Nordic Hydrology, 8, 1977, 33-46

Estimation of Parameters of a Uniformly Nonlinear Surface Runoff Model

Vijay P. Singh
New Mexico Institute of Mining and Technology
Socorro, NM, NM-87801, U.S.A.

The surface runoff model, based on the concept of uniform nonlinearity (Dooge 1967), has three parameters. It is shown that these parameters can be estimated from the informaion that can be easily obtained from a topographic map of the watershed. By applying the model to natural agricultural watersheds the reliability of estimating parameters in the proposed manner is reexamined.

Introduction

Models of watershed surface runoff, both linear and nonlinear, abound in hydrological literature (Chow 1964; Dooge 1973; Singh 1976). A serious problem which the designer faces in the use of these models is how to estimate their parameters from information that is ordinarily available on a given watershed. One way to get around this problem has been to optimize the model parameters over a set of rainfall-runoff events by a suitable optimization algorithm in conjunction with an objective function. These optimized parameter values are then used in the model for prediction of surface runoff for rainfall events not used in the optimization. This method does not basically solve the problem and has other serious limitations:

(1) These optimized parameter values will change as soon as the optimization set of events is changed. Thus these values represent the system only for the events used in the optimization and not for any other events. This implies that these values cannot be used in prediction of runoff for different events.

(2) For lack of data this method cannot possibly be used on ungaged watersheds.

(3) Optimization is an expensive operation. The parameter values, obtained by optimization, cannot be extrapolated.

One way to tackle this problem basically is to relate the model parameters to information that is commonly available on a given watershed and thus develop equations for the parameters in terms of that information. This method will not suffer from the above mentioned limitations and will be readily applicable to watersheds having no rainfall-runoff record. In the present study we attempt to estimate by this method the parameters of a uniformly nonlinear surface runoff model proposed by Dooge (1967). An implicit assumption in the proposed method is that no changes take place in watershed physiography over the period in question. If changes do take place and are pronounced in scale, the determining equations will have to be rederived. Also, the watersheds, grouped together, must have similar geologic and physiographic features.

A Uniformly Nonlinear Runoff Model

A simple uniformly nonlinear model may consist of n storage elements (or reservoirs) combined in series as shown in Fig. 1. The governing equations for a nonlinear

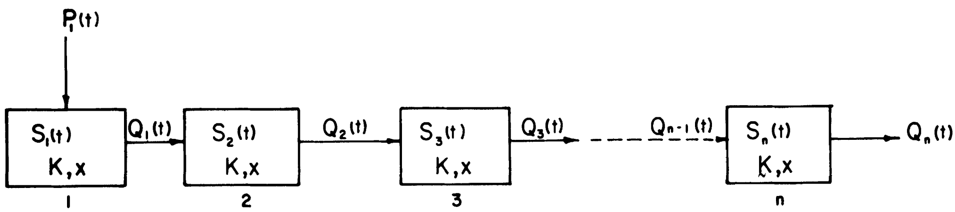

Fig. 1. A uniformly nonlinear runoff model.

storage element consist of a spatially lumped form of continuity equation and a nonlinear storage-discharge relationship which can be written respectively as:

$$P \equiv Q + \frac{dS}{dt} \qquad (1)$$

$$Q = KS^{x} \qquad (2)$$

where P is inflow to the element, Q outflow from the element, S storage in the element, dS/dt rate of change of storage in the element, t time, and K and x characteristic parameters of the element. In a uniformly nonlinear model the parameters K and x do not vary from one storage element to another. In the model considered here, input P_1 (or rainfall-excess) is fed into the first element and the output from the first element becomes input to the second element and so on. We are interested in outflow Q_n from the n th element. By combining Eqs. (1) and (2) we obtain a single differential equation for a storage element relating storage (hence outflow) to inflow. Thus for n storage elements n differential equations can be written. Using matrix notation, symbolized by bold-faced letters, the model can be written mathematically as:

$$\dot{S} = P + KBS \qquad (3)$$

$$Q = KCS \qquad (4)$$

where

$$\dot{S} = \begin{Bmatrix} \frac{dS_1}{dt} \\ \frac{dS_2}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dS_n}{dt} \end{Bmatrix} ; \quad P = \begin{Bmatrix} P_1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} ; \quad S = \begin{Bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ S_n \end{Bmatrix} ;$$

