Ray Theory for a Pre-strained Medium

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Summary

Because of the similarity between the equations of motion governing infinitesimal vibrations due to a small perturbing force superimposed on an already existing state of finite strain and the equations of linear anisotropic elasticity, methods of analysis used in one may be extended to the second.

In particular, in this paper, the technique of ray expansions is considered. The analysis is found to be virtually identical in the two cases although there are differences in physical interpretation.

1. Introduction

In Walton (1973), equations of motion were established for infinitesimal vibrations due to a small perturbing force superimposed on an already existing state of finite strain in an elastic medium. Because of their similarity in form to the equations for infinitesimal anisotropic elasticity, one may expect that methods of analysis used in such problems may also be used in the present case. In particular, we shall consider the so-called ray method for calculating amplitudes of displacements near wave fronts.

The analysis involved is virtually identical to that of anisotropic elasticity (Babich 1961; Vlaar 1968; Červený 1972). We follow most closely the approach used by Dr R. Burridge in lectures for the Mathematical Tripos at Cambridge 1971 and because of the similarity between the present work and the infinitesimal anisotropic case, the basic theory is given only briefly in Section 2.

One section of the present paper is devoted to the interpretation of the transport equation. The corresponding equation in infinitesimal anisotropic elasticity has been interpreted by Burridge in terms of energy conservation in a ray tube (for the case of isotropy, see Červený & Ravindra 1971). In Section 3, a similar result is obtained here, but in terms of incremental energy conservation.

Babich (1961) showed that the bicharacteristic rays satisfy the criterion of stationary time paths. In Section 4, an alternative proof of this result is given. Both proofs are valid for the present problem and for that of infinitesimal anisotropic elasticity.

Throughout the work, the perturbing force is assumed to be a point impulse with a view towards seismological applications. This also enables the result of Walton (1973) to be used in order to find initial values of the various unknown quantities. This is done in Section 5.

In the final section, the particular example of a Hadamard–Green material is considered in detail. This material has the property that the characteristic equation
factorizes regardless of the pre-strain imposed; that is, the ray equations for one
particular wave can be found explicitly.

The main purpose of the present paper is to lay down the mathematical found-
ations for a study of the seismological implications of pre-straining in the Earth, and,
in particular, in the neighbourhood of an earthquake source. These will be considered
in a further paper.

2. Ray theory for a pre-strained medium

Prior to the application of the pre-strain, the medium is assumed to be both
isotropic and homogeneous and at rest under no forces. A material point in this
state is denoted by \( \mathbf{X} \). After the application of a body force \( \mathbf{F}(\mathbf{X}) \), when the medium
is once more at rest, a material particle initially at the point \( \mathbf{X} \) is at the point \( \mathbf{x} \), where \x and \X
are both taken relative to some fixed Cartesian frame of reference.

This pre-strain is described by the deformation gradient \( A_{pq} \) which is defined by

\[
A_{pq} = \frac{\partial x_p}{\partial X_q}.
\]

Moreover, it is assumed that the material is hyperelastic, that is, it possesses a strain
energy function \( W \) which, by virtue of the assumed homogeneity and isotropy, will
be of the form

\[
W = W(I_1, I_2, I_3)
\]

where the \( I_n \) are the invariants of the tensor \( g_{pq} = A_{pq} A_{qs} \) (summation over repeated
suffices is to be understood, unless stated otherwise) and are given by

\[
I_1 = g_{pp}, I_2 = \frac{1}{2}(I_1^2 - g_{pq} g_{pq}), I_3 = \det(g_{pq}).
\]

The pre-strained state is assumed to be one of static equilibrium and hence, if \( \sigma_{pq} \)
denotes the Cauchy stress in this state, we have

\[
\sigma_{pq, q} + \rho F_{p,q} = 0
\]

where \( \rho(x) \) denotes the density in this state and where the convention \( B_{pq} = \partial B/\partial x_p \)
has been adopted.

Moreover, we assume that this pre-stress and the pre-strain are related by the
constitutive law of perfect elasticity for finite deformations

\[
\sigma_{pq} = 2I_3^{-1} \{ (W_1 + I_1 W_2) g_{pq} - W_2 g_{pq} g_{qr} + I_3 W_3 \delta_{pq} \}
\]

where \( W_a = \partial W/\partial I_a \).

