Data analysis of continuous gravitational wave: Fourier transform – II

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ABSTRACT

In this paper we obtain the Fourier Transform of a continuous gravitational wave. We have analysed the data set for (i) a 1-yr observation time and (ii) an arbitrary observation time, for an arbitrary location of detector and source, taking into account the effects arising due to the rotational as well as orbital motion of the Earth. As an application of the transform we considered spin-down and N-component signal analysis.

Key words: gravitational waves – methods: data analysis – pulsars: general.

1 INTRODUCTION

Detection of gravitational waves (GW) mainly depends on the efficient Fourier analysis of the data of the output of the detector such as LIGO/VIRGO. In an earlier paper we presented Fourier analysis of a 1-d observation data set of the response of a laser interferometer (Srivastava & Sahay 2002a). Hereafter the reference to this paper and its results are cited as Paper I. We have seen that the amplitude and frequency modulations result in a large number of side bands about the signal frequency \( f_o \). Consequently, the maximum power lies in the frequency \( f_o + 2f_m \) with amplitude reduction by 74 per cent to what one would have expected due to increased data interval. Here, \( f_m \) is the rotation frequency of the Earth. Hence, for GW detection it is desirable to obtain Fourier transform (FT) for longer observation data. In the next section we investigate 1-yr observation data. Incidently, there exists correspondences and (FT) for longer observation data. In the next section we investigate 1-yr observation data. Incidently, there exists correspondences and (FT) for longer observation data.

To facilitate analogous modifications we introduce corresponding quantities with a tilde viz., \( \tilde{C} \) and \( \tilde{D} \) in place of \( C \) and \( D \). In Section 3 we obtain generalizations of the results for an arbitrary observation time. As an application of the results obtained we consider spin-down and \( N \)-component signal analysis in Sections 4 and 5. The last section contains the discussion and the conclusions of the paper.

2 FOURIER TRANSFORM FOR 1-YR INTEGRATION

The phase of the GW signal of frequency \( f_o \) at time \( t \) is given via Paper I equation (25), i.e.

\[
\Phi(t) = 2\pi f_o t + Z \cos(w_{ob}t - \phi) + N \cos(w_{rot}t - \delta) - R - Q
\]  

where

\[
\begin{align*}
\mathcal{P} &= 2\pi f_o \frac{Z}{c} \sin \alpha (\cos \beta_o (\sin \theta \cos \epsilon + \cos \theta \sin \epsilon) + \cos \beta_o \sin \theta \cos \epsilon), \\
\mathcal{Q} &= 2\pi f_o \frac{Z}{c} \sin \alpha (\sin \beta_o (\sin \theta \cos \epsilon + \cos \theta \sin \epsilon) + \cos \beta_o \sin \theta \cos \epsilon), \\
\mathcal{N} &= \sqrt{\mathcal{P}^2 + \mathcal{Q}^2}, \\
Z &= 2\pi f_o \frac{Z}{c} \sin \theta, \\
R &= \mathcal{Z} \cos \phi,
\end{align*}
\]

\[
\begin{align*}
\delta &= \tan^{-1} \frac{\mathcal{P}}{\mathcal{Q}}, \\
\phi' &= w_{ob} t - \phi, \\
\xi_{tot} &= w_{ob} t = a \xi_{rot}, \\
\alpha &= w_{ob} / w_{rot} \approx 1/365.26, \\
\xi_{rot} &= w_{rot} t
\end{align*}
\]

where \( R_o, R_{ob}, w_{ob} \) and \( w_{ob} \) represent respectively the Earth’s radius, the average distance of the Earth’s centre from the origin of Solar system barycentre (SSB) frame, and the rotational and the orbital angular velocity of the Earth. \( \theta \) and \( \phi \) denote the celestial colatitude and celestial longitude of the source. \( \epsilon \) and \( \epsilon \) represent the obliquity of the ecliptic and the velocity of light. \( \alpha \) is the colatitude of the detector. Here \( t \) represents the time in s elapsed from the instant the Sun is at the Vernal Equinox and \( \beta_o \) is the local sidereal time at that instant, expressed in radians.

The FT for a 1-yr observation time, \( T_{obs} \), is given as

\[
\left[ \tilde{h}(f) \right] = \int_0^{aT} \cos[\Phi(t)]e^{-i2\pi f't} \, dt;
\]

\[
a = a^{-1} = \frac{w_{ob}}{w_{rot}}; \quad T = \text{one sidereal day};
\]

\[
T_{obs} = \bar{a}T \simeq 3.14 \times 10^7 \text{ s}.
\]
This splits, similar to Paper I equations (33–35), into two terms as
\[
[h(f)]_y = I_{h_+} + I_{h_-};
\]
\[
I_{h_\pm} = \frac{1}{2w_\text{orb}} \int_0^{2\pi} e^{i(\xi(t) + Z \cos(\phi) + N \cos(\theta) - \phi - \phi)} d\xi,
\]
(8)