$$B = \begin{Bmatrix} S_1^{x-1} & 0 & 0 \cdots\cdots\cdots & 0 & 0 \\ S_1^{x-1} & -S_2^{x-1} & 0 \cdots\cdots\cdots & 0 & 0 \\ 0 & S_2^{x-1} & -S_3^{x-1} \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & S_{n-1}^{x-1} & -S_n^{x-1} \end{Bmatrix} ; \text{ and}$$

$$C = \begin{Bmatrix} S^{x-1} & 0 & 0 \cdots\cdots\cdots & 0 & 0 \\ 0 & S^{x-1} & 0 \cdots\cdots\cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & S_{n-1}^{x-1} & 0 \\ 0 & \cdots\cdots\cdots & \cdots\cdots\cdots & 0 & S_n^{x-1} \end{Bmatrix}$$

From Eqs. (3) and (4) it is clear that the model has three parameters n, K and x. These parameters must be specified before we can hope to use the model.

Parameter Estimation

It will be useful to establish if any of the parameters has physical significance. The parameter n signifies the number of storage elements in the model. The value of n must be greater than one to obtain proper hydrograph shape, and will depend upon the topographic complexity of a watershed. This can essentially be called a shape parameter. A natural watershed entails a network of channels and overland flow planes. The combined action of a channel and a plane is being simulated here by a nonlinear storage element. It would then seem that a very large number of storage elements will be needed to simulate the action of a network of channels and planes and, in turn, the runoff response of a watershed. Fortunately, it so happens that only a small number of storage elements will suffice. This is because the planes and channels having more or less similar hydraulic behavior can be combined and then their combined action can be simulated by a single storage element. The exact value of n will vary from one watershed to another, but it seems plausible that n will more or less be the same for watersheds in a certain area range having the same order of drainage evolution.

The parameter x quantifies the nonlinearity of surface runoff process. Although the value of x may change throughout the development of a runoff hydrograph, may change from one event to another and from one watershed to another, it is plausible that x can be fixed for watersheds in a certain area range and that this fixed value of x will provide a good aproximation to the degree of nonlinearity in surface runoff.

The precise physical significance of the parameter K is not clear. It appears that it accounts for both translation and attenuation effects, and consequently it may change considerably from one watershed to another. The topographic characteristics of a watershed seem to be dominant factors affecting the value of K. Although K will most likely change from event to event on the same watershed but this change may not be large. Thus it seems plausible that K can be expressed in terms of topographic characteristics.

These plausible hypotheses were tested on a natural watershed SW-17, Riesel (Waco), Texas. This is a small watershed of 1.2 ha in areal extent. Its description in detail can be found elsewhere (e.g., USDA 1963 and subsequent publications). Nine rainfall-runoff events were available on this watershed in the USDA publications. These events, after subtracting infiltration (Philip 1957), were divided into two sets. One set, called optimization set, consisted of five events; another set, called prediction set consisted of four events. These two sets did not have any events in common. Then, using the objective function (Singh 1975)

$$F = \sum_{j=1}^{M} [Q_{p_o}(j) - Q_{p_e}(j)]^2 \Rightarrow \min \qquad (5)$$

(where $Q_{p_o}(j)$ is observed hydrograph peak for j th event, Q_{p_e} (j) estimated hydrograph peak for j th event, and M number of events) the parameters K and x were optimized by the Modified Rosenbrock algorithm (Rosenbrock 1960; Palmer, 1969; Himmelblau 1972) over the optimization set of events for various values of n. The values of F, K and x obtained from optimization are given in Table 1. It is interesting

Table 1 - Values of objective function and optimized parameters for watershed SW-17, Riesel (Waco), Texas.

Case number	Number of storage elements n	Objective function F	Optimal values of parameters x	K (cm^{-x}/hr)
1	2	2.692	3.10	0.26
2	3	0.162	1.4	0.22
3	4	0.168	1.36	0.27
4	5	0.172	1.36	0.32
5	6	0.174	1.35	0.38
6	7	0.175	1.35	0.42
7	8	0.175	1.34	0.46
8	9	0.175	1.34	0.51
9	10	0.175	1.35	0.55

to note that (1) F does not change for $n > 7$, (2) x remains fixed for $n < 6$, (3) n changes little for $n < 6$, (4) x decreases as n increases, (5) K increases as n increases, and (6) for $n = 3$, F is minimum. From these observations it appears that n could be fixed at 3 and x at 1.4. If it can be shown that these parameter values are reasonable then the model will have only one parameter K left to be specified. For further examination, hydrographs were predicted by using various sets of parameter values for the prediction set of events. Tables 2 and 3 compare observed and predicted hydrograph peak and its time. It is clear from these tables that the values of $n = 3$ and $x = 1.4$ are reasonable. It must be pointed out that a higher value of n may lead to equally good prediction, but it will increase computation and is, hence, undesirable.

