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Discussions

Discussion: ''Estimating the Probability Distribution of von Mises Stress for Structures Undergoing Random Excitation'' [ASME J. Vibr. Acoust., 122, No. 1, pp. 42–48 (2000)][1]

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In their recent paper [1] Segalman and coauthors estimate the probability distribution for the von Mises stress resulting from Gaussian random loadings of zero mean. Although they do not explain clearly what use is intended for this ''important result for reliability of structures,'' we anticipate that it must be somehow related either to fatigue or to the estimation of the extreme value over some return period (peak factor). The problem with the von Mises stress when it is defined in the time domain as it is in this paper is that:

- it is always positive and does not reduce itself to the applied alternating stress in the simple uniaxial case.
- its frequency content is not consistent with the frequency content of the stress components and the natural frequencies of the structure.

These drawbacks which are related to the quadratic form of the von Mises stress can be removed by an alternative definition in the frequency domain [2,3]: If σ is the stress vector, the von Mises stress σ_c is defined in the time domain by

$$\sigma_c^2=\sigma^T Q\sigma=\text{Trace}\{Q[\sigma\sigma^T]\} \tag{1}$$

where Q is a constant matrix. Taking the expected value, we get

$$E[\sigma_c^2]=\text{Trace}\{QE[\sigma\sigma^T]\} \tag{2}$$

We recognize $E[\sigma\sigma^T]$ as the covariance matrix of the stress vector, related to the PSD matrix of the stress vector $\Phi_{\sigma\sigma}(\omega)$ by

$$E[\sigma\sigma^T]=\int_{-\infty}^{\infty}\Phi_{\sigma\sigma}(\omega)d\omega \tag{3}$$

From Eq. (2) and (3), we have

$$E[\sigma_c^2]=\int_{-\infty}^{\infty}\Phi_c(\omega)d\omega = \text{Trace}\left\{Q\int_{-\infty}^{\infty}\Phi_{\sigma\sigma}(\omega)d\omega\right\} \tag{4}$$

where $\Phi_c(\omega)$ is the PSD of the equivalent von Mises stress. Equation (4) is exact and does not involve any assumption. Next, we can *define* a Gaussian random process of zero mean by

$$\Phi_c(\omega)=\text{Trace}\{Q\Phi_{\sigma\sigma}(\omega)\} \tag{5}$$

We call it ''equivalent von Mises stress.'' It is obvious that Eq. (4) is satisfied. Unlike the definition in the time domain as in [1], the foregoing random process is alternating, it reduces itself to the component stress in the uniaxial case and it is consistent with the frequency content of the stress components. From the PSD defined by Eq. (5), classical uniaxial random fatigue life prediction methods can be applied [2,3] and the peak factor formulae can also be used [3]. The foregoing formulation in the frequency domain has been found extremely fast and useful in predicting the critical areas of structures subjected to random loading. It is worth pointing out that the spectral formulation can be extended to the multiaxial rainflow method [4] and to the Crossland failure criterion [5].

References

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Closure to ''Discussion of 'Estimating the Probability Distribution of von Mises Stress for Structures Undergoing Random Excitation'' [ASME J. Vibr. Acoust., 122, No. 1, pp. 42–48 (2000)]

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The core issue raised by Preumont and Pitoiset is the question of why one would be interested in von Mises stress—as ordinarily defined—since it appears to have features that make it difficult to apply to fatigue analysis.

The answer is that this work, as well as most other work that evaluates the von Mises stress, is focused on other forms of structural and material failure than fatigue. In problems of yield and

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ductile fracture, the features of von Mises stress that Preumont and Pitoiset find off-putting are believed to capture the physics reasonably well. In such problems, the statistics of von Mises stress are standard tools to assess reliability.

In their final comments Preumont and Pitoiset advocate the utility of the autospectral density function of the stress vector in studying fatigue problems. To us, those comments suggest the interesting notion of extending the probabilistic methods presented in our paper to frequency space.