From Walton (1973, equation (2.31)), the displacement \( u(x, t) \) of a particle from
its position \( x \) in the pre-strained state, due to a perturbing force \( f(x, t) \) satisfies the
equation

\[
(d'_{pqrs} u_{r,s})_{,q} + \rho F_{p,q} u_q + f_p = \rho u_p.
\]

The tensor \( d'_{pqrs} \) is analogous to the modulus tensor of linear elasticity and is given by

\[
d'_{pqrs} = d_{pqrs} + \sigma_{qs} \delta_{pr}
\]

\[
d_{pqrs} = 2I_3^{-1} \{ 2I_3(W_3 + I_3 W_{33}) \delta_{pq} \delta_{rs} - I_3 W_3(\delta_{pr} \delta_{qs}) + W_2\}
\]

\[
+ 2I_3 [(W_1 + I_1 W_{23})(g_{pq} \delta_{rs} + g_{rs} \delta_{pq}) - W_{23}(g_{pm} \delta_{qr} + g_{mr} \delta_{qm})]
\]

\[
- W_2(g_{ps} g_{qr} + g_{pr} g_{qs})
\]

\[
+ 2(W_2 + W_1 + 2I_1 W_{12} + I_1^2 W_{22}) g_{pq} g_{rs}
\]

\[
- 2(W_1 + I_1 W_{22})(g_{rs} g_{pq} g_{qm} + g_{pq} g_{mr} g_{sm}) + 2W_{22} g_{pm} g_{qm} g_{rm} g_{sm} \}
\]

(8)
where $W_{nm} \equiv \partial^2 W/\partial I_n \partial I_m$. We note that $d_{pqrs}$ possesses the same symmetries as in the case of linear elasticity (zero pre-strain), yet $d_{pqrs}^2$ possesses only the $(pq) \leftrightarrow (rs)$ symmetry.

Equation (6) is our starting point and we see that this equation is very similar to that governing the infinitesimal motion of an inhomogeneous, anisotropic elastic medium. The main differences are that the equation contains the extra term $(\rho F_{p,q} u_q)$ and that the $d_{pqrs}$ do not possess the full symmetries of the corresponding tensor in linear elasticity.

Because of the similarity between the two problems, one may expect that approximate methods available for one can be extended to the second. One particular method used to solve problems of wave propagation in homogeneous media is the technique of ray expansions and it is the purpose of this paper to develop and interpret this method when applied to pre-strained media.

The technique was first used in the study of anisotropic media by Babich (1961) and subsequently by Vlaar (1968) and Červeny (1972). We follow most closely the approach of Burridge’s Cambridge lectures and will merely reproduce the main points of the derivation. For a more detailed account, see any of the above references.

We shall only consider point impulsive perturbing forces $f(x, t)$ acting at the origin. Equation (6) may be replaced by

$$ (d_{pqrs} u_{r,s})_{q} + \rho F_{p,q} u_q - \rho u_p = 0 \quad \text{for} \quad t > 0, \; x \neq 0 $$

(9)

together with suitable initial conditions, about which more will be said later.

To solve this equation, a ray series solution of the following form is assumed

$$ u_p(x, t) = \sum_{n=0}^{\infty} A_n(x) E_n(t - S(x)) $$

(10)

where $dE_n(r)/dr = E_{n-1}(r)$. The surface $S(x) = t$ corresponds to the wave front.

Following the standard procedure of inserting this expansion into equation (9) and equating coefficients of $E_n$ to zero, we obtain

$$ (d_{pqrs} h_q h_s - \rho \delta_{pr}) A^{n+2}_{r,s} - d_{pqrs} (h_q A^{n+1}_{r,s} + h_s A^{n+1}_{r,q} + h_s A^{n+1}_{q,s}) - d_{pqrs} h_q A^{n+1}_{r,s} $$

$$ + (d_{pqrs} A^{n+1}_{r,s} + d_{pqrs} A^{n+1}_{r,q} + \rho F_{p,q} A^{n+1}_q) = 0 \quad \text{for} \quad n = -2, -1, 0, \ldots $$

(11)

where $A^{-2} = A^{-1} = 0$ by definition and where $h = \nabla S$. This equation thus represents a system of equations for the coefficients $A_n$.

The first of these is given by taking $n = -2$. Equation (11) reduces to

$$ (d_{pqrs} h_q h_s - \rho \delta_{pr}) A^0_r = 0. $$

(12)

For this to have a non-zero solution for $A^0$, we require

$$ G(h, x) \equiv \det (D_{pr}) = 0 $$

(13)

where we have defined

$$ D_{pr}(h, x) = d_{pqrs} h_q h_s - \rho \delta_{pr}. $$

(14)

Equation (13) is known as the characteristic equation and is directly analogous to the equation for the slowness surface as defined for the case of a homogeneous pre-strain (Walton 1973).