Now the question is whether the model with $x = 1.4$ and $n = 3$ produces hydrographs with appropriate shape. For two sample events predicted and observed hydrographs are shown in Figs. 2 and 3. From these figures it is evident that the hydrograph shape is well preserved. These results confirm that the model, with $x = 1.4$ and $n = 3$, represents the runoff process well at least for the watershed in question. It remains, however, to show whether $x = 1.4$ and $n = 3$ are adequate for other watersheds. This question will be resolved in concurrence with the determination of K from watershed topography.

Twenty one natural agricultural watersheds were selected from two geographically distinct regions: 5 near Hastings, Nebraska, and 16 near Riesel (Waco), Texas. These

Table 2 - Observed and predicted hydrograph peak on watershed SW-17, Riesel (Waco), Texas, using various sets of optimized parameter values.

Date of Event	Observed hydrograph peak (Qp_o) cm/hr	Number of reservoirs used: 2		3		4		5		6		7		8		9		10	
		Q_{pe}	EQ	Q_{pe}	EQ	Q_{pe}	EQ	Q_{pe}	EQ	Q_{pe}	EQ	Q_{pe}	EQ	Q_{pe}	EQ	Q_{pe}	EQ	Q_{pe}	EQ
4-24-1957	7.366	8.405	-0.141	7.507	-0.019	7.478	-0.015	7.519	-0.021	7.543	-0.024	7.548	-0.025	7.558	-0.026	7.567	-0.027	7.584	-0.030
5-13-1957	4.420	5.138	-0.163	4.884	-0.105	4.894	-0.107	4.924	-0.114	4.944	-0.119	4.961	-0.123	4.974	-0.126	4.981	-0.127	4.986	-0.128
6-9-1962	9.627	7.186	0.254	7.123	0.260	7.162	0.256	7.206	0.251	7.234	0.249	7.252	0.247	7.267	0.245	7.271	0.245	7.275	0.244
3-29-1965	6.196	9.131	-0.474	8.058	-0.300	7.953	-0.283	7.971	-0.287	7.984	-0.288	7.973	-0.287	7.977	-0.287	7.989	-0.289	8.019	-0.294

Q_{pe} = Estimated hydrograph in cm/hr, $EQ = (Q_{po} - Q_{pe})/Q_{po}$

Table 3 - Observed and estimated hydrograph peak time on watershed SW-17, Riesel (Waco), Texas, using various sets of optimized parameter values.

Date of Event	Observed hydrograph peak time t_{po} (min)	Number of Reservoirs used: 2		3		4		5		6		7		8		9		10	
		t_{pe}	Et	t_{pe}	Et	t_{pe}	Et	t_{pe}	Et	t_{pe}	Et	t_{pe}	Et	t_{pe}	Et	t_{pe}	Et	t_{pe}	Et
4-24-1957	35	31	0.114	34.5	0.014	36.6	-0.046	38.0	-0.086	39.3	-0.123	40.6	-0.160	41.7	-0.191	42.7	-0.22	43.6	-0.240
5-13-1957	26	29.4	-0.131	31.3	-0.196	32.4	-0.246	33.2	-0.277	34.0	-0.308	35.0	-0.346	36.0	-0.385	37.1	-0.427	38.0	-0.460
6-9-1962	33	10.3	0.688	25.5	0.227	26.2	0.206	26.4	0.200	27.3	0.173	28.4	0.139	29.4	0.109	30.4	0.079	31.2	0.050
3-29-1965	110	68.1	0.381	70.3	0.361	72.2	0.344	73.5	0.332	74.8	0.320	76.1	0.308	77.2	0.298	78.2	0.289	79.1	0.280