obtained for 1-d observation data reveals that the integrand of the
\[
D
\]
and
\[
J
\]
may, therefore, employ the results obtained there by introducing
\[D\]
and \[J\], in place of \[\tilde{B}\] and \[\tilde{D}\] in place of \[\bar{B}\] and \[\bar{D}\], leaving \[A\] unchanged. Hence
\[
\int h(f) \, d\xi \sim \frac{\bar{v}}{2w_\text{orb}} \sum_{k=\infty}^{\infty} \sum_{m=\infty}^{\infty} e^{iA} \tilde{B}[-i\bar{D}];
\]
(13)

where \[\bar{D}\] is associated with \[\tilde{B}\] and \[\tilde{D}\] with \[\bar{B}\]. We may, therefore, employ the results obtained there by introducing
\[\bar{A}\] and \[\bar{D}\] in place of \[A\] and \[D\], leaving \[\tilde{B}\] and \[\tilde{D}\] to be considered depends on the arguments of Bessel functions i.e. \[Z\] \[N\] and \[Q\]. Referring to equation (2) it is found that
\[
Z_{\text{max}} = 3133215 \left( \frac{1}{134} \right)
\]
and \[\gamma = \pi/4\].

The FT of a frequency-modulated (FM) signal for
\[
f_0 = 50 \text{ Hz}, \quad h_{\phi_0} = h_{\phi_1} = 1\]
\[
\alpha = \pi/4, \quad \beta = 0, \quad \gamma = \pi/4, \quad \theta = \pi/18, \quad \phi = 0, \quad \psi = \pi/4
\]
is shown in Fig. (1). The spectrum resolution is kept equal to \(1/134\) Hz. We have convinced ourselves by plotting the FT at higher resolutions that the resolution of \(1/134\) Hz is sufficient to represent the relevant peaks. We notice that the drop in amplitude at the source frequency is about 98 per cent which may be attributed to the presence of a large number of side bands.

The complete response \(R(t)\) of the detector and its FT can be
obtained similar to the one presented in section 4 of Paper I. Taking the results straight away from Paper I equations (43, 44, 45) one may get
\[
\tilde{R}(f) = [\tilde{R}_1(f)]_y + [\tilde{R}_2(f)]_y
\]
\[
= \frac{1}{2} \left\{ e^{\imath 2\phi_0} [h(f + 2f_\text{rot})]_y \right. \]
\[
\times \left[ h_{\phi_0} \left( F_{1_x} + i F_{1_y} \right) + h_{\phi_1} \left( F_{2_x} - i F_{2_y} \right) \right]
\]
\[
+ e^{\imath 2\phi_1} [h(f - 2f_\text{rot})]_y \right. \]
\[
\times \left[ h_{\phi_0} \left( F_{3_x} + i F_{3_y} \right) + h_{\phi_1} \left( F_{4_x} + i F_{4_y} \right) \right]
\]
\[
+ e^{\imath \phi_0} [h(f + f_\text{rot})]_y \right. \]
\[
\times \left[ h_{\phi_0} \left( F_{6_x} + i F_{6_y} \right) + h_{\phi_1} \left( F_{7_x} + i F_{7_y} \right) \right]
\]
\[
+ e^{\imath \phi_1} [h(f - f_\text{rot})]_y \right. \]
\[
\times \left[ h_{\phi_0} \left( F_{2_x} - i F_{2_y} \right) + h_{\phi_1} \left( F_{4_x} - i F_{4_y} \right) \right]
\]
\[
\left. + 2i[h(f)]_y \left[ h_{\phi_0} F_{5_x} - i h_{\phi_1} F_{5_y} \right] \right\}
\]
(19)

Fig. (2) shows the power spectrum of the complete response of the Doppler modulated signal. We have kept here all the parameters same as in FM. In this case we also observe that maximum power is associated with \(f_0 + 2f_\text{rot}\) and the minimum lies in \(f_0 - f_\text{rot}\).

3 FOURIER TRANSFORM FOR AN ARBITRARY OBSERVATION TIME

It is important to obtain the FT for arbitrary observation time. The results obtained will be employed to outline how the spin-down of a pulsar due to the gravitational radiation back reaction, or due to some other mechanism, can be taken into account.

The FT for a data of observation time \(T_0\) is given via
\[
\tilde{h}(f) = \int_0^{T_0} \cos(\Phi(t)) e^{-i2\pi f t} dt,
\]
(20)

which may be split into
\[
\tilde{h}(f) = I_{h_+} + I_{h_-};
\]
(21)

\[
I_{h_-} = \frac{1}{2w_\text{orb}} \int_0^{T_0} e^{i\xi(t) + Z \cos(\phi) + N \cos(\theta) - \phi - \phi} d\xi,
\]
(22)

After performing the integration and proceeding in a straightforward manner we have

\[ I_{\nu} = \frac{1}{2w_{\text{int}}} \int_{0}^{\infty} e^{-i[\nu + 2 \cos(\alpha \phi + \lambda')] \xi} \, d\xi, \tag{23} \]

\[ v_{\phi} = \frac{f_{0} + f}{f_{\text{int}}}, \quad \xi_{\theta} = w_{\text{int}} T_{\nu}; \quad \xi = \xi_{\text{int}} = w_{\text{int}} t, \tag{24} \]

[refer to Paper I equations (33–35)].

