$$\omega = \int_0^{\bar{t}} \frac{\left[\frac{1}{\alpha_1}(1 - e^{-\alpha_1 t}) + v_1^0\right] e^{-\alpha_2 t} - \left[\frac{1}{\alpha_2}(1 - e^{-\alpha_2 t}) + v_2^0\right] e^{-\alpha_1 t}}{\left[\frac{1}{\alpha_1}(1 - e^{-\alpha_1 t}) + v_1^0\right]^2 + \left[\frac{1}{\alpha_2}(1 - e^{-\alpha_2 t}) + v_2^0\right]^2}\, dt$$

$$+ 2\int_{\bar{t}}^{0} \frac{\left[\frac{1}{\alpha_1}(1 - 2e^{-\alpha_1 t} + e^{-\alpha_1 \bar{t}}) + v_1^0\right] e^{-\alpha_2 t} - \left[\frac{1}{\alpha_2}(1 - 2e^{-\alpha_2 t} + e^{-\alpha_2 \bar{t}}) + v_2^0\right] e^{-\alpha_1}}{\left[\frac{1}{\alpha_1}(1 - 2e^{-\alpha_1 t} + e^{-\alpha_1 \bar{t}}) + v_1^0\right]^2 + \left[\frac{1}{\alpha_1}(1 - 2e^{-\alpha_2 t} + e^{-\alpha_2 \bar{t}}) + v_2^0\right]^2} \cdot dt$$

$$+ \int_{\bar{t}}^{0} \frac{\left[\frac{1}{\alpha_1}(1 - e^{-\alpha_1 t}) - v_1^0\right]^2 e^{-\alpha_2 t} - \left[\frac{1}{\alpha_2}(1 - e^{-\alpha_2 t}) - v_2^0\right] e^{-\alpha_1 t}}{\left[\frac{1}{\alpha_1}(1 - e^{-\alpha_1 t}) - v_1^0\right]^2 + \left[\frac{1}{\alpha_2}(1 - e^{-\alpha_1 t}) - v_2^0\right]^2}\, dt, \quad (19)$$

and
$$\omega = \begin{cases} 2\pi, & \text{if } (v_1^0, v_2^0) \text{ is in } V^+, \\ 0, & \text{if } (v_1^0, v_2^0) \text{ is outside of } V^+. \end{cases} \quad (20)$$

(In the first two cases, the contribution from the second integral, which is the change in angle of the line from (v_1^0, v_2^0) to a point on the curve $t_2 = \bar{t}$ as the curve is traced, is seen to approach π as $t \to \infty$.)

Unfortunately, formula (19) is too complicated for practical application in such a simple system. We hope that further approximate analysis would reduce the formula to one which is easily mechanized.

Conclusion

The discussion demonstrates that there are ways of deciding which initial relay position will give time-optimal response for each set of initial conditions without explicitly solving the transcendental switching equations. The result presented is not a practical solution in its present development.

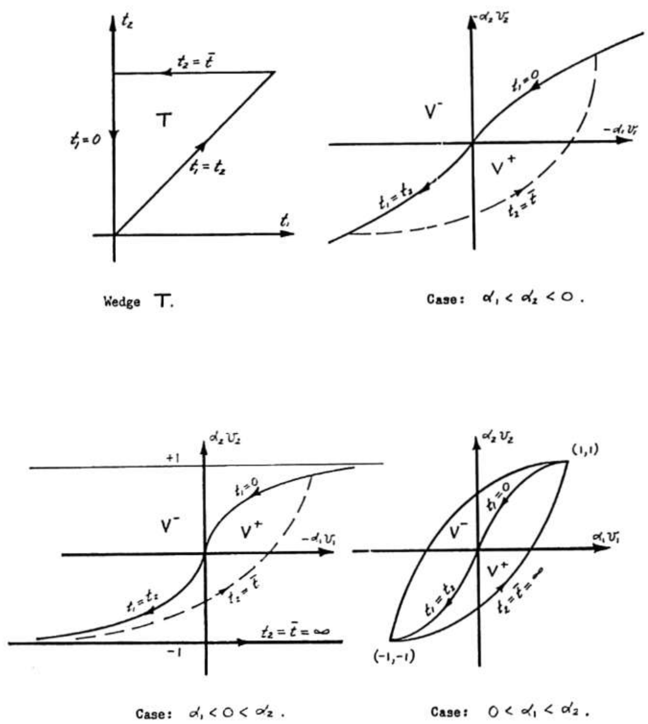

Fig. 1

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DISCUSSION

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The problem of determining the initial relay position for all initial states is exactly that of determining time-optimal feedback control. One wishes to obtain the control as a function of the state of the system—not $u(t)$ but $u(v_1, v_2, \ldots, v_n)$. Rang's idea is a good one but unfortunately deciding whether the surface integral is zero or not does not seem to be a simple matter. Even for the case $n = 2$ where it is not difficult to determine the switching curve equation (19) is convincing evidence that as yet this is not a practical method.

Let me make one minor point. In order that the system have the normal form (5) it must be that no component of the vector $P^{-1}b$ vanishes. This is equivalent to assuming that the system is controllable (proper).

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