t_{pe} = Estimated hydrograph peak time min, $Et = (t_{po} - t_{pe})/t_{po}$

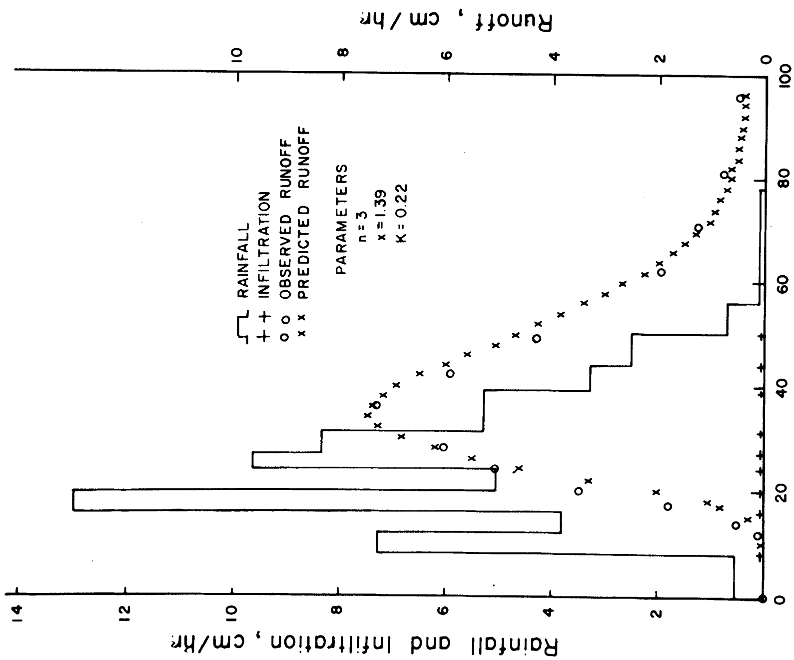

Fig. 2. Hydrograph prediction by the model for rainfall event of 4-24-1957 on watershed SW-17, Riesel (Waco), Texas.

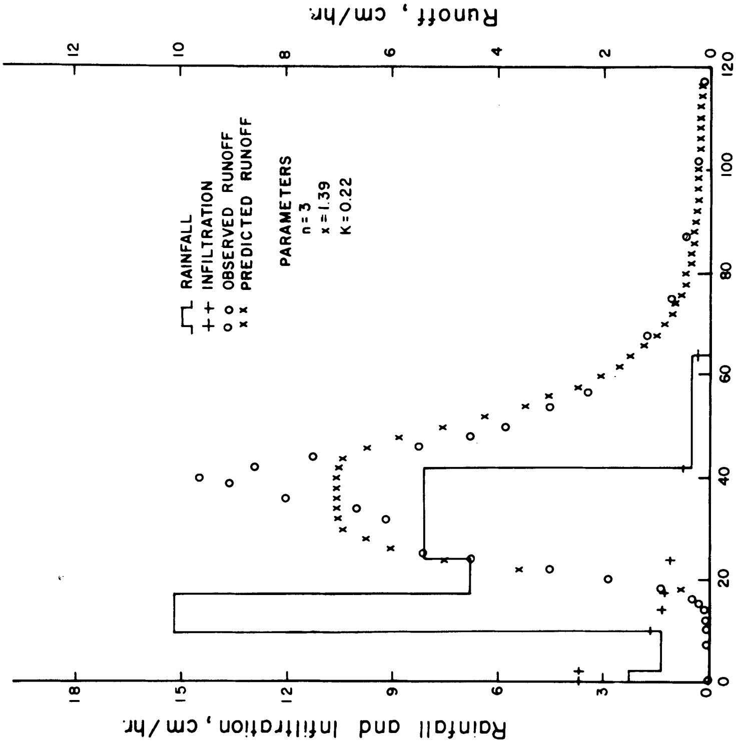

Fig. 3. Hydrograph prediction by the model for rainfall event of 6-9-1962 on watershed SW-17, Riesel (Waco), Texas.

watersheds vary in area from 1.2 to 1720 ha. Their detailed description can be found in the USDA publications. Rainfall events of each watershed were divided, as before, into optimization set and prediction set. Infiltration was estimated by Philip's equation (Philip 1957) and subtracted from rainfall to yield rainfall-excess. Then K was optimized with $n = 3$ and $x = 1.4$ for optimization set of events on each watershed by the modified Rosenbrock algorithm in conjunction with Eq. (5). Topographic characteristics, selected for correlating them with K, included area, width, length of the main stream, weighted slope and shape factor (Chorley Malm and Pagorzelski 1957). These characteristics and optimized K are given for each watershed in Table 4. Area, width and length of mainstream were given in the USDA publications. The shape factor (Shape) of Chorley, Malm and Pagorzelski (1957) was utilized in this study:

$$Shape = \frac{\pi L^2}{4A}$$

Table 4 - Watershed characteristics.

