coupled to \( \delta \), the final expression shows that it is not. In particular, the expressions above show that \( \delta \) is distributed as a Gaussian, and \( r^2 \) is an independent variate drawn from a chi-square distribution with five degrees of freedom, \( \chi_5^2(\sigma) \). The fact that the overdensity and the square of the angular momentum are independent does not seem to have been noticed before. A \( \chi_5^2(\sigma) \) distribution is rather similar in shape to a lognormal, so this provides a simple way to see why spins of peaks in Gaussian random fields are also approximately lognormal (Heavens & Peacock 1988).

The above results can be used to generalize the excursion set algorithm studied in the main text. Set \( n = 0 \) and \( y_i(n) = 0 \) for \( 1 \leq i \leq 6 \). Thereafter, at each step labelled by \( n \), choose six, rather than one, independent Gaussian random variates \( g_i \), each with variance \( s \). For each \( i = 1, 6 \) set \( y_i(n) = y_i(n-1) + g_i \), and use these to compute \( \delta \) and \( r^2 \). These can be used to give the values of the overdensity and the angular momentum for the scale on which the variance is \( \sigma^2 \propto ns \) (recall that \( n \) is the number of steps taken).

Now check to see if \( \delta = -y_1(n) \) exceeds a critical value, say \( \delta_{\text{crit}}(\sigma, r^2) \). If it does, the six-dimensional walk stops at this scale. If not, the walk continues to smaller scales. Because each of the \( g_i \) is chosen independently of the values of the \( y_i \) or of \( \sigma \), the walk takes independent steps in the six-dimensional space; it is in this sense that this algorithm generalizes the one-dimensional excursion set random walk studied in the main body of this paper.

The analysis above shows clearly that the one-dimensional random walk approach of Sheth et al. (2001) corresponds to the following approximation. Replace the dependence of \( \delta_{\text{crit}}(\sigma, r^2) \) on the random variate \( r^2 \) by a dependence on its average value \( \langle r^2 \rangle \propto \sigma^2 \). This means that the critical density for collapse is a function of \( \sigma \) alone, \( \delta_{\text{crit}}(\sigma) \). As a result, the random walk in six-dimensions can be reduced to a walk in one-dimension only, thereby greatly reducing the complexity of the problem.

Recently, just such a six-dimensional random walk algorithm has been used by Chiueh & Lee (2001), although they did not notice the considerable simplifications which follow from the algebra presented above. They simulated an ensemble of six-dimensional random walks, and set the parameters of the barrier to be crossed by requiring that the resulting first-crossing distribution give the unconditional mass function. In particular, they showed that \( \delta_{\text{crit}} = 1.5[1 + (2r^2/3)^2/0.15]^{1.5} \) provided a good fit to the required critical value of the overdensity.

The algebra above allows one to see what such an approach implies. To do this, suppose the barrier is \( \delta_{\text{crit}} = 1.686(1 + r^2) \). The dependence on \( r^2 \) means that if the particle has walked to \( \delta = 1.686 \), it will still not have crossed, because \( r^2 \) is always certainly greater than zero. So, to cross, the particle needs to have some \( \delta > 1.686 \). How much greater? This depends on the typical value of \( r^2 \). Because \( r^2 \) is drawn from a \( \chi_5^2 \) distribution, \( \langle r^2 \rangle \sim \sigma^2 \). Now suppose that the \( \chi_5^2 \) distribution is very sharply peaked at its mean value (it is quite well peaked, but taking the extreme case helps to see the argument). This means that the barrier shape is something like \( \delta_{\text{crit}} = 1.686(1 + \sigma^2) \). The fact that a \( \chi_5^2 \) distribution is not very sharply peaked at its most probable value simply means that sometimes when \( \delta = 1.686(1 + \sigma^2) \) the particle will still be less than \( \delta_{\text{crit}} \), so the walk must go on. Of course, sometimes \( r^2 < \sigma^2 \), and in this case the walk will stop even if \( \delta < 1.686(1 + \sigma^2) \). So this means that we can think of the dependence on \( r^2 \) as making the critical value of the boundary height, when expressed as a function of \( \sigma^2 \) (the way Sheth et al. 2001 did) a little fuzzy. So, provided the \( \chi_5^2 \) distribution is not too broad, the considerably simpler one-dimensional random walk
approach of Sheth et al. (2001) should be a reasonable approximation.

Fig. B1 shows the result of doing this for two choices of the barrier shape, both of which produce first-crossing distributions which, when inserted into equation (3), give mass functions of bound objects which are similar to the one in simulations of hierarchical clustering. The upper panel was constructed using a barrier whose height increased linearly with $r^2$, and the lower panel shows results for the barrier shape used by Chiueh & Lee (2001), which increases as $r^4$. The crosses show the values of $\delta$ and $\sigma$ at which each six-dimensional random walk crossed the barrier. The solid curve shows the approximation used by Sheth et al. (2001); it provides a reasonable description of the increase of $\delta_{\text{crit}}$ with $\sigma$.

Determining the barrier shape by requiring agreement with the clustering simulations is unsatisfying, especially in view of the fact that the two different boundaries given above provide equally adequate approximations to the mass function. For this reason, the main text shows the result of combining the six-dimensional walk described above with the ellipsoidal collapse model of Bond & Myers (1996). This was relatively easy to do, because a simple fitting function for how the critical collapse boundary associated with this ellipsoidal collapse depends on the initial shear field has been given by Sheth et al. (2001, their equation 3 and their fig. 1).

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