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Discussion on the Gaussian assumption in flow rate measurements using a primary weighing method

A. Silva Ribeiro, C. Oliveira Costa, J. Sousa Lopes and J. Duarte Henriques

ABSTRACT

Flow rate measurement is a common task in many hydraulic infrastructures included in systems with a large impact on the economy. The quality requirements that such measurement must fulfil imply the best knowledge of the measurement results (estimates and measurement uncertainties). Methods such as those given by the *Guide for the Expression of Uncertainty in Measurement* (GUM) have been widely used as tools to evaluate measurement uncertainties. However, such methods have implicit assumptions on the nature of the mathematical models and the applicability conditions, which are not often taken into account by their users, who apply them regardless of the specific nature of the actual metrological problems. One such assumption is that the output probability function is Gaussian, which is true only if some input conditions are met. In practice, many metrological problems are described by mathematical models with non-ideal conditions, the measurement uncertainty solutions thus being quite different from those predicted by the GUM method. The development of metrological studies has shown that the Monte Carlo method is suitable to deal with non-ideal problems and has several advantages. One such advantage is particularly useful for the specific problem of flow rate measurement using a primary weighing method: the ability to give information on the output quantity probability function. In this way, it is possible not only to obtain the output quantity estimate but also to test the normality of the output measurement uncertainty interval, which in fact has a non-Gaussian shape.

Key words | flow rate, measurement uncertainty, Monte Carlo method

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INTRODUCTION

The modern approach of measurement based on a probabilistic concept (Leaning & Finkelstein 1979) implies that the statement of measurement results should include two parameters: the best estimate of the measurand and the related measurement uncertainty.

The need to develop methods to perform the evaluation of measurement uncertainties led to the ISO edition, in 1995, of the *Guide for the Expression of Uncertainty in Measurement* (GUM), which has been adopted, since then, for that purpose.

In many cases it had plenty of success. However, with the increasing complexity of measurement systems, some difficulties started to arise and alternatives began to be studied and discussed in international scientific events, the Monte Carlo method (MCM) being one of the most promising.

In fact, it is now understood that GUM provides solutions to metrological problems under a significant set of assumptions and restrictions, one of them being the assumption that the probability distribution of the output quantity, irrespective of the mathematical model used, is Gaussian.

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In a particular case related to flow rate measurement based on one of two primary weighing systems (the relevant weighing system is part of the mass flow rate measuring system–flow rate primary standard–referred to as MM1 and briefly described in section 4.2), developed and installed at the so-called 'hydraulic test rig' (hereafter briefly named 'test rig') of the Laboratório de Ensaios Hidráulicos (LEHid–Hydraulic Testing Laboratory, which belongs to the Portuguese National Laboratory of Civil Engineering, LNEC, located in Lisbon, Portugal) where flow meter calibration tests are frequently carried out, the measurement uncertainties were evaluated using both methods (GUM and MCM) leading to different results. Analysis of the results clearly shows that the probability distribution of the measurand (mass flow rate) obtained using MCM has a significant departure from normality, colliding with the Gaussian assumption of GUM. This departure should be expected considering the nonlinear contributions found in the mathematical model that supports the measurement results.

In order to promote a better understanding of this problem, the following sections will describe the basics of GUM and MCM methods, putting some emphasis on the GUM constraints and MCM advantages and the weighing system, pointing out its major components and their contributions to the measurement uncertainty evaluation. Afterwards, a discussion of the results is presented, including the layout of output quantity (mass flow rate) probability distribution function and the departure of measurement uncertainties obtained by comparing both approaches.

MEASUREMENT UNCERTAINTY EVALUATION USING MCM

Measurement uncertainty can be evaluated according to different approaches. The most accurate is the analytical approach, applying the mathematical convolution of distribution functions related to the mathematical model (Dietrich 1991). However, this approach becomes too complex to apply to most mathematical models related to measurement processes.

The GUM method (ISO 1995), based on the development of the 1st order Taylor series to functional relations of the type:

$$Y = f(X_1, \ldots, X_N) \tag{1}$$

is known to have several constraints, namely:

- Input quantities must have symmetric probability distributions.
- The central limit theorem is a condition and implies that the output quantity must have a Gaussian distribution function.
- The mathematical model should be linear or have only small nonlinearity effects (the procedure of applying higher order derivatives often increases the complexity significantly).
- The mathematical model must be differentiable.
- The typical mathematical model must have explicit form, with a single output quantity.