Watershed	Area (ha)	Width (m)	Length of main stream. (m)	Slope (%)	Shape	Parameter K (cm)-0.4 sec
Riesel (Waco)						
C	234.21	1402	2366	2.04	1.8761	0.0245
D	449.22	1892	3567	2.10	2.2246	0.0382
G	1772.59	2592	7829	2.05	2.7160	0.0134
Y	125.05	915	1537	2.40	1.4830	0.0400
Y-2	53.42	854	1000	2.58	1.4702	0.0794
Y-4	32.30	595	610	2.85	0.9031	0.0700
Y-6	6.50	259	338	3.22	1.0634	0.0800
Y-7	16.19	381	543	1.86	1.4289	0.1340
Y-8	8.418	183	244	1.94	0.5550	0.1667
Y-10	7.53	381	338	2.37	1.0584	0.1620
W-1	71.23	610	1646	2.18	2.9887	0.1594
W-2	5.26	823	945	2.55	1.3335	0.0989
W-6	17.12	457	445	2.02	0.9090	0.1312
W-10	7.97	305	323	.162	1.0289	0.2100
SW-12	1.20	119	116	3.95	0.8770	0.3021
SW-17	1.21	122	116	1.83	0.8712	0.1975
Hastings						
2-H	1.21	76	189	6.13	2.0395	0.2501
4-H	1.47	107	162	5.96	1.3921	0.4462
W-3	194.66	1207	2720	5.30	2.9861	0.0700
W-8	844.20	1811	7953	5.50	5.8850	0.0179
W-11	1412.40	2012	11673	5.09	7.5763	0.0005

where L is the length of the mainstream and A area of the watershed. This shape factor is a dimensionless parameter, and quantifies the watershed shape.

To correlate K with topographic characteristics a multiple linear regression analysis was used. K was obviously the dependent variable in the analysis. The linear regression analysis yielded a correlation coefficient of 0.9208 and a standard error of estimate of 0.0746 where slope was most highly correlated with a correlation coefficient of 0.8080, then were length of main stream, area and shape factor respectively. The regression equation can be written as:

$$K = 0.00044\,Area - 0.00014\,Length + 0.03828\,Slope + 0.08511\,Shape \qquad (6)$$

To check the reliability of Eq. (6) residuals between optimized K and K estimated from Eq. (6) were computed for all the watersheds. As evident from Fig. 4 there is considerable scattering of points around the regression-fit line, and we would naturally like to minimize this scattering.

In the hope of improving the correlation all variables, dependent as well as independent, were transformed logarithmically to the base 10. Henceforth, we will deal with these transformed variables only. Then the regression analysis was performed. A correlation coefficient of 0.9890 and a standard error of estimate of 0.1827 were obtained. This time length of mainstream provided the highest correlation with a correlation coefficient of -0.9763, then did area, shape factor and slope respectively. The regression equation can be written as:

$$\text{Log}\,K = -0.30871\,\text{Log}\,Area - 0.21608\,\text{Log}\,Length - 0.08328\,\text{Log}\,Slope + 0.30379\,\text{Log}\,Shape \qquad (7)$$

To check the reliability of Eq. (7) residuals between optimized K and K estimated from Eq. (7) were computed. As shown in Fig. 5, the scattering of points is considerably reduced and consequently the relationship for K is much improved.

In the multiple linear regression analysis the independent variables are assumed to be independent in a statistical sense; they are seldom so, as clearly seen form the partial correlation matrix for the transformed variables given in Table 5. It then appears that a fewer number of independent variables may suffice to develop a reasonable equation for K. To accomplish this, shape factor was removed from independent variables, and then regression analysis was performed. A correlation coefficient of 0.9885 and a standard error of estimate of 0.1757 were obtained. Now width was most highly correlated with a correlation coefficient of -0.972, and then was area. The regression equation can be written as:

$$\text{Log}\,K = -0.27271\,\text{Log}\,Area - 0.24309\,\text{Log}\,Width \qquad (8)$$

Table 5. Partial correlation matrix for variables after transformation.