Discussion: ''On Stability of Time-Varying Multidimensional Linear Systems'' [ASME J. Vibr. Acoust., 121, No. 4, pp. 509–511 (1999)][1]

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Formal study of the stability of systems started more than a century ago by the pioneering work of Routh, Poincaré, and Lyapunov. Since then many researchers have been vigorously developing the theory of the stability of systems. As a result, presently there is a vast body of literature on this subject. Thus, it may be impossible for an individual researcher to know a good portion of the available results. This can possibly lead to the establishment or rediscovery of stability results whose stronger versions have been already obtained. Such is more or less the case for the stability result in [1].

In [1], the authors study the stability of the system

$$\ddot{x}(t)+D\dot{x}(t)+K(t)x(t)=\theta_n, \quad x(0)=:x_0, \quad \dot{x}(0)=:\dot{x}_0, \tag{1}$$

for all $t\geqslant 0$, where $x(t)\in\mathbb{R}^n$ and θ_n is the zero vector in $\mathbb{R}^n$. In (1), the constant damping matrix $D\in\mathbb{R}^{n\times n}$ is symmetric and positive definite and so is the stiffness matrix $K(t)$ for all $t\geqslant 0$. Moreover, the function $t\mapsto K(t)$ is continuously differentiable and there exist positive constants k_0 and k_1 such that

$$k_0 I_n \leqslant K(t) \leqslant k_1 I_n, \tag{2}$$

for all $t\geqslant 0$, where I_n is the $n\times n$ identity matrix.

The authors let $\sigma(D)=\{d_1,d_2,\ldots,d_n\}$ denote the set of eigenvalues (spectrum) of the matrix D where the elements of this set are ordered as $0<d_1\leqslant d_2\leqslant \ldots \leqslant d_n$. They denote the maximum eigenvalue of a symmetric matrix H by $\lambda_{\max}(H)$.

With this setup, the stability result in [1] is:

Theorem 1. Let

$$\alpha := \max_{t\geqslant 0}\{\lambda_{\max}(K^{-1/2}(t)\dot{K}(t)K^{-1/2}(t))\}, \tag{3}$$

$$\hat{D} := D - d_1 I_n. \tag{4}$$

If (i)

$$\hat{D}K(t)+K(t)\hat{D}\geqslant 0, \tag{5}$$

for all $t\geqslant 0$ and (ii)

$$d_1 > \frac{\alpha}{2}, \tag{6}$$

then the system (1) is exponentially stable. □

It should be noted that for a general function $t\mapsto K(t)$, it is straightforward to derive $t\mapsto \dot{K}(t)$ in the closed form; however, it is not as such for the function $t\mapsto K^{-1/2}(t)$. Therefore, when Theorem 1 is used, α in (3) should be computed numerically by computing $\lambda_{\max}(K^{-1/2}(t)\dot{K}(t)K^{-1/2}(t))$ at instances $t=0$, t_1, $t_2,\ldots,t_i,\ldots$, which are separated from each other by a suitable step size h. Moreover, when Theorem 1 is used, the eigenvalues of the damping matrix D should be computed, the matrix $\hat{D}$ be formed, and the truth of inequality (5) be verified (numerically).

A stability result for the system (1), even when the damping matrix is time varying is given in Gil' (1998, p. 70). This result is:

Theorem 2. Consider the system

$$\ddot{x}(t)+D(t)\dot{x}(t)+K(t)x(t)=\theta_n, \quad x(0)=:x_0, \quad \dot{x}(0)=:\dot{x}_0, \tag{7}$$

for all $t\geqslant 0$, where $x(t)\in\mathbb{R}^n$. Let the function $t\mapsto D(t)$ be piecewise continuous, the matrix $K(t)$ be symmetric and positive definite for all $t\geqslant 0$, and the function $t\mapsto K^{1/2}(t)$ be differentiable. If