It also makes use of subjective information, namely that related to the evaluation of the expansion intervals by the Welch-Satterthwaite formula (ISO 1995) which depends on the degrees of freedom that, in some cases (type B evaluation), can be arbitrarily defined.

Because many measurement systems are complex, nonlinear, having implicit and multifunctional relations, input asymmetric probability functions and highly correlated variables, among other characteristics, the use of GUM becomes inappropriate to evaluate measurement uncertainty in such systems.

The increasing demands and requirements related to measurement systems, the development of computational capabilities and the restrictions related to the use of GUM, enhanced the opportunity to study the application of numerical approaches to this type of metrological problem, the Monte Carlo method being one interesting approach (Cox et al. 2001). It should be pointed out that MCM has been already accepted, in the context of the *Guide to the Evaluation of Uncertainties Framework* (GUF), as an appropriate method to perform this evaluation, which is described in the draft annex 1 of GUF (JCGM 2008).

The nature of the measurement analysis is similar to other stochastic problems that combine distribution

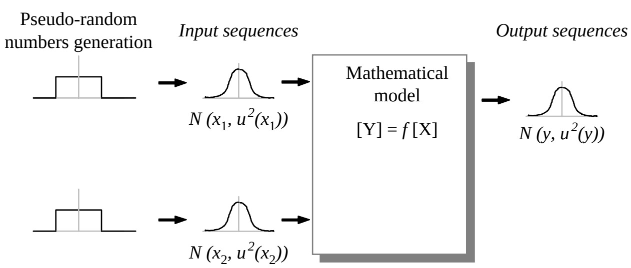


Figure 1 | The propagation of distributions using MCM.

functions of input quantities according to a mathematical model in order to obtain output distribution functions, as presented in Figure 1.

The iterative process of MCM requires the use of validated computational tools. The typical procedure has the following steps:

a. Definition of the input quantities.
b. Definition of an initial numerical sequence size.
c. Generation and validation of pseudo-random number draws (obtaining uniform probability density functions, PDFs, in the interval [0,1]).
d. Conversion of uniform PDF draws into other PDFs of interest (namely, Gaussian, uniform, triangular and arcsin) related to the input quantities.
e. Insertion of the PDF sequences into the mathematical models in order to obtain the PDFs of the output quantities.
f. Ordering of the output PDFs to obtain uncertainty intervals related to the output quantities and evaluate the simulation accuracy.
g. Comparison of the simulation accuracy with the required accuracy, in order to accept the results or increase the sequence size mentioned in b.
h. Report the results.

MCM has several advantages that can be pointed out when compared with GUM, some of the most relevant are (Ribeiro 2006): absence of requirements to the input PDFs and to the evaluation of partial derivatives; use of nonlinear, implicit, output multivariable and complex mathematical models; evaluation of expanded uncertainties without requiring the number of degrees of freedom; evaluation of the correlation as a result and not as a part of the evaluation process; information available at the output, estimates, uncertainties and PDFs; the possibility of using asymmetric input PDFs and obtaining information related to non-Gaussian output PDFs.

In particular, the assumption that the output is a Gaussian PDF can lead to incorrect solutions of some problems where nonlinear behaviour can create deviations of the output estimates or to the uncertainty intervals. In fact, GUM does not provide information regarding these effects because it gives only estimates of the measurement uncertainty under specific conditions including the Gaussian symmetry of the output PDF. The advantages of using MCM are basically related to the fact that it provides the PDFs within all the information it contains. This is the main support for the discussion presented herein.