From equation (12), we conclude that $A^0$ must be a null-vector of $D_{pr}(h, x)$. By the symmetry of $d_{pqrs}$, $D_{pr}$ must also be symmetric and hence right and left eigenvectors of $D_{pr}$ will be identical. We denote by $B^m(h, x)$, the three unit eigenvectors of $D_{pr}(h, x)$ and by $\lambda^m(h, x)$, the corresponding eigenvalues, regarded as functions of $h$ and $x$; that is

$$ D_{pr}(h, x) B^m_{r}(h, x) = \lambda^m(h, x) B^m_{p}(h, x) \quad \text{for} \quad m = 1, 2, 3. $$

(15)
The condition for $\mathbf{B}^m(h, x)$ to be a null-vector is then

$$\lambda^m(h, x) = 0.$$  \hspace{1cm} (16)

These, of course, are the three roots of equation (13).

For simplicity, we assume that the $\lambda^m$ are distinct. Since $A^0$ must be in the direction of one of the $\mathbf{B}^m$, we may write

$$A^0 = \sigma^0 \mathbf{B}$$  \hspace{1cm} (17)

where the superscript $m$ has been dropped. $\sigma^0$ is the amplitude of $A^0$ and to calculate this, we return to equation (11) and consider the case when $n = -1$, which may be written

$$(d'_{pqr} h_q h_s - \rho \delta_{ps}) A_r^1 - d'_{pqr}(h_q A^0_{r,s} + h_s A^0_{r,q} + h_q, s A^0_r) - d'_{pqr}, q h_s A^0_r = 0.$$  \hspace{1cm} (18)

Eliminating $A^1$ by contracting equation (18) with $B_p$ and replacing $A^0$ by $\sigma^0 \mathbf{B}$, we obtain

$$2d'_{pqr} B_p B_r h_q \sigma^0_{,s} + \sigma^0(d'_{pqr} B_p B_r h_s), q = 0$$  \hspace{1cm} (19)

taking into account the symmetry of $d'_{pqr}$. This first order partial differential equation for $\sigma^0$ can be simplified by the introduction of the bicharacteristic rays.

To define these, let us write equation (16) in the form

$$H(h, x) = 1$$  \hspace{1cm} (20)

where $H(h, x)$ is a second degree homogeneous function of $h$. It can be seen from the form of $D_{pr}(h, x)$ that this is always possible. The bicharacteristic rays are then defined by

$$\frac{dx_p}{d\xi} = \frac{1}{2} \frac{\partial H}{\partial h_p}$$  \hspace{1cm} (21)

where $\xi$ is some measure along the ray.

An immediate consequence (Courant & Hilbert 1962) of this is that equation (20) is equivalent to the equation

$$\frac{dh_p}{d\xi} = -\frac{1}{2} \frac{\partial H}{\partial x_p}.$$  \hspace{1cm} (22)

From the homogeneity of $H(h, x)$, a second consequence is

$$\frac{dS}{d\xi} = 1.$$  \hspace{1cm} (23)

Thus we may identify $\xi$ with $S$ and hence, on the wave front $S(x) = t$, we may identify $\xi$ with $t$, thus enabling $dx/d\xi$ to be interpreted as the local wave speed (that is, the wave velocity along the ray).

We shall now simplify equation (19) by showing that the direction of differentiation in the equation is along a bicharacteristic ray. We consider the two equations

$$B_p (d'_{pqr} h_q h_s - \rho \delta_{ps}) B_r = 0$$  \hspace{1cm} (24)

$$H(h, x) = 1.$$  \hspace{1cm} (25)

Since these both represent the same surface in $h$-space, we may obtain two expressions for the normal to this surface at the point $h$ and hence for some scalar quantity $\phi$,

$$\phi \frac{\partial H}{\partial h_q} = 2d'_{pqr} B_p B_r h_s.$$  \hspace{1cm} (26)
To calculate $\phi$, we contract this equation with $h_q$ and utilize the homogeneity of $H$ to obtain
\[ \phi = d_{pqrs} B_p B_r h_q h_s = \rho. \] (27)

The ray derivative may now be written
\[ \frac{d}{d\xi} = \nabla \cdot \nabla \] (28)

where
\[ \rho V_s = \frac{dX_s}{d\xi} = d_{pqrs} B_p B_r h_q \] (29)

$V$ may be interpreted as the wave velocity along the ray.

Equation (19) for the amplitude $\sigma^0$ now reduces to the simple form
\[ \frac{d\sigma^0}{d\xi} + \frac{\sigma^0}{2\rho} (\nabla \cdot \rho \nabla) = 0. \] (30)

This equation can be easily integrated to give
\[ \sigma^0(\xi) = \sigma^0(\xi_0) \exp \left\{ - \int_{\xi_0}^\xi \frac{\nabla \cdot \rho \nabla}{2\rho} d\xi \right\} \] (31)

where the integral is taken along the ray.

In particular, we note that because of the exponential dependence of $\sigma^0$, $A^0$ will be zero at one point on a ray if and only if it is zero at all points on that particular ray.

Equations (12), (21), (22) and (31) thus, in theory, give us the equations for the rays, the equation for the wave front and the first term in the ray expansion (10) provided certain initial conditions are known.