	Area	Width	Length	Slope	Shape	*K*
Area	1.00	0.900	0.913	0.732	0.796	-0.954
Width		1.000	0.998	0.899	0.660	-0.972
Length			1.000	0.905	0.699	-0.976
Slope				1.000	0.667	-0.843
Shape					1.000	-0.717
K						1.000

To evaluate the goodness of Eq. (8) the residuals of K were plotted as shown in Fig. 6. It is clear that the relation for K is nearly as good as Eq. (7).

To find out a different combination of independent variables that will give an equally good relationship for K, shape factor and width were deleted from independent variables and then regression analysis was performed. A correlation coefficient of 0.9882 and a standard error of estimate of 0.1834 were obtained. Length of main stream alone gave a correlation coefficient of -0.9763, and it was further improved by area and slope respectively. The regression equation can be written as:

$$\text{Log}\, K = -0.22889\, \text{Log}\, Area - 0.26395\, \text{Log}\, Length + 0.10079\, \text{Log}\, Slope \qquad (9)$$

Again, residuals of K were computed to determine the reliability of Eq. (9), as shown in Fig. 7. This provides, as clear from the figure, just as good a relationship for K.

Thus we have three different relationships for K given by Eqs. (7) - (9) which are comparable. Any one of the three relationships can be used to estimate K. However, one may prefer to choose Eq. (8) or Eq. (9) because of fewer variables involved therein. Since the ultimate objective of the model is to predict surface runoff, we would like to see how good these estimates of the parameters are. Hydrographs were then predicted for the prediction set of events of each of the 21 watersheds, utilizing $x = 1.4$, $n = 3$, and K estimated by Eq. (9). For four sample watersheds observed and predicted hydrograph peak characteristics are given in Table 6. Figs. 8 and 9 show observed and predicted runoff hydrographs for two sample events. It is clear from the tables and the figures that observed and predictd hydrographs are in close agreement. This confirms that it is reasonable to take $x = 1,4$, $n = 3$ and K estimated by Eq. (9), and that with these values of the parameters the model predicts surface runoff well.

It must be remarked that it would be more desirable to test the reliability of the proposed method on entirely a different group of watersheds than the ones used to develop the method. This was not done, for only a small number of watersheds were available.

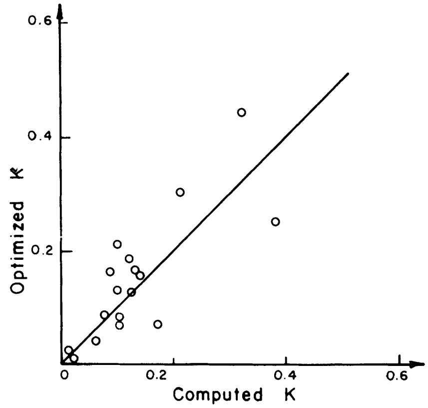

Fig. 4. Optimized K versus computed K using Eq. (6).

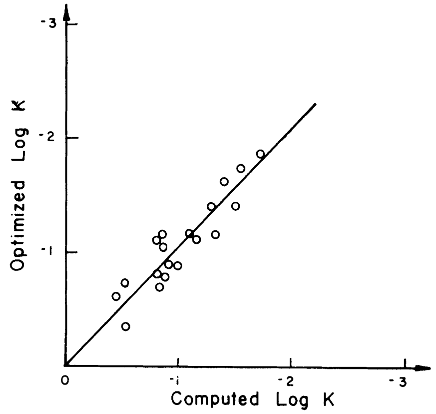

Fig. 5. Optimized K versus computed K using Eq. (7).

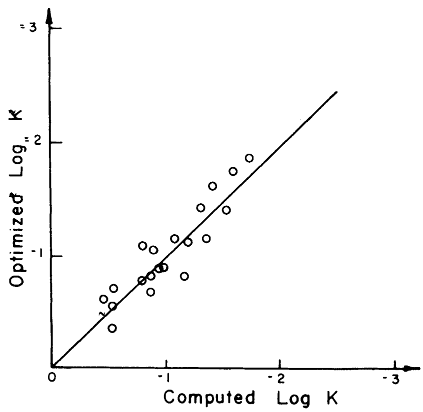

Fig. 6. Optimized K versus computed K using Eq. (8).