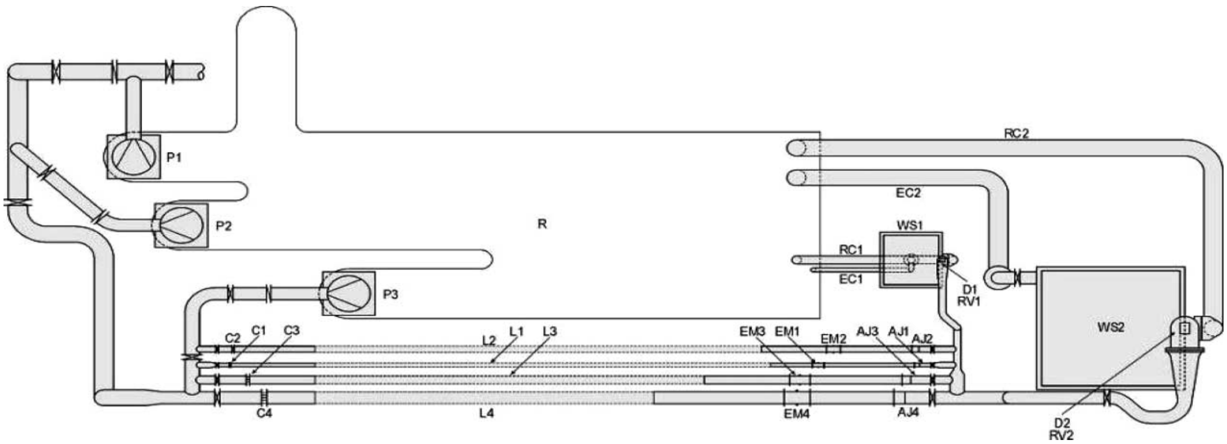
BRIEF PRESENTATION OF LEHID TEST RIG FOR FLOW METER CALIBRATION

Test rig general description

Figure 2 shows a partial view of the LEHid test rig, while a simplified plan view is depicted in Figure 3, where the actual location and identification of the test rig relevant components are presented.

As regards flow meter calibration, every test is run under automatically controlled pumping in closed loop: the water pumped from the reservoir R flows through the selected hydraulic circuit and, by means of one of the RC or EC conduits, returns to R afterwards.

Figure 2 | Partial view of the test rig.



Legend:
AJ*i* - adjustable joint # *i* (*i* = 1, 2, 3, 4)
C*i* - flow conditioner # *i* (*i* = 1, 2, 3, 4)
D*j* - flow diverter # *j* (*j* = 1, 2)
EC*j* - weighing tank emptying conduit # *j* (*j* = 1, 2)
EM*i* - primary device of electromagnetic flowmeter # *i* (*i* = 1, 2, 3, 4)
L*i* - test line # *i* (*i* = 1, 2, 3, 4)
P*k* - pump # *k* (*k* = 1, 2, 3)
R - water supply tank
RC*j* - return conduit # *j* (*j* = 1, 2)
RV*j* - pressure and flow regulating valve # *j* (*j* = 1, 2)
WS*j* - weighing system # *j* (*j* = 1, 2)

Figure 3 | Simplified plan view of the test rig.

The water supply tank has a net capacity of about 340 m^3 and the water depth varies normally between 4.3 m and 4.0 m in the course of a test. The pumping control system adequately deals with such a variation in order to maintain a steady-state flow in the test line in use during each test step.

Each pump has a variable speed drive based on an electronic frequency inverter, thus allowing the output flow rate to be easily set to any required value within its total range. P1 and P2 can operate either separately or in parallel, while P3 can only operate alone.

The four test lines (L1, … , L4) run parallel at different levels and their longitudinal axes are horizontal. They include hot deep galvanised steel pipes and fittings, connected by grooved end quick couplings that greatly facilitate installation of any flow meter to be calibrated and its subsequent removal. Every pipe has one or more pressure tappings, depending on its length.

The test lines have different values of the basic nominal diameter (i.e. the test line nominal diameter when it is constant throughout the entire test line length), DN*, as shown in Table 1, but the same total length of about 15 m. Besides shut-off valves at both ends and an adjustable joint (AJ1, … , AJ4) connected to the downstream valve, each test line is equipped with a flow conditioner (C1, … , C4) installed upstream and an electromagnetic flow meter with primary device (EM1, … , EM4) conveniently located downstream, all of them with a nominal diameter DN equal to DN*.

By using adequate fittings (concentric streamlined reducers and expanders), pipes and couplings, a portion of each test line (drawn with dashed lines in Figure 3) can be converted into a conduit with DN other than DN* as mentioned in Table 1, thus allowing flow meters of different (two or three) nominal sizes to be installed for calibration in the same test line. In every possible case, the straight lengths of upstream and downstream pipes connected to the flow meter submitted to calibration, although dependent on its type, are in general greater than 10*D* and 5*D*, respectively, *D* being the pipe internal diameter.