$$P(t)+P^T(t)\leqslant 0, \tag{8}$$

for all $t\geqslant 0$, where

$$P(t) := -K^{-1/2}(t)[(K^{1/2}(t))^{\cdot}+D(t)K^{1/2}(t)], \tag{9}$$

then the system (1) is stable. □

When Theorem 2 is used, the matrix $P(t)$ in (9) should be computed numerically at instances $t=0,t_1,t_2,\ldots,t_i,\ldots$, which are separated from each other by a suitable step size h, by the formula

$$P(t_i) = -K^{-1/2}(t_i)\left[\frac{K^{1/2}(t_{i+1})-K^{1/2}(t_i)}{t_{i+1}-t_i}+D(t_i)K^{1/2}(t_i)\right]. \tag{10}$$

Having computed $P(t)$ at $t=0$, t_1, $t_2,\ldots,t_i,\ldots$, the truth of inequality (8) can be examined (numerically).

It appears that in terms of the computational efficiency the stability test in Theorem 1 is not superior to that in Theorem 2. Thus, what should be examined is the applicability of Theorems 1 and 2 in establishing the stability of systems. In the following, these theorems are applied to an example.

Consider the system

$$\begin{bmatrix}\ddot{x}_1(t)\\ \ddot{x}_2(t)\end{bmatrix}+\begin{bmatrix}0.6 & 0.05\\ 0.05 & 0.5\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}+\begin{bmatrix}\left(\frac{1+2t}{1+t}\right)^2 & 0\\ 0 & \left(\frac{1+2t}{1+t}\right)^2\end{bmatrix}\times\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\theta_2, \tag{11}$$

for all $t\geqslant 0$. Identifying the matrices D and $K(\cdot)$ in (11), the following can be obtained

$$d_1 = 0.4793, \quad d_2 = 0.6207, \tag{12}$$

$$K^{-1/2}(t)\dot{K}(t)K^{-1/2}(t)=\begin{bmatrix}\frac{2}{(1+2t)(1+t)} & 0\\ 0 & \frac{2}{(1+2t)(1+t)}\end{bmatrix}, \tag{13}$$

for all $t\geqslant 0$. From (13), it is clear that $\alpha=2$, and hence inequality (6) does not hold. That is, Theorem 1 cannot determine the sta-

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bility of the system (11). Roughly speaking, Theorem 1 could have established the stability of the system (11) if the system were more damped, i.e., if d_1 were larger.

Next, using (9), it is concluded that for the system (11),

$$P(t) = -\begin{bmatrix} \frac{1}{(1+2t)(1+t)} & 0 \\ 0 & \frac{1}{(1+2t)(1+t)} \end{bmatrix} - \begin{bmatrix} 0.6 & 0.05 \\ 0.05 & 0.5 \end{bmatrix}, \tag{14}$$

for all $t \geqslant 0$. Having $P(\cdot)$ in (14), it is easily concluded that inequality (8) holds. Thus, by Theorem 2, the system (11) is stable. In fact, it is clear from (14) that for any positive definite matrix D, no matter how small its eigenvalues are, inequality (8) holds, and the stability of the system (11) is guaranteed. Roughly speaking, with a very small damping (dissipation), the system (11) is stable.

In conclusion, the stability result in Gil' (1998, p. 70) is stronger than that in [1], because (i) it can handle time varying damping matrices; (ii) it can establish the stability of systems whose stability cannot be determined by the result in [1], as it was shown by an example.

References

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Closure to ''Discussion of 'On Stability of Time-Varying Multidimensional Linear Systems''' [ASME J. Vibr. Acoust., 121, No. 4, pp. 509–511 (1999)]

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Dr. Shahruz claims that Theorem 2 in [1] is stronger than Theorem 1 in [2]. Since Dr. Shahruz proves his claim by showing an example, not a mathematical derivation, I feel his argument is not convincing. Also it indicates clearly in [2], that the main purpose of the Theorem in [2] is to use damping to eliminate the parametric resonance. So there is no need for considering a time-varying damping.

References

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