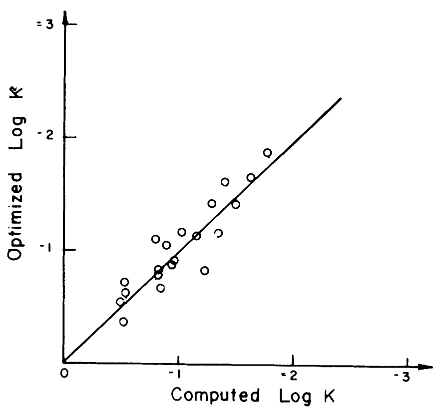

Fig. 7. Optimized K versus computed K using Eq. (9).

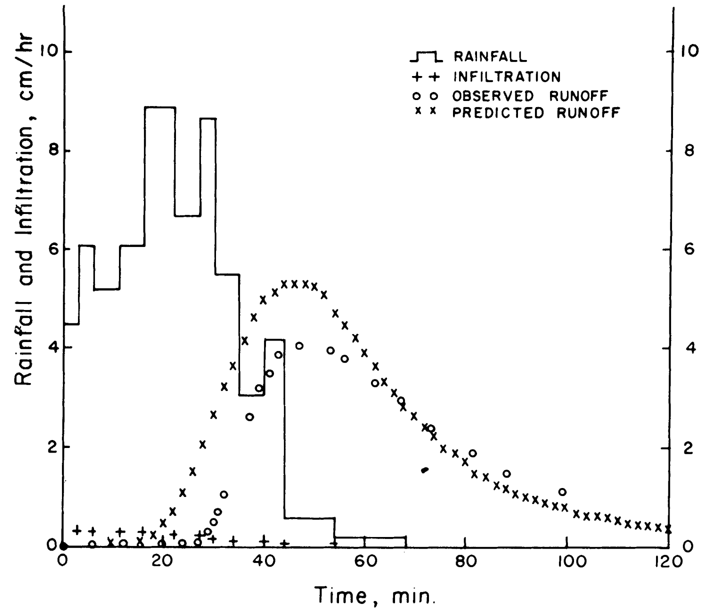

Fig. 8. Hydrograph prediction by the model for rainfall event of 5-13-1957 on watershed Y-2, Riesel (Waco), Texas.

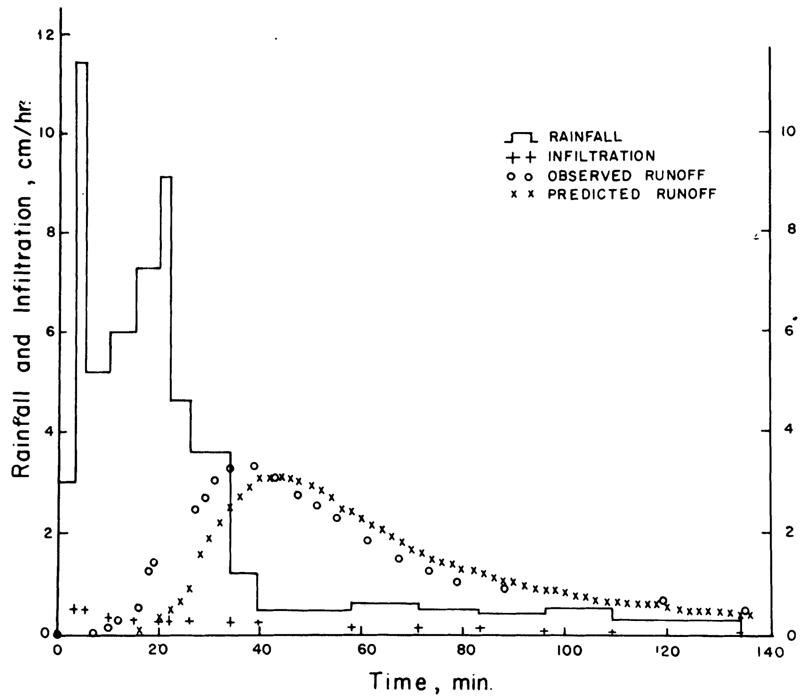

Fig. 9. Hydrograph prediction by the model for rainfall event of 4-24-1957 on watershed Y-2, Riesel (Waco), Texas.

Table 6 - Hydrograph peak predictions on natural agricultural watersheds.