Table 1 | Basic and converted nominal diameters of the test lines

	Nominal diameter	
Test line	**Basic**	**Converted**
L1	DN 100	DN 80
L2	DN 150	DN 125
L3	DN 200	DN 250
L4	DN 300	DN 350, DN 400

In addition to pump speed variation, two spear regulating valves (RV1 and RV2) can be used for fine setting of pressure and flow rate.

Primary flow rate measuring systems

The test rig is equipped with two mass flow rate measuring systems (flow rate primary standards), MM1 and MM2, of quite different ranges: MM1–50 $kg\,s^{-1}$; MM2–350 $kg\,s^{-1}$. Each one of them–MMj ($j = 1, 2$)–has the following components:

- a flow diverter (Dj)
- a weighing system (WSj), composed of a weighing platform and a weighing tank (WTj) placed on it
- a time interval meter (hereafter referred to as 'chronometer').

Each flow diverter operation is synchronized with the chronometer, which is actuated by a half-turn electric motor driven by a dedicated frequency inverter.

As regards the weighing systems, their safe (i.e. with no risk of weighing tank overflow for any practicable flow rate value) measuring ranges are the following: WS1–1,700 kg and WS2–17,600 kg. Each one of them–WSj ($j = 1, 2$)–measures the apparent mass of water m_a (not the conventionally true mass m) put into the weighing tank WTj during the time interval Δt (measured by the chronometer) elapsed between the initial instant when the flow (necessarily under steady-state condition) is diverted to WTj and the instant of its diversion to the conduit RCj that returns it to the water supply tank R.

During any calibration test, the water temperature is measured at two flow sections of the test line (one located upstream and the other downstream) and, whenever applicable, inside the weighing tank.

The mathematical model

The mathematical model applied is based on the principle of gravimetric measurement of mass flow rate, the expression of the functional relation being given by:

$$Q_m = \frac{\Delta m}{\Delta t} = \frac{m_f - m_0}{t_f - t_0} = \frac{m_{cf} - m_{c0}}{t_f - t_0} \cdot \frac{\left(1 - \frac{\rho_a}{\rho_p}\right)}{\left(1 - \frac{\rho_a}{\rho}\right)} \quad (2)$$

where Q_m is the mass flow rate, m the mass of weighed liquid (water), m_c the conventional mass of that liquid, Δt the time interval of weighing tank filling, ρ_a the air density, ρ_p the reference density of the standard weights, ρ the liquid density, and the indexes '0' and 'f' meaning the initial and final quantity observations in the test.

Equation (2) may appear in some documents modified to (3), where the buoyancy constant, $C_b = \left(1 - \frac{\rho_a}{\rho_p}\right) \cdot \left(1 - \frac{\rho_a}{\rho}\right)^{-1}$, is included:

$$Q_m = \frac{m_{cf} - m_{c0}}{t_f - t_0} \cdot \frac{\left(1 - \frac{\rho_a}{\rho_p}\right)}{\left(1 - \frac{\rho_a}{\rho}\right)} = \frac{\Delta m_c}{\Delta t} \cdot C_b \quad (3)$$

The mathematical model to be used in the evaluation of uncertainty should also take into account other systematic effects that influence the measurand. This is the case of the mass difference, δm, expressing the variation in mass related to the time delay of the flow diverter operation. In order to input this influence in the mathematical model, δm was added to Equation (2), thus leading to Equation (4), used as the functional relation in the evaluation of measurement uncertainties:

$$Q_m = \frac{(\Delta m_c + \delta m)}{\Delta t} \cdot \frac{\left(1 - \frac{\rho_a}{\rho_p}\right)}{\left(1 - \frac{\rho_a}{\rho}\right)} \quad (4)$$

Table 2 presents the contributions to the uncertainty budget related to each input quantity present in the model above.

Contributions to the output measurement uncertainty

The study carried out (Ribeiro 2006) was based on a set of input quantities and on the assumption of probability distributions that, according to the experimental data, instrumentation manuals and reference data, characterize such input quantities. Table 2 summarizes the quantities, their PDFs and the parameters used.

The set of input quantities presented in Table 2 does not include quantities studied without significant influence in the results: namely, repeatability, linearity, mobility, electrostatic and magnetic influences, reversibility and levelling of the weighing platform, quantities related to the weighing instruments and voltage variation due to influence between

Table 2 | Description of sources of uncertainty and their PDF data related to the input quantities

	Sources of uncertainty		
Input quantity	**Designation**	**Symbol**	**PDF**
m_{cf}, 1,700 kg	Resolution of weighing instrument	$\delta m_{c,res}$	R $(0 \pm 2.5 \times 10^{-2})$ kg
	Excentricity	$\delta m_{c,exc}$	R $(0 \pm 1 \times 10^{-2})$ kg
	Calibration uncertainty	$\delta m_{c,cal}$	N $(0 \pm 3.9 \times 10^{-2})$ kg
	Zero drift	$\delta m_{c,0}$	R $(0 \pm 1 \times 10^{-2})$ kg
	Temperature influence	$\delta m_{c,T}$	T $(0 \pm 1.7 \times 10^{-2})$ kg
	Long-term drift	$\delta m_{c,d}$	R $(0 \pm 5 \times 10^{-2})$ kg
	Data processing	$\delta m_{c,p}$	R $(0 \pm 5 \times 10^{-2})$ kg
m_{c0}, 0 kg	Resolution of weighing instrument	$\delta m_{c0,res}$	R $(0 \pm 2.5 \times 10^{-2})$ kg
	Excentricity	$\delta m_{c0,exs}$	R $(0 \pm 1 \times 10^{-2})$ kg
	Zero drift	$\delta m_{c0,0}$	R $(0 \pm 1 \times 10^{-2})$ kg
	Temperature influence	$\delta m_{c0,T}$	T $(0 \pm 1.7 \times 10^{-2})$ kg
	Long-term drift	$\delta m_{c0,d}$	R $(0 \pm 5 \times 10^{-2})$ kg
	Data processing	$\delta m_{c0,p}$	R $(0 \pm 5 \times 10^{-2})$ kg
δm	Process mass loss	$\delta m_{c,\Delta m}$	U $(0; 5 \times 10^{-3})$ kg
	Flow diverting process	$\delta m_{c,def}$	U $(-Q_m \times \Delta t_{def}; 0)$ kg
Δt	Indicator resolution (A/D conv. 12 bit)	δt_{res}	R $(0 \pm 2.5 \times 10^{-4}\ \Delta t)$ s
Notes: $\Delta t = 34$ s (for $Q_m = 50\ \mathrm{kg\,s^{-1}}$) and $\Delta t = 170$ s (for $Q_m = 10\ \mathrm{kg\,s^{-1}}$)	Repeatability	δt_{rep}	R $(0 \pm 5 \times 10^{-5}\ \Delta t)$ s
	Calibration uncertainty	δt_{cal}	R $(0 \pm 1 \times 10^{-7}\ \Delta t)$ s
	Signal triggers	δt_{trig}	R $(0 \pm 1 \times 10^{-5}\ \Delta t)$ s
	Signal stability	δt_{e}	R $(0 \pm 1 \times 10^{-4}\ \Delta t)$ s
	Temperature influence	δt_{T}	R $(0 \pm 4 \times 10^{-4}\ \Delta t)$ s
	Drift	δt_{d}	R $(0 \pm 3.6 \times 10^{-5}\ \Delta t)$ s
	Method reproducibility	δt_{reprod}	R $(0 \pm 1 \times 10^{-7}\ \Delta t)$ s
	Data processing	δt_{p}	R $(0 \pm 1 \times 10^{-4}\ \Delta t)$ s
ρ	Water density	ρ	R $(1{,}000 \pm 2)\ \mathrm{kg\,m^{-3}}$
ρ_a	Air density	ρ_a	N $(1{,}193 \pm 0.032)\ \mathrm{kg\,m^{-3}}$

channels or electromagnetic interferences, related to the time measurement. Regarding the data presented, some additional information should be given.

The data related to input quantities of time measurement are multiplied by Δt, meaning the time interval measured, because the numerical values presented were obtained under relative form.

The liquid diverting into the weighing tank and back to the reservoir results from the forward and backward movement of the flow diverter, which includes a wheel and an arm that control a baffle plate pivoting around a horizontal axis. The chronometer mentioned above, which can detect the start and stop triggering of the flow diverter, is connected to the flow diverter and measures the correspondent time interval.

The flow diverter operation should be balanced in order to provide equivalent amount of liquid diverted into the weighing tank and back to the reservoir at the baffle transitions (ISO 4185 1990). In an ideal process, these transitions are instantaneous but in the actual process each of these movements has its own duration, as shown in Figure 4.

In the practical case under study, each of these steps was developed in a half turn of the wheel, leading to a time

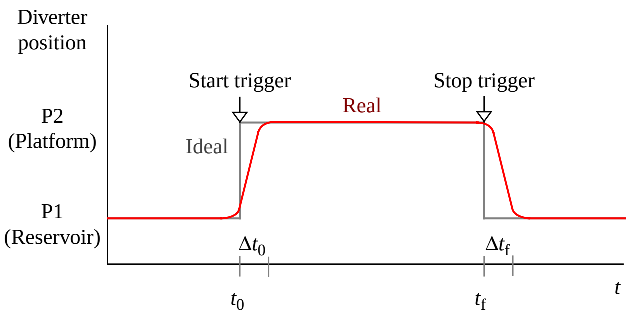


Figure 4 | Ideal and actual behaviour of flow diverting signal.

delay in each one of the transitions of the process. This time delay was measured experimentally using optical sensors, allowing measurement of the time differences, Δt_{def} (see Figure 5), which are within the interval [0 ms; 5 ms], the values near 5 ms being the most probable.

Therefore, to consider this effect as an input quantity, an asymmetric PDF was adopted as adequate to the physical representation of its behaviour. This was achieved using a half-arcsin PDF (Ribeiro 2006), which can give this type of asymmetric probability function, its expression being:

$$f(\xi; a) = \begin{cases} 0 & \text{if } \xi < 0 \\ \frac{2}{\pi\sqrt{a^2 - \xi^2}} & \text{if } 0 \le \xi < a \\ 0 & \text{if } \xi \ge a \end{cases} \quad (5)$$

the corresponding graphic of which, in the case of the flow diverter input quantity, is shown in Figure 6.

A similar behaviour was observed in the measurement of the input quantity related to the mass loss during the weighing tank filling process. In fact, the evaluation of this input quantity shows that there is an asymmetric

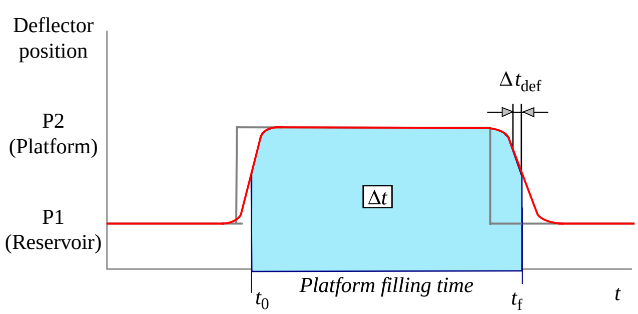


Figure 5 | Time delay due to the final flow diverting process.

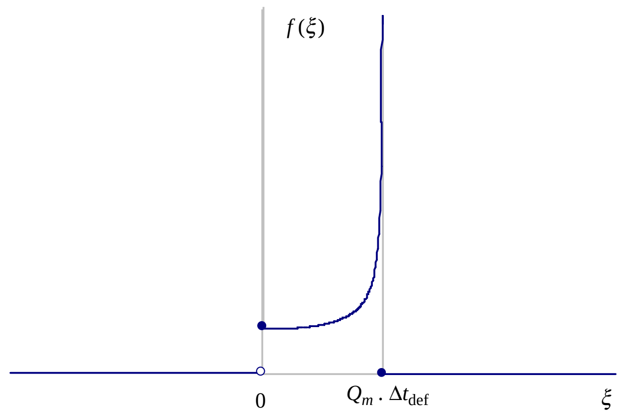


Figure 6 | Half-arcsin probability density function applied to the flow diverting input quantity.

distribution from 0 (no mass loss) to a maximum value of 5×10^{-3} kg, the most probable values being those near the maximum. Again, the half-arcsin PDF is a good option to provide a solution instead of a symmetric distribution centred on zero, which gives physically meaningless negative values.

EVALUATION OF THE MASS FLOW RATE MEASUREMENT UNCERTAINTY AND DISCUSSION OF THE GAUSSIAN ASSUMPTION

The MCM numerical simulation aiming at the evaluation of flow rate measurement uncertainty was developed considering two nominal values of mass flow rate of 10 kg s^{-1} and 50 kg s^{-1}.

The use of resources properly validated is a major condition to achieve results with appropriate quality; therefore, the algorithms implemented were previously tested, and are based on well-known algorithms. The solutions adopted were the following:

- Mersenne twister pseudo-random number generator (Matsumoto & Nishimura 1998).
- Box-Muller algorithm to obtain Gaussian PDF and inverse function method to obtain the other PDFs (Knuth 1998).
- QuickSort algorithm to order the output sequences (Press *et al.* 1986).
- Accuracy criteria based on Cox *et al.* (2001).

Another major condition to apply MCM is to assure that the accuracy obtained is within the required accuracy based on the proposed objective. In this case, the accuracy required should be better than 0.005%.

The simulation can be developed as an iterative process, with a comparison in each iterative step or, as was the case here, a constant size of the numerical sequences could be established in order to ensure that the accuracy requirement was fulfilled. Preliminary tests showed that sizes of 10^6 values in these numerical sequences would create higher accuracy than that required, as can be seen in Table 3.

The numerical sequences related to each input quantity presented in Table 2 were combined according to the mathematical model (Equation (4)) for the nominal values of mass flow rate ($10\,kg\,s^{-1}$ and $50\,kg\,s^{-1}$), leading to the MCM evaluation of the output quantity: mass flow rate. The generated output PDF of the mass flow rate allows us to obtain the statistical parameters of interest: namely, the mean value, the experimental standard deviation and the standard and expanded measurement uncertainty intervals, $w(Q_m)$ and $W(Q_m)$, respectively.

A comparison between MCM and other methods enhances an important advantage of MCM: its output provides information on the probability density functions while the others only give the estimates and the uncertainty interval. In fact, besides other features, MCM returns information on function shape, symmetry and skewness. This fundamental issue is essential to the purpose of this paper. The study carried out using MCM allowed us to build the PDFs presented in Figure 7.

At a first glance, the visual analysis of these PDFs would predict a Gaussian shape; however, differences were obtained from the comparison of the statistical parameters:

Table 3 | Results obtained using MCM for two nominal mass flow rates

Parameters (mass flow rate)	$Q_m = 10\,kg\,s^{-1}$	$Q_m = 50\,kg\,s^{-1}$
Mean value	$10.0107\,kg\,s^{-1}$	$50.0571\,kg\,s^{-1}$
Standard deviation	±0.019%	±0.020%
$w(Q_m)$	±0.020%	±0.021%
$W(Q_m)$	±0.037%	±0.038%
Accuracy (95%)	±0.0008%	±0.003%

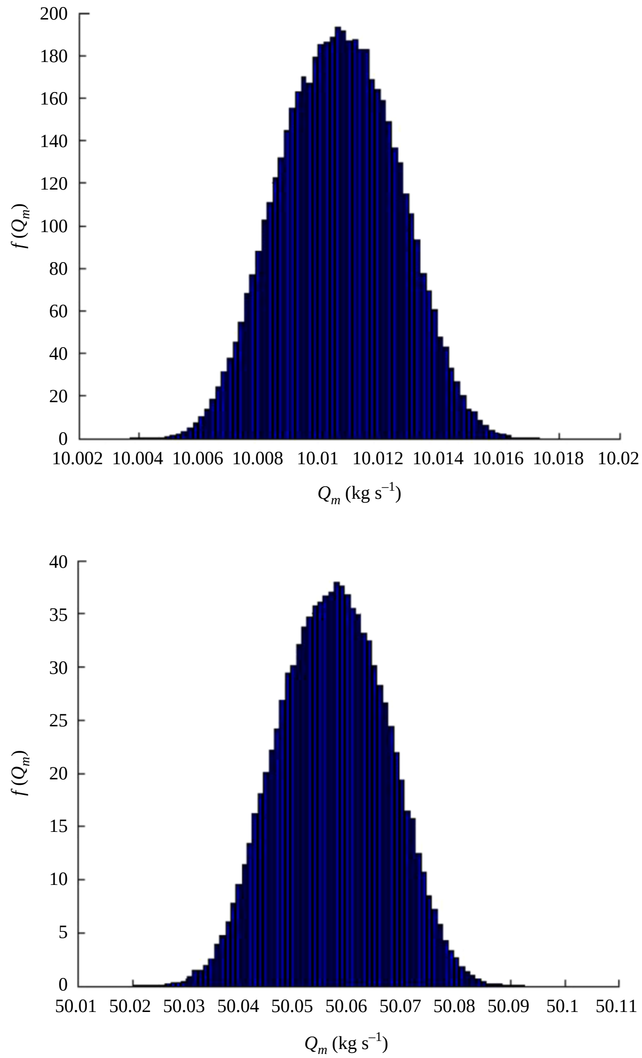


Figure 7 | Output PDFs related to the two mass flow rate measurements studied.

standard deviation vs. standard measurement uncertainty, and the values of 1.96 times the standard measurement uncertainty vs. the 95% expanded measurement uncertainty.

Because of the deviations found, it was decided to study the other moments of the probability distribution, especially those concerning symmetry location and variability related to the Gaussian shape; that is, *skewness* and *excess kurtosis*. Both parameters are well known (Mood *et al.* 1974), and are defined by the following equations:

$$\text{skewness} = \mu_3 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^3}{(n-1)s^3} \tag{6}$$

Table 4 | Skewness and excess kurtosis parameters obtained from the output PDFs

	Mass flow rate	
Parameters	$Q_m = 10\,\mathrm{kg\,s^{-1}}$	$Q_m = 50\,\mathrm{kg\,s^{-1}}$
Skewness	−0.008	−0.003
Excess kurtosis	−0.41	−0.38

$$\text{excess kurtosis} = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^4}{(n-1)s^4} - 3 \tag{7}$$

In these formulas, $\bar{y}$ is the average value of the quantity Y, n is the sample size and s the standard deviation of the sample.

Regarding the parameters mentioned, it should be remembered that, in the first case, a zero value means a symmetric shape while a negative or positive value means that the data are left or right skewed, respectively. In the second case, the zero value means a Gaussian shape while a negative or positive value means that the PDF is more flat or peaked than the Gaussian shape, respectively.

The analysis of the output values led to the results presented in Table 4, showing a small effect of asymmetry due to the use of the half-arcsin asymmetric PDF (the evaluation of uncertainty without considering the two asymmetric PDFs as input data produced output PDFs with skewness of about 10 times lower than those that include the asymmetry effect, presented in Table 4) and also revealing a 'flatness' of the PDF with a significant departure from the Gaussian shape. This would explain the differences obtained and mentioned above (between standard deviation, standard uncertainty and expanded uncertainty) enhancing this very important feature of the Monte Carlo method.

The measurement uncertainties were also evaluated using mainstream GUM approach. The results obtained have significant differences of about 20% lower for the measurement uncertainty interval and a 3% bias for the flow rate estimates.

CONCLUSIONS

The diversity and complexity of many measurement systems require the development of new solutions related to the metrological characterization of those systems. The Monte Carlo method is a simple and simultaneously robust solution to promote a better understanding of the measurement results.

This robustness comes from many aspects, one of the most relevant being perhaps the fact that MCM uses directly the input information (quantity values) in the mathematical models without constraints, while approaches such as GUM require intermediate operations as model differentiation and evaluation of correlation between quantities, creating difficulties in many applications.

The study and the discussion carried out is a contribution to show how important the additional information provided by the MCM can be. In fact, the ability to obtain probability distributions rather than only the uncertainty intervals can give a better assessment of the measurement results, thus allowing discussion of important issues such as whether the output estimate has a symmetric and Gaussian shape and, also, to decide which parameter would be the best estimate.

Much of the discussion of this study could be applied to any other similar metrological problem, which is consistent with the definition of metrology and, therefore, the contribution given has a wider application. Nevertheless, it is an important contribution in this specific domain of hydraulics, in its metrological perspective, because it provides, simultaneously, a means to obtain measurement uncertainties in calibration processes using gravimetric flow rate standards and also a robust MCM solution to the evaluation of measurement uncertainties, overcoming a particular influential behaviour that is 'invisible' to other approaches.

The MCM approach is a robust numerical tool that provides convergent solutions for complex measurement problems, thus being able to validate the use of mainstream GUM. In fact, MCM can generally be used with this purpose, allowing the user to decide whether mainstream GUM is advisable or not. As shown in the case above, GUM provides a solution that underestimates measurement uncertainties and gives biased flow rate estimates. Therefore, it is strongly recommended that the uncertainties of flow rate measurements provided by this type of test rig should be obtained using MCM instead of the mainstream GUM.

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