In most cases, it is sufficient to consider only the first term in detail. However, the remaining terms can be calculated in an analogous manner to that given above.

It is worth noting that, as yet, the extra term $\rho F_{pq} u_q$ in equation (9) has played no part. However, it will be effective in the calculation of the remaining coefficients.

The only difference, so far, between the present problem and that of anisotropic elasticity has therefore been the replacement of the tensor of elastic moduli by the tensor $d_{pqrs}$. However, there are certain differences in physical interpretation.

### 3. The transport equation

Equation (2.30) is called the transport equation and in linear elasticity, it may be interpreted in terms of conservation of energy. That is, the energy flux of the first term of the ray series expansion is constant along a ray tube (for isotropic elasticity see Červený & Ravindra 1971). An analogous result may be obtained here in terms of incremental energy.

Firstly, we note that equation (2.30) may be rewritten, with the aid of equation (2.28) in the form
\[ \nabla \cdot [\rho(\sigma^0)^2 \nabla] = 0 \] (1)

and hence for any surface $S$ with normal $\mathbf{n}$,
\[ \int_S \rho(\sigma^0)^2 \nabla \cdot \mathbf{n} dS = 0. \] (2)

In particular, if $S$ is the surface of a ray tube, $\nabla$ is orthogonal to $\mathbf{n}$ on the sides of the tube and hence there is no contribution to the integral from this part of $S$. Thus only the ends contribute and we conclude that the quantity $\{\rho(\sigma^0)^2 \nabla \cdot \mathbf{n} dS\}$ is constant along an elemental ray tube with cross-section $dS$. 
It is this result we wish to interpret in terms of conservation of energy. We consider an arbitrary volume \( V \) of the material when it is in the pre-strained state (before the perturbing force is applied). Let this volume be denoted by \( V_0 \) prior to the application of the finite pre-strain. After the application of the perturbing force, the total strain energy \( \omega \) of the volume is then given by

\[
\omega = \int_{V_0} W(A_{px} + \partial u_p/\partial X_x) \, dV_0
\]

(3)

where \( W \) is the strain energy function as defined in equation (2.2). Differentiating equation (3) with respect to time, we find that the rate of change of the total strain energy is given by

\[
\dot{\omega} = \int_{V_0} \frac{\partial W}{\partial A_{px}} (A_{q\beta} + \partial u_q/\partial X_{\beta}) \frac{\partial \dot{u}_p}{\partial X_x} \, dV_0.
\]

(4)

Now, since \( \partial u_q/\partial X_\beta \) is an infinitesimal quantity, equation (4) may be expanded to give

\[
\dot{\omega} = \int_{V_0} \left\{ \frac{\partial W}{\partial A_{px}} (A_{q\beta}) + \frac{\partial u_r}{\partial X_\gamma} \frac{\partial^2 W}{\partial A_{px} \partial A_{\gamma\beta}} (A_{q\beta}) \right\} \frac{\partial \dot{u}_p}{\partial X_x} \, dV_0.
\]

(5)

Utilizing the expressions

\[
\begin{align*}
\frac{dV_0}{\partial X_x} &= u_{p,q} A_{px} \\
\sigma_{pq} &= I_3^{-\frac{1}{2}} A_{q\beta} \frac{\partial W}{\partial A_{px}} (A_{r\beta})
\end{align*}
\]

(6)

we may write equation (5) as an integral over the volume \( V \), in the form

\[
\dot{\omega} = \int_{\tilde{V}} \left\{ \sigma_{pq} + I_3^{-\frac{1}{2}} u_{r,s} A_{q\beta} \frac{\partial^2 W}{\partial A_{px} \partial A_{r\beta}} \right\} \dot{u}_{p,q} \, dV.
\]

(7)

Finally, it follows from Walton (1973) that

\[
\sigma_{pq} = I_3^{-\frac{1}{2}} A_{q\beta} \frac{\partial W}{\partial A_{px}} (A_{r\beta})
\]

(8)

and hence, equation (7) becomes

\[
\dot{\omega} = \int_{\tilde{V}} (\sigma_{pq} + \sigma_{pq}' u_{r,s}) \dot{u}_{p,q} \, dV.
\]

(9)

To calculate the rate of change of the total energy \( E \), we also require the kinetic energy \( K \). This is given by

\[
K = \frac{1}{2} \int_{\tilde{V}} \rho \dot{u}^2 \, dV.
\]

(10)

Thus, the rate of change of the total energy \( \dot{E} = \omega + K \), is given by

\[
\dot{E} = \int_{\tilde{V}} \left\{ \sigma_{pq} \dot{u}_{p,q} + \sigma_{pq}' u_{r,s} \dot{u}_{p,q} + \rho \dot{u}_p \dot{u}_p \right\} \, dV.
\]

(11)
Finally, using the equations of motion (2.9), provided the volume $V$ does not contain the origin, we may write equation (11) in the form

$$\dot{E} = \int_S \sigma_{pq} \dot{u}_p \dot{u}_q dS + \int \rho \dot{u}_p (F_p + F_{p,q} u_q) dV + \int d^{pqrs}_{\text{pars}} u_r \dot{u}_s dS$$

making use of the divergence theorem. The first integral in this equation may be interpreted as the rate of working of the surface forces on $S$ due to the pre-strain and the second integral as that of the body force causing the pre-strain. There remains the final integral and this may be interpreted as the rate of change of the incremental energy; that is, the quantity $-(d^{pqrs}_{\text{pars}} u_r \dot{u}_s)$ corresponds to the incremental energy flux.

Evaluating this expression when $\dot{u}$ is given by the first term in the ray expansion for one particular wave, we obtain

$$\dot{d^{pqrs}_{\text{pars}} u_r \dot{u}_s} = \rho (\sigma^0)^2 V_q (E_0')^2.$$  

Comparing this with equation (2), we may deduce the result that the incremental energy flux due to a single wave through a ray tube is constant moving along with the wave front.

4. Fermat's principle

Babich (1961) showed that the bicharacteristic rays satisfied the criterion of stationary time paths or, more correctly, that the ray paths are extremals of a certain line integral. The proof is also valid for pre-strained media but here we present an alternative proof of the same result. The approach follows that used by Červený & Ravindra (1971) for isotropic elasticity.

The problem is to show that the line integral

$$I = \int_{x_0}^{x_1} ds$$

is stationary when the path of integration connecting the fixed end-points $x_0$ and $x_1$ is along a bicharacteristic ray; $s$ is the distance measured along the path and $V(x, t)$ (where $t = dx/ds$, the tangent to the curve) is the magnitude of the wave velocity in the direction $t$.

We consider an arbitrary curve $\Gamma$ connecting $x_0$ and $x_1$, and denote a point on $\Gamma$ by $y$ and the tangent to $\Gamma$ at the point $y$ by $t(y)$. To calculate $V(x, t)$, we consider the bicharacteristic ray through $y$ in the direction $t(y)$. The direction of the corresponding $h(y)$ is then obtained from equation (2.21). The magnitude of $h(y)$ is then known from equation (2.20). With $h(y)$ known, we may calculate $V(y, t(y))$ from equation (2.29).

Writing $x = (x, y, z)$ and considering $x$ as the ray variable, we may write equation (1) in the form

$$I = \int_{x_0}^{x_1} [V(x, t)]^{-1} (1 + y'^2 + z'^2)^{\frac{1}{2}} dx$$

where $y' = dy/dx$ and $z' = dz/dx$ as measured along $\Gamma$.

The conditions for such an integral to be stationary for variations in $\Gamma$ are given by Euler's equations

$$\frac{d}{dx} \left\{ \frac{y'}{V(1+y'^2+z'^2)^{\frac{1}{2}}} - \frac{(1+y'^2+z'^2)^{\frac{1}{2}} \partial V}{V^2} \frac{\partial}{\partial y} \left( \frac{1}{V} \right) \right\} = (1+y'^2+z'^2)^{\frac{1}{2}} \frac{\partial}{\partial y} \left( \frac{1}{V} \right)$$

$$\frac{d}{dx} \left\{ \frac{z'}{V(1+y'^2+z'^2)^{\frac{1}{2}}} - \frac{(1+y'^2+z'^2)^{\frac{1}{2}} \partial V}{V^2} \frac{\partial}{\partial z} \left( \frac{1}{V} \right) \right\} = (1+y'^2+z'^2)^{\frac{1}{2}} \frac{\partial}{\partial z} \left( \frac{1}{V} \right)$$

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These two equations may be combined and written in the following vector form

\[
\frac{d}{ds} \left\{ \frac{1}{V} t - \frac{1}{V^2} \frac{DV}{Dt} \right\} = \nabla \left( \frac{1}{V} \right) \tag{4}
\]

where \( D/Dt \) represents the normal derivative and is defined by

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} - t \left( \frac{\partial}{\partial t} \right).
\]

We remark that since \( t \) is a unit vector \( \partial/\partial t \) is not well defined however \( D/Dt \) is well defined.

Equation (4) represents three equations. Two of these follow automatically from equations (3). The third can be proved from the definition of \( V \).

We require equation (4) in terms of \( h \) and \( H \) instead of \( V \) and \( t \). Now, for a given \( x \) and \( t \), \( h(x, t) \) and \( V(x, t) \) are defined by equations (2.20) and (2.21).

\[
2Vt = \frac{\partial H}{\partial h} \tag{6}
\]

\[
H(h, x) = 1 \tag{7}
\]

and hence, by the homogeneity of \( H(h, x) \),

\[
[t, h] V = 1. \tag{8}
\]

Since these are valid for all \( x \) and all unit vectors \( t \), we may differentiate equation (7) with \( D/Dt \) to obtain

\[
t_q \frac{Dh_q}{Dt_p} = 0 \tag{9}
\]

and equation (8) to obtain

\[
\frac{DV}{Dt} = Vt - V^2 h \tag{10}
\]

where we have made use of equation (9).

Equation (4) can now be much simplified and written

\[
\frac{dh}{ds} = \nabla \left( \frac{1}{V} \right). \tag{11}
\]

To eliminate \( V \) from this equation, we again use equation (8) to obtain

\[
\nabla_p \left( \frac{1}{V} \right) \equiv \left[ \frac{\partial}{\partial x_p} \left( \frac{1}{V} \right) \right]_t = t_q \left( \frac{\partial h_q}{\partial x_p} \right)_t. \tag{12}
\]

Also from equation (7),

\[
0 = \left( \frac{\partial H}{\partial x_p} \right)_t = \left( \frac{\partial H}{\partial x_p} \right)_h + 2Vt_q \left( \frac{\partial h_q}{\partial x_p} \right)_t. \tag{13}
\]

Finally, then, by combining equations (11)-(13) and utilizing the relationship \( ds = V \, d\xi \), we obtain

\[
\frac{dh}{d\xi} = -\frac{1}{2} \left( \frac{\partial H}{\partial x} \right)_h. \tag{14}
\]

This equation we immediately recognize as equation (2.22) which is satisfied if and only if the curve \( \Gamma \) is a bicharacteristic ray. Thus we have shown that the bi-
characteristic rays are extremals of the integral $I$ and hence the travel time between two fixed points is stationary along a ray.

5. Application of the initial conditions

In Section 2, the first term in the ray expansion was calculated for just one of the waves. There will, of course, be three such terms in the full solution, one corresponding to each of the eigenvectors $B^m$, that is, one corresponding to each wave. Hence the full solution takes the form

$$u(x, t) = \sum_{m=1, 2, 3} \sigma^{m0}(x) B^m(x) E_0(t - S^m(x))$$

where the three wave fronts are given by $S^m(x) = t$.

To obtain an explicit expression for the above we require initial values of $\sigma^{m0}$ and $h^m$. Now since we are only considering point impulsive perturbing forces at the origin, let us consider a small region $\Omega[|x| < \epsilon, t < \epsilon_t]$. The actual solution and the one obtained by replacing the values of the pre-strain in $\Omega$ by the value at the origin will be asymptotically equal as $\epsilon$ and $\epsilon_t$ tend to zero.

In $\Omega$, the pre-strain is constant (with $d_{pqrs} = d_{pqrs}^0$, $h = h^0$, say). From Walton (1973), equation (2.97), we know that for the case of a homogeneous pre-strain, the first term in the ray expansion is given by

$$u_p(x, t) = \sum_{m=1, 2, 3} \frac{(A^m_{q} K_{qs} \eta^m_{s}) A^m_p}{2\pi r |V_q G^m| \sqrt{|K^m|}} \delta(t + \eta^m(x))$$

where

$$G^m = \det(D^{m0})$$

$$D^{m0}_{pr} = d^{0}_{pqrs} \eta^m_q \eta^m_s - \rho^0 \delta_{pr}$$

$K^m$ is the Gaussian curvature on the surface $G^m = 0$ at the point $\eta^m$ and $A^m$ is the null-vector of $D^{m0}_{pq}$. The tensor $K_{qs}$ depends on the particular source function. Finally, the summation is over the three points $\eta^m$, one on each sheet of the slowness surface $G = 0$, satisfying the condition that the normal to this surface at the point $\eta^m$ is in the direction $x$ and such that $\eta^m(x) < 0$.

Comparison of equations (1) and (2) then yields, in the limit $\epsilon \to 0$, $\epsilon_t \to 0$

$$h^m \sim -\eta^m$$

$$\sigma^{m0} \sim \frac{(A^m_q K_{qs} \eta^m_s)}{2\pi r |V_q G^m| \sqrt{|K^m|}}$$

for $m = 1, 2, 3$. (5)

and moreover that $E_0(r) = \delta(r)$. In deriving the first of equations (5) we have used the fact that $\eta^m(x)$ is a zeroth degree homogeneous function of $x$ (Walton 1973).

Thus, the relevant initial conditions may be calculated and hence, in theory, the first term in the ray expansion found. For a given pre-strain, $d_{pqrs}^0$ may be calculated, provided the form of the strain energy function is known. Once this is known, the problem of a homogeneous pre-strain can be solved as described in Walton (1973). From this solution, initial values for the ray equations can be obtained. With these, the ray equations can be solved and the final solution obtained.

In practice, algebraic difficulties prevent explicit solutions being found, except in certain simple cases, although numerical methods may be used. In the following section, we shall consider the example of a Hadamard–Green material since this material has an extremely simple form of strain energy function.

However, before doing this, we consider the nodal surfaces of solution (1). Nodal surfaces are defined as surfaces on which the wave displacement has zero amplitude. They play an important role in practical seismology; in particular, in the
fault–plane solutions of earthquakes (Scheidegger 1957) and hence their shape in a pre-strained medium is of interest.

It was shown in Section 2 that the wave amplitude is zero at one point on a ray if and only if it is zero at all points on that particular ray. Thus, the nodal surface for a particular wave is made up of bicharacteristic rays. Moreover, from equation (5) the following condition must also hold at the origin

\[ A_m K_{qs} \eta_m^q = 0. \]  

This equation may be thought of as a restriction on \( \eta^m \) at the origin or together with the ray equation (2.21) as a restriction on the initial directions of the rays at the origin. The surface generated by the rays satisfying this restriction is the nodal surface of the particular wave under consideration.

6. Hadamard–Green materials

In Walton (1973), the example of a Hadamard–Green material was considered in detail. This material was chosen for two reasons. Firstly, it has a particularly simple form of strain energy function and secondly, it has the property that the characteristic equation always factorizes, regardless of the pre-strain imposed. So as an example of the preceding theory we shall again consider a Hadamard–Green material.

A material is defined as Hadamard–Green if and only if it possesses a strain energy function \( W \) of the form (Ogden 1970)

\[ W = AI_1 + BI_2 + f(I_3) \]  

where the \( I_n \) are defined in equation (2.3), \( A \) and \( B \) are constants and \( f \) is an arbitrary function subject to certain conditions at \( I_3 = 1 \). These conditions are found to be (Walton 1973)

\[
\begin{align*}
3A + 3B + f(1) &= 0, \\
A + 2B + f'(1) &= 0 \\
4[f''(1)+f'(1)+B] &= \lambda, \\
2(A+B) &= \mu
\end{align*}
\]  

where \( \lambda \) and \( \mu \) are the Lamé constants.

With this form for the strain energy function \( W \), equation (2.14) for the quantity \( D_{pr}(h,x) \) becomes.

\[
D_{pr}(h,x) = (h_q \sigma_{qs} h_s - 2I_3^{-\frac{1}{2}}f'(I_3) h_q h_q - \rho) \delta_{pr} + 2I_3^{-\frac{1}{2}}(f'(I_3)+2I_3 f''(I_3)) h_p h_s + 2I_3^{-\frac{1}{2}} B(g_{pq} h_q g_{rs} h_s - g_{pr} h_q g_{qs} h_s). 
\]  

The next step is to find the eigenvalues and eigenvectors of this matrix. Clearly \( h \) will always be an eigenvector of this regardless of the form of \( \sigma_{pq} \) and \( g_{pq} \). It was shown in Walton (1973) that for the case of a homogeneous pre-strain, this eigenvector corresponds to the quasi P-wave and hence it will also correspond to that here.

The particular root of the characteristic equation corresponding to this eigenvector is given by

\[ H(h,x) = \frac{1}{\rho} \{ h_q \sigma_{qs} h_s + 4I_3^{\frac{1}{2}}f''(I_3) h_q h_q \} = 1. \]  

When there is no pre-strain, this equation reduces to

\[
\left( \frac{\lambda+2\mu}{\rho} \right) h_q h_q = 1
\]  

which we immediately recognize as the equation for the P-wave slowness surface and which confirms that equation (4) corresponds to the quasi P-wave.
Inserting the expression (2.21), we find that the bicharacteristic rays for the quasi P-wave are given by

\[
\frac{dx_p}{d\xi} = \frac{1}{\rho} \left( \sigma_{pq} h_q + 4I_3 \frac{\delta}{\delta h} f''(I_3) h_p \right)
\] (6)

and also, the second ray equation (2.22) becomes

\[
\frac{dh_p}{d\xi} = -\frac{1}{2} \left( \frac{\sigma_{pq}}{\rho} \right) h_q h_s - 2(I_3 \frac{\delta}{\delta h} f''(I_3))_p h_q h_q.
\] (7)

In general, these may not be integrated to give explicit expressions for the rays although in several simple cases, it is possible to obtain equations for the rays in integral form. One of these of particular interest is that of the inhomogeneous shear pre-strain.

Thus, let us consider the shear pre-strain defined by the deformation gradient

\[
A_{ps} = \delta_{ps} + \tau(z) \delta_{p1} \delta_{s3}.
\] (8)

This corresponds to a shear in the x-direction which is a function of only the coordinate z.

For such a pre-strain in a Hadamard–Green material, the Cauchy pre-stress is given by

\[
\sigma_{pq} = \tau \mu (\delta_{p1} \delta_{q3} + \delta_{p3} \delta_{q1}) + \tau^2 \mu \delta_{p1} \delta_{q1} + 2\tau^2 B \delta_{p2} \delta_{q2}.
\] (9)

Equation (6) for the bicharacteristic rays for the quasi P-wave then reduces to

\[
\rho \frac{dx}{d\xi} = (\lambda + 2\mu) h_1 + \tau^2 \mu h_1 + \tau \mu h_3
\]

\[
\rho \frac{dy}{d\xi} = (\lambda + 2\mu) h_2 + 2\tau^2 B h_2
\]

\[
\rho \frac{dz}{d\xi} = (\lambda + 2\mu) h_3 + \tau \mu h_1
\] (10)

and the second ray equation (7) yields

\[
\frac{dh_1}{d\xi} = \frac{dh_2}{d\xi} = 0.
\] (11)

This implies that the quantities \(h_1\) and \(h_2\) are constant along any ray.

Finally, equation (4) becomes

\[
(\lambda + 2\mu) h_4 h_4 + 2\tau \mu h_1 h_3 + \tau^2 \mu h_1^2 + 2\tau^2 B h_2^2 = \rho.
\] (12)

Regarding \(z\) as the ray variable, we may combine equations (10) and integrate to obtain

\[
x = \int_0^z \frac{(\lambda + 2\mu) h_1 + \tau^2 \mu h_1 + \tau \mu h_3}{(\lambda + 2\mu) h_3 + \tau \mu h_1} \, dz
\] (13)

\[
y = \int_0^z \frac{(\lambda + 2\mu) h_2 + 2\tau^2 B h_2}{(\lambda + 2\mu) h_3 + \tau \mu h_1} \, dz.
\] (14)

Equations (13) and (14) are the equations for the rays. A particular ray is parameterized by values of \(h_1\) and \(h_2\). Then, if \(\tau(z)\) is known, \(h_3(z)\) can be found from
equation (12) and equations (13) and (14) can then, in theory, be integrated. In practice, this may be done numerically. Once the rays are known, it is a routine procedure to find the first term in the ray expansion (2.10).

Returning to the case of a general pre-strain and considering the other roots of the characteristic equation, we find that they may be written in the form

\[ 2I_3^{-\frac{1}{2}} \{ BQ^n + Ah_q g_{qs} h_s \} = \rho \quad n = 1, 2 \]  

(15)

where the \( Q^n \) are the roots of the quadratic equation

\[ Q^2 + Q(h_p g_{pq} g_{qr} h_r - I_1 h_q g_{qs} h_s) + I_3 (h_q h_d)(h_p g_{pq} h_r) = 0. \]  

(16)

These correspond to the quasi S-waves and, in general, are distinct. In fact, it can be shown that the only case when they are not distinct is when the pre-strain is hydrostatic.

Since, in general, equation (16) does not factorize (not even in the case of simple shear), the ray equations cannot be written down explicitly.

In conclusion, we state that very little progress can be made analytically for a general elastic medium in a general state of pre-strain. Even when we consider a material as simple as a Hadamard–Green material in a state of simple shear pre-strain, the bicharacteristic rays for the quasi P-wave can only be found in integral form and those for the quasi S-wave can only be found in terms of the roots of the quadratic equation (16).

If we wish to make further progress, we are posed with two alternatives. The first of these is to use numerical methods. The second is to use a fact that has virtually been ignored in the present paper and that is, the magnitude of the pre-strain. Although this is not infinitesimal, it is small and hence expansion techniques may be used.

In a further paper (Walton 1973, in preparation) on the seismological applications of the present work, use is made of this fact in order to obtain some explicit results on the effect of pre-straining. Two problems are considered; they are the fault-plane solutions for the radiation pattern of earthquakes and S-wave dispersion.

In the first of these, under the assumption that the pre-strain in the neighbourhood of an earthquake source is predominantly pure shear, the ray equations (2.22) and (2.23) reduce to integrable form and hence an analytic expression for the P-wave nodal surface can be found. This may be compared with the familiar pair of orthogonal planes (one in the plane of the fault and one orthogonal to the direction of slip) predicted by the theory for an unstrained material.

When the S-waves are considered, it is found that the roots of the characteristic equation corresponding to these waves can only be obtained in the form of roots of a certain quadratic equation, as is the case when a Hadamard–Green material is considered. However it is possible to obtain from this equation an approximate expression for the difference in arrival times of the two S-waves and hence, to calculate the amount of dispersion due to pre-straining.

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