Watershed	Date of Event	Observed hydro-graph peak (cm/hr)	Estimated hydro-graph peak (cm/hr)	Relative error* in peak prediction	Observed hydro-graph peak time (min)	Estimated hydro-graph peak time (min)	Relative error in peak time prediction
Riesel (Waco), Texas							
Y-2	4-24-1957	4.267	4.346	-0.018	37	49.6	-0.341
	5-13-1957	3.150	3.165	-0.005	35	44.4	-0.269
	6- 4-1957	4.547	3.423	0.247	34	41.8	-0.229
	3-29-1965	5.975	5.837	0.023	82	115.4	-0.407
Y-7	4-24-1957	5.994	5.352	0.107	31	37.0	-0.194
	5-13-1957	5.156	4.203	0.185	33	40.6	-0.230
	6-23-1958	4.470	3.398	0.240	57	114.0	-1.000
	3-29-1965	5.778	6.491	-0.124	79	77.8	0.015
Y-10	4-24-1957	6.858	6.321	0.078	35	37.0	-0.08
	5-13-1957	4.851	4.184	0.138	26	35.6	-0.369
	6- 9-1962	1.001	1.300	-0.299	38	39.4	-0.037
	3-29-1965	6.925	7.618	-0.100	69	82.4	-0.194
W-2	4-24-1957	5.182	5.882	-0.135	56	42.8	0.236
	5-13-1957	3.912	4.545	-0.162	56	37.2	0.336
	3-29-1965	4.653	5.771	-0.240	131	75.0	0.428

* Relative error = (observed quantity - estimated quantity)/observed quantity.

Conclusions

The parameters of a uniformly nonlinear model proposed by Dooge (1967) have been estimated. It has been shown that two of the parameters x and n, signifying number of storage elements and degree of nonlinearity respectively, can be fixed for watersheds in a certain area range. The remaining parameter K can be reliably estimated from the topographic information commonly known about a watershed. Thus the uniformly nonlinearly model can be completely specified, and is a promising tool in watershed hydrology.

Acknowledgment

The work upon which this study is based was supported in part by funds provided through the New Mexico Water Resources Research Institute by the Department of Interior, Office of Water Research and Technology, as authorized under the Water Resources Research Act of 1964, Public Law 88-379 as amended, under project number 3109-206.

References

Chorley, R. J., Malm, D. E. G. and Pogorzelski, H. A. (1957) A new standard for estimating drainage basin slope. *Am. J. Sci. 255,* 138-141.

Chow, V. T. (1964) Runoff. Section 14 in Handbook of Applied Hydrology edited by V. T. Chow, pp. 14-1 - 14-54, McGraw-Hill Book Company, New York.

Dooge, J. C. I. (1967) A new approach to nonlinear problems in surface water hydrology: hydrologic systems with uniform nonlinearity. IASH Pub. 76 409-413.

Dooge, J. C. I (1973) Linear theory of hydroogic systems. Technical Bulletin Nok. 1468, 327 p., Agricultural Research Service, U. S. Department of Agriculture, Washington, D.C.

Himmelblau, D. M. (1972) Applied nonlinear programming. pp. 158-167, McGraw-Hill Book Company, New York.

Palmer, J. R. (1969) An improved procedure for orthogonalizing the search vectors in Rosenbrock's and Swann's direct search optimization methods. *Computer Journal 12,* 67-71.

Philip, J. R. (1957) The theory of infiltration: 1. the infiltration equation and its solution. *Soil Science 84,* 257-264.

Rosenbrock, H. H. (1960): An automatic method for finding the greatest or lest falue of a function. *Computer Journal 4,* 175-184.

Singh, V. P. (1975) A laboratory investigation of surfce runoff. *J. Hydrology 25(2),* 187-200.

Singh, V. P. (1976) Studies on rainfall-runoff modeling: 2. a distributed kinematic wave model of watershed surface runoff. WRRI Report No. 065, 153 p. New Mexico Water Resources Research Institute, New Mexico State University, Las Cruces, New Mexico.

United States Department of Agriculture (1963) Hydrologic data for experimental agricultural watersheds in the United States. U.S. Government Printing Office, Washington, D. C.

First received: 22 June, 1976
Revised version received: 6 September, 1976

Address:
Department of Geoscience,
New Mexico Tech,
Socorro,
NM-87801, U.S.A.