As \( I_{\nu} \) contributes very little to \( \hat{h}(f) \), we drop \( I_{\nu} \) and write \( v \) in place of \( \nu \). Using the identity

\[ e^{i\kappa \cos \theta} = J_{\nu}(\pm \kappa) + 2 \sum_{l=1}^{\infty} i^{l} J_{l}(\pm \kappa) \cos l\theta, \tag{25} \]

we obtain

\[ \hat{h}(f) \simeq \frac{1}{2w_{\text{int}}} e^{-(\mathcal{R} - \mathcal{Q})} \times \int_{0}^{\infty} e^{i\nu} \left[ J_{\nu}(\mathcal{Z}) + 2 \sum_{k=1}^{\infty} J_{k}(\mathcal{Z}) i^{k} \cos k(\alpha \phi - \phi) \right] \times \left[ J_{\nu}(N) + 2 \sum_{m=1}^{\infty} J_{m}(N) i^{m} \cos m(\xi - \delta) \right] d\xi. \tag{26} \]

After performing the integration and proceeding in a straightforward manner we have

\[ \hat{h}(f) \simeq \frac{v}{2w_{\text{int}}} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{iA} B[C - iD]; \tag{27} \]

\[ A = \frac{i(k + m)\pi}{2} - \mathcal{R} - \mathcal{Q} \]

\[ B = \frac{k \xi_{\text{int}}(N)}{v^{2} \int \xi_{\text{int}} + m \phi} \]

\[ C = \sin v_{\xi_{\text{int}}} \cos(ak\xi_{\text{int}} + m\xi_{\text{int}} - k\phi - m\delta) \]

\[ - \frac{ak + m}{v} [\cos v_{\xi_{\text{int}}} \sin(ak\xi_{\text{int}} + m\xi_{\text{int}} - k\phi - m\delta) \]

\[ + \sin(k\phi + m\delta)] \]

\[ D = \cos v_{\xi_{\text{int}}} \cos(ak\xi_{\text{int}} + m\xi_{\text{int}} - k\phi - m\delta) \]

\[ + \frac{ka + m}{v} \sin v_{\xi_{\text{int}}} \cos(ak\xi_{\text{int}} + m\xi_{\text{int}} - k\phi - m\delta) \]

\[ - \cos(k\phi + m\delta) \]

The FT of the two polarization states of the wave can now be written as

\[ h_{+}(f) = h_{0} \hat{h}(f) \simeq \frac{v h_{0}}{2w_{\text{int}}} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{iA} B[C - iD]; \tag{29} \]

\[ h_{\times}(f) = -i h_{0} \hat{h}(f) \simeq \frac{v h_{0}}{2w_{\text{int}}} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{iA} B[D - iC]. \tag{30} \]

Now it is simple matter to obtain the FT of complete response. One gets
In this case most of the power of the signal also lies in the frequency

\[ T / 2 \]

is plotted in Fig. (3) with a resolution of 1 Hz.

The FT of FM signal of a detector for a data of 120 d for

\[ f_0 = 25 \text{ Hz,} \quad h_{a0} = h_{\phi0} = 1 \]

is plotted in Fig. (3) with a resolution of 1/2T \(_0\) \( \approx 9.67 \times 10^{-8} \) Hz.

The power spectrum of the complete response is plotted in Fig. (4).

In this case most of the power of the signal also lies in the frequency

\[ f_0 + 2f_{\text{rot}} \] and least with \( f_0 - f_{\text{rot}} \).

4 SPIN-DOWN

Pulsars lose their rotational energy by processes such as electromagnetic braking, emission of particles and emission of GW. Thus, the rotational frequency is not completely stable, but varies over a time-scale which is of the order of the age of the pulsar. Typically, younger pulsars have the largest spin-down rates. Current observations suggest that spin-down is primarily due to electromagnetic braking (Manchester 1992; Kulkarni 1992). Over the entire observing time, \( T_0 \), the frequency drift would be small but it may be taken into account for better sensitivity. To account this aspect we consider the evaluation of FT in a sequence of time windows by splitting the interval \( 0 - T_0 \) in \( M \) equal parts, each of interval \( \Delta t (T_0 = M \Delta t) \) such that the signal over a window may be treated as monochromatic. The strategy is to evaluate the FT over the window and finally to add the result. This process has been suggested by Brady & Creighton (2000) and Schutz (1998) