Large t Limit of Compton Scattering of a Scalar Photon by a Scalar Nucleon in Perturbation Expansion

Jun OTOKOZAWA and Hiroshi SUURA

Department of Physics
and
Atomic Energy Research Institute
College of Science and Engineering, Nihon University
Kanda-Surugadai, Tokyo

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Ladder diagrams for Compton scattering of a massless scalar photon by a scalar nucleon are evaluated in the large t limit. The leading term in the 2n-th order is found to be \((\ln t)^{2n-2}/t\), compared to \((\ln t)^{n-1}/t\) for the case of scattering of two scalar particles of finite masses. In terms of partial waves, the 2n-th order diagram gives a leading singularity \(1/(l+1)^{2n-1}\) at \(l = -1\), compared to \(1/(l+1)^n\) for the finite mass case. Regge poles generated by summing ladder diagrams and the corresponding large t behavior are discussed.

§ 1. Introduction and summary

The large t (momentum transfer squared) limit of scattering amplitudes for spinless particles in perturbation expansion, or corresponding singularities in the l plane have been investigated by a great number of authors. As far as the leading Regge pole is concerned, any simple method leads to the same result. A straightforward evaluation of ladder diagrams in the large t limit by Federbush-Grisaru\(^1\) and Polkinghorne\(^2\) (hereafter referred to as FGP) gives \((\ln t)^{n-1}/t\) as the leading term of 2n-th order in the coupling constant and the summation of these leading terms leads to the Regge pole behavior \(t^{a(l+1)}\). Alternatively, one can examine singularities of partial wave amplitudes in the l plane by the N/D method\(^3\) or by the Bethe-Salpeter equation.\(^4\) In each case, one obtains the same leading Regge pole as in FGP, and one finds that the leading term in each order in the large t limit corresponds to the highest pole singularities at \(l = -1\), namely, \(1/(l+1)^n\) in 2n-th order. More realistic but vastly more complicated cases of Compton scattering involving intermediate vector mesons and spin 1/2 or scalar nucleons have been analysed in more or less similar approximation by Gell-Mann et al.\(^5\) They examined especially in which case the leading singularity at \(l = 0\) can generate the nucleon Regge pole. These authors have been careful enough to mention that their results hold only for finite mass intermediate vector mesons, implying that the same results may not
be drawn for the case of electrons interacting with the electromagnetic field. Indeed we should expect a difference in large \( t \) behaviors of these two cases. For the Regge trajectory \( \alpha(s) \) obtained in these evaluations assuming a finite mass \( \lambda \) for intermediate bosons is logarithmically divergent as \( \lambda \to 0 \), while we should expect no infrared divergence for ladder diagrams except for the 4-th order term\(^7\). Then we expect that the two limiting orders, namely, \( t \to \infty \) first, then \( \lambda \to 0 \), and \( \lambda \to 0 \) first, then \( t \to \infty \), must give different results. Our aim is to confirm this conjecture.

In order to answer this problem we have adopted the Compton scattering of a massless scalar photon by a scalar nucleon and examined ladder diagrams which are most singular in the large \( t \) limit in each order. In a more realistic case of scattering via a massless vector meson, we have numerators involving integration momenta, which may change the large \( t \) behavior derived from the consideration of the denominators alone, which are the same as for the scalar-scalar scattering. Also, as pointed out in reference 6), in the case of a vector meson, there are certain cancellations of singular terms from various diagrams in each order, which may alter drastically the conclusions drawn for the case of scalar-scalar scattering\(^8\). However, if we ever obtain different situations for zero versus non-zero mass intermediate vector mesons, it must come from the structure of denominators involved, which are the same as for the scalar-scalar scattering. Thus the investigation of the scalar-scalar case will be very useful in detecting the possible origin for this difference.

In \( \S \, 2 \), we evaluate the Feynmann parametric integral for the 2\( n \)-th order ladder diagram (Fig. 1). We consider the same boundary region of integration parameters as in FGP. This region contributes the leading term in large \( t \) limit, or transformed into partial waves, contributes the leading pole at \( l = -1 \). However, we retain certain terms neglected in the FGP method, which are essential to avoid the fictitious infrared divergence. We obtain a compact expression which can be used for finding either large \( t \) limit or singularities at \( l = -1 \). For \( \lambda \neq 0 \), and \( t \to \infty \), we reproduce the conventional result, i.e.

\[
F^{(n)} \sim e^2 \left( \frac{1}{(n-1)!} \right) (\pm t)^{1/2} \left[ \frac{\alpha(s) \ln(\pm t)}{16 \pi^2} \right]^n,
\]

with

\[
\alpha(s) = \frac{e^4}{16 \pi^2} \int_0^1 \frac{d\eta}{m^2 \eta - \eta (1-\eta) s + (1-\eta) \lambda^2}.
\]

Where \( s \) is the square of c.m.s. energy. Notice the logarithmic divergence of \( \alpha(s) \) as \( \lambda \to 0 \), which we mentioned above. The \( \pm \) sign in front of \( t \) stands for \( n \) = odd and \( n \) = even, respectively, the former giving exchange and the latter

\(^8\) The \( \ln^2 t \) cancellation in the 4-th order photoproduction diagrams observed in reference 6) is connected with the Gauge invariance. In fact, with a suitable choice of Gauge for the external photon, we can avoid the appearance of \( \ln^2 t \) in any diagram.
direct amplitudes. (Hereafter, the same double sign convention will be used.)
For \( \lambda = 0 \), we find as the leading term
\[
F^{(n)} \sim \frac{4e^2}{(2n-2)!} \frac{1}{|t|} \left[ A(s) \ln^2(\pm t) \right]^{n-1}
\]  
(3)
with
\[
A(s) = \frac{e^2}{16\pi^2} \frac{1}{m^2 - s}.
\]  
(4)
This term comes from the contribution of an “infrared region”, namely \( \gamma \sim 0 \)
in Eq. (2). For \( \lambda \neq 0 \), this region gives \( (e^2/16\pi^2)(m^2 - s)^{-1}\ln(1/\lambda^2) \). Apart from
a numerical factor, we find that Eq. (3) is reproduced from Eq. (1) by replac­
ing \( \lambda^2 \) by the inverse power of \( t \).

In § 3, we examine the partial wave amplitudes of the perturbation series.
For \( \lambda \neq 0 \), we have for \( l+1 \sim 0 \)
\[
f_l^{(n)} \sim (\mp 1)^l \frac{e^2}{2\rho^2} \frac{1}{l+1} \left( \frac{\alpha(s)}{l+1} \right)^{n-1},
\]  
(5)
where \( \rho \) is the center-of-mass momentum in the s channel. For \( \lambda = 0 \), we find
\[
f_l^{(n)} \sim (\mp 1)^l \frac{e^2}{2\rho^2} \frac{4}{l+1} \left[ \frac{A(s)}{(l+1)^2} \right]^{n-1}
\]  
(6)
as the highest singularity, which corresponds to the leading term (3). The
appropriate sums of Eq. (5) generate a Regge pole of +signature at \( l = -1 + \alpha \)
and a –signature pole at \( l = -1 - \alpha \). For \( \lambda = 0 \), on the other hand, we obtain
two Regge poles of +signature on both sides of \( l = -1 \) and two poles of –signa­
ture on the left of \( l = -1 \), or on the imaginary axis at \( l = -1 \). Furthermore,
we are left with the fixed single and double poles at \( l = -1 \), as the \( n=1 \) (Born)
and \( n=2 \) (4-th order) terms cannot be expressed by Eq. (6). When discussing
a high energy behavior, we have to take into account the overall in­
fared factor\(^{10}\) coming from soft photon radiative corrections. However, our aim
is not drawing physical conclusions from this rather unphysical case of scalar
photon interaction, but establishing the essential feature of the vanishing mass
case.

§ 2. Ladder diagram of order 2n

We consider a 2n-th order ladder diagram for Compton scattering given in
Fig. 1.
Large \( t \) Limit of Compton Scattering of a Scalar Photon

There are \( n \) nucleon lines exchanged, which carry the total momentum \( p_1 - p_2 \) for even \( n \) (\( t \) cut) and \( p_1 - k_2 \) for odd \( n \) (\( u \) cut). We choose an arbitrary one of these exchanged lines except for the end ones and assign a parameter \( x_0 \) to it. The other lines are assigned Feynmann parameters as shown in Fig. 1. The standard parametric representation of Feynmann integrals gives

\[
F^{(n)} = e^{2 \left( \frac{-e^2}{16\pi^2} \right)^{n-1} (n-1)!} \times \prod \delta \left( 1 - \sum x_i - \sum y_i - \sum z_i - x_0 \right) \prod d\gamma_i \prod dy_i \prod dz_i \prod dx_0 ,
\]

(7)

where \( D_v \) and \( D_N \) are given by the following determinants:

\[
D_v = \det A ,
\]

(8)

where we use the notation

\[
\gamma_i = x_i + y_i + z_i ,
\]

(9)

also

\[
D_N = \begin{bmatrix}
\gamma_1 + x_1 & x_1 & \cdots & x_1 \\
\gamma_2 + x_0 & x_0 & \cdots & x_0 \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_N + x_{N-2} & x_{N-2} & \cdots & x_{N-2} \\
\end{bmatrix} ,
\]

(10)

where

\[ p = p_1 + k_1 \; ; \; \; \; p_1^2 = -m_1^2 \; , \; k_1^2 = -k^2 \; , \]

and

\[
C = \left[ y_1 + \cdots + y_{N-1} + x_2 + \cdots + x_{N-2} + x_0 \right] (m^2 + p^2)
\]

\[ + (z_1 + \cdots + z_{N-1}) \lambda^2 . \]

(11)

We define as usual

\[
s = -\vec{p}^2 \; , \; \; t = - (p_1 - p_2)^2 .
\]

(12)

\( D_N \) contains \( t \) in the form \( \pm x_0 \; x_1 \cdots x_{N-1} \; t \). In the limit \( t \rightarrow \infty \), the boundary region at \( x_i \ll \varepsilon \ll 1 \) contributes most. As is readily seen, the same region con-
tributes to singularities of the partial wave amplitude of \( F^{(0)} \) in the \( l \) plane. In the FGP method, therefore, all \( x_i \)'s are put zero everywhere in \( D_0 \) and \( D_N \) except for those in front of \( t \). However, we cannot do this for \( \lambda = 0 \), as it leads to logarithmic divergences from the lower end of \( y_i \) integration. If we set \( \lambda = 0 \) and all \( y_i = 0 \), we are left with polynomials of \( \gamma_i \) and \( x_i \), and we retain lowest order terms in \( x_i \) necessary to avoid the divergence at \( y_i = 0 \). It develops that the linear terms of \( x_i \) in \( C \), Eq. (11), are sufficient to prevent the divergences at \( y_i = 0 \) for \( i = 2, \cdots, n-2 \). Since \( x_1 \) and \( x_{n-1} \) are lacking in \( C \), we have to retain \( x_1^2 m^2 \) and \( x_{n-1}^2 m^2 \) coming from the square of \( x_1 p_1 \) and \( x_{n-1} p_2 \) in the end lines of Eq. (10), in order to prevent the divergences at \( y_0 = 0 \) and \( y_{n-1} = 0 \). Introducing a new integration variables by

\[
\begin{align*}
  x_i &= \gamma_i \xi_i, \\
  y_i &= \gamma_i \eta_i, \\
  z_i &= \gamma_i (1 - \xi_i - \eta_i), \\
  d x_i \, d y_i \, d z_i &= \gamma_i^2 d \xi_i \, d \eta_i \, \theta (1 - \xi_i - \eta_i),
\end{align*}
\]

we obtain under this approximation

\[
D_0 = \gamma_1 \gamma_2 \cdots \gamma_{n-1},
\]

\[
D_N = \gamma_1 \gamma_2 \cdots \gamma_{n-1} \prod_{i=1}^{n-1} \gamma_i \left[ \sum_{i=1}^{n-1} \gamma_i \left[ F_i + x_0 F_0 \right] \right],
\]

where

\[
\begin{align*}
  F_i &= (m^2 - s) (\gamma_i + \xi_i^2 s + \lambda^2 (1 - \eta_i)), \quad i = 2, \cdots, n-2, \\
  F_i &= (m^2 - s) \eta_i + \gamma_i s + m^2 \xi_i^2 + \lambda^2 (1 - \eta_i), \quad i = 1, n-1, \\
  F_0 &= \pm \xi_1 \cdots \xi_{n-1} t + (m^2 - s).
\end{align*}
\]

By this transformation we can de-parameterize the integral (7),

\[
F^{(0)} = e^2 \left( \frac{e^2}{16 \pi^2} \right)^{n-1} (n-1)! \int \cdots \int (1 - \sum \gamma_i x_0) \, dx_0 \, \prod d \gamma_i
\]

\[
\times \left[ \prod \left( d \xi_i \, d \eta_i \right) \theta (1 - \xi_i - \eta_i) \right]^{n-1} \left[ \sum_{i=1}^{n-1} \gamma_i F_i + x_0 F_0 \right]^{n-1}
\]

\[
= e^2 \left( \frac{e^2}{16 \pi^2} \right)^{n-1} \int \cdots \int (1 - \xi_i - \eta_i) F_i F_0 \gamma_1 \gamma_2 \cdots \gamma_{n-1}
\]

\[
= e^2 \left( \frac{e^2}{16 \pi^2} \right)^{n-1} \int \cdots \int \left( F_i \right)^{n-1} \gamma_1 \gamma_2 \cdots \gamma_{n-1}
\]

\[
\int \cdots \int \left( \frac{1}{F_0} \right)^{n-1} \gamma_1 \gamma_2 \cdots \gamma_{n-1}
\]

Actually, we should have restricted the region of \( x_0 \) integration to \( x_0 < \varepsilon \), where our approximation is justified. However, the region \( x_0 > \varepsilon \) does not contribute to the leading singularity in any case, so that Eq. (15) is justified as far as the leading singularities are concerned.\(^*\) Denoting

\[
f_i (\xi_i) = e^2 \left( \frac{e^2}{16 \pi^2} \right)^{n-1} \int \theta (1 - \xi_i - \eta_i) \, d \eta_i \]

we write

\(^*\) The contribution from the region \( x_0 \) gives only terms of the order \( 1/t \ln^{n-2} t \).
Large $t$ Limit of Compton Scattering of a Scalar Photon

\[ F^{(n)} = e^t \left( \ldots \right. \prod \left( \xi_i \right) \left( f_1(\xi_1) \cdot f_2(\xi_2) \ldots f_{n-1}(\xi_{n-1}) \right) \left( \frac{\xi_1 \xi_2 \ldots \xi_{n-1} t}{(m^2 - s)} \right) \cdot \left. \right) . \tag{17} \]

For $\lambda \neq 0$, we can safely put $\xi_i = 0$ in $f_i(\xi_i)$, and from Eqs. (2) and (4) we see

\[ f_i(\xi_i = 0) = a(s) \quad (\lambda \neq 0). \tag{18} \]

From Eq. (17), Eq. (18) and an asymptotic relation

\[ \left( \ldots \right. \prod \left( \xi_i \right) \left( \frac{1}{(m^2 - s) \pm \xi_1 \xi_2 \ldots \xi_{n-1} t} \right) \sim \frac{1}{(n-1)!} \frac{1}{(\pm t)} \ln^{n-1}(\pm t), \tag{19} \]

we immediately obtain the FGP result, Eq. (1). For $\lambda = 0$, however, we cannot put $\xi_i = 0$ in $f_i(\xi_i)$. From Eqs. (14) and (16) we have

\[ f_i(\xi_i) = A(s) \left( \ln \xi_i + \ln \frac{m^2-s}{m^2} \right) \times \left\{ \begin{array}{ll} 2, & i = 1, n-1, \\
1, & i = 2, \ldots, n-2, \end{array} \right. \quad (\lambda = 0, \xi_i \sim 0), \tag{20} \]

where $A(s)$ is defined by Eq. (4). The logarithmic singularity of $f_i$ at $\xi_i = 0$ boosts the power of $\ln t$ as we see from an asymptotic formula

\[ \int \ldots \int \left( \prod \left( \xi_i \right) \right) \frac{\ln \xi_i \cdot \ln \xi_{n-1}}{m^2 - s \pm \xi_1 \xi_2 \ldots \xi_{n-1} t} \sim (2n-2)! \times \frac{1}{\pm t} \ln^{n-2}(\pm t). \tag{21} \]

Thus, taking only the $\ln \xi_i$ term of $f_i(\xi_i)$ in Eq. (20), we obtain the leading term (3). Roughly speaking, for $\lambda = 0$ the infrared divergence at $\xi_i = 0$ is cut by $\xi_i \sim (1/t)$ instead of $\lambda$, so that each $\ln^2 \xi$ in $a(s)$ is replaced by $\ln t$, resulting in the doubling of the power of $\ln t$.

§ 3. Partial wave amplitudes and Regge poles

We define the partial wave amplitudes by

\[ F = \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) f_l. \tag{22} \]

From Eq. (17), we have

\[ f_i^{(n)} = e^{2t} \int \left( \ldots \right. \prod \left( \xi_i \right) f_1(\xi_1) \cdot f_2(\xi_2) \ldots f_{n-1}(\xi_{n-1}) \left( \frac{2p^2 \xi_1 \xi_2 \ldots \xi_{n-1}}{1 + \frac{m^2-s}{2p^2 \xi_1 \xi_2 \ldots \xi_{n-1}}} \right), \tag{23} \]

where

\[ t = -2p^2(1 - \cos \theta), \quad p^2 = (m^2 - s)^2/4s. \]

For small $\xi_i$, $Q_i$ can be expanded in inverse powers of the argument,
Higher order terms in $\xi_i$ contribute poles at $l = -2, -3, \ldots$. The singularity at $l = -1$ arises from $\Gamma'(l+1)$ in Eq. (24), and also from the $\xi_i$ integration,

$$\int d\xi_i \xi_i^l f_i(\xi_i) = \begin{cases} 
\xi_i^{l+1} \frac{\alpha(s)}{l+1} \alpha(s) & (\lambda \neq 0), \\
A(s) \xi_i^{l+1} \frac{l+1}{l+1} \left( 1 + \ln \left( \frac{m^2 - s}{m^2} \right) \right) \times \left\{ \begin{array}{ll} 2, & i = 1, n-1, \\
1, & i = 2, \ldots, n-2, \\
(\lambda = 0). \end{array} \right. 
\end{cases}$$

Again the logarithmic singularity of $f_i(\xi_i)$ gives the higher singularity for $\lambda = 0$. Near $l = -1$,

$$f_i^{(\lambda)} = (\pm 1)^l \frac{e^2}{2p^2} \frac{1}{l+1} \left( \frac{\alpha(s)}{l+1} \right)^{n-1} \alpha(s) \frac{1}{l+1} \left( 1 + \ln \left( \frac{m^2 - s}{m^2} \right) \right)^{n-1} \times \left\{ \begin{array}{ll} 2, & i = 1, n-1, \\
1, & i = 2, \ldots, n-2, \\
(\lambda = 0). \end{array} \right.$$  \hspace{1cm} (26)

$n = \text{even}$ terms are the so-called direct amplitudes and $n = \text{odd}$ terms are the exchange amplitudes. The Born term $n = 1$ cannot be incorporated into Eq. (26) in case $\lambda = 0$ because of the factor $4$. The 4-th order term $n = 2$ is infrared divergent and we must keep $\lambda$ finite. Hence Eq. (26) does not apply for $n = 2$. If we sum Eq. (26) over all $n$ even and odd respectively, then we have to subtract the $n = 1$ and $n = 2$ terms, which remain as fixed poles at $l = -1$. This is an unsatisfactory situation, but in the following we will forget these fixed poles and examine the sum of Eq. (26) just to see what happens. In Eq. (26), we have kept lower order poles, the meaning of which may be doubtful, because in our process of approximation, we may have neglected singularities of the same orders. So, for the moment we will keep only the leading pole to each order $n$, and discuss the effect of inclusion of lower order poles later. The direct and exchange amplitudes are then

$$f_{i \text{dir}}^{(\lambda)} = \sum_{n \text{ even}} f_i^{(\lambda)} = \begin{cases} 
eq 0, \\
(\alpha(s)) \frac{1}{l+1} \left( 1 + \ln \left( \frac{m^2 - s}{m^2} \right) \right)^{n-1} \times \left\{ \begin{array}{ll} 2, & i = 1, n-1, \\
1, & i = 2, \ldots, n-2, \\
(\lambda = 0). \end{array} \right. 
\end{cases}$$

and

$$f_{i \text{exch}}^{(\lambda)} = \sum_{n \text{ odd}} f_i^{(\lambda)} = \begin{cases} 
eq 0, \\
(-1)^l \frac{e^2}{2p^2} \frac{l+1}{(l+1)^2 - \alpha^2(s)} \alpha(s) \frac{1}{l+1} \left( 1 + \ln \left( \frac{m^2 - s}{m^2} \right) \right)^{n-1} \times \left\{ \begin{array}{ll} 2, & i = 1, n-1, \\
1, & i = 2, \ldots, n-2, \\
(\lambda = 0). \end{array} \right. 
\end{cases}$$

(27)

(28)
Large $t$ Limit of Compton Scattering of a Scalar Photon

We form the Regge poles of $\pm$ signature by

$$ f^\pm = \frac{1}{2} [ f_i^{\pm} \pm (-1)^l f_i^{\mp}] . $$

(29)

From Eqs. (27) and (28)

$$ f^\pm = \begin{cases} \frac{e^2}{2 p^2} \frac{1}{l+1 \mp \alpha(s)} & (\lambda \neq 0), \\
\frac{e^3}{4 (l+1)^2} \frac{A(l+1)}{\mp A(s)} & (\lambda = 0). \end{cases} $$

(30)

Thus for $\lambda \neq 0$, we have the Regge poles of $\pm$ signature at $l = -1 \pm \alpha(s)$. These poles determine the large $t$ limit of the scattering amplitudes as

$$ F \sim \frac{e^2}{2} [ t^{a-1} \mp (-t)^{a-1} + t^{a-1} - (-t)^{a-1} ] \ (\lambda \neq 0). $$

(31)

The first two terms coming from the $+$ signature pole dominates over the last two. For $\lambda = 0$, on the other hand, we have two poles of $+$ signature at $l = -1 \pm \sqrt{A(s)}$, and two poles of $-$ signature at $l = -1 \pm i \sqrt{A(s)}$. For the physical $t$ channel, $s < 0$, and $A(s) > 0$. Hence the $-$ signature poles contribute to $F$ terms like

$$ t^{\pm 1 \mp i \sqrt{A(s)}} \mp (-t)^{\pm 1 \mp i \sqrt{A(s)}}, $$

which are dominated by the $+$ signature poles. The latter give

$$ F \sim \frac{e^2}{2} [ t^{1+\sqrt{A}} \mp (-t)^{1+\sqrt{A}} + t^{1-\sqrt{A}} \mp (-t)^{1-\sqrt{A}}], $$

(32)

in which the first two dominate over the last two. The same result is obtained, of course, by the direct summation of Eq. (3).

If we include lower order poles as in Eq. (26), we obtain poles at

$$ l = -1 + \frac{1}{2} \left[ A \ln \frac{m^2 - s}{m^2} \pm \sqrt{A^2 \ln \left( \frac{m^2 - s}{m^2} \right) - A} \right] \ (+ \text{signature}) $$

and

$$ l = -1 + \frac{1}{2} \left[ -A \ln \frac{m^2 - s}{m^2} \pm \sqrt{A^2 \ln \left( \frac{m^2 - s}{m^2} \right) - A} \right] \ (-\text{signature}). $$

(33)

The $-$ signature poles are to the left of $l = -1$ and do not contribute the leading term in the large $t$ limit. One of the $+$ signature poles which is to the right of $l = -1$ gives the leading term.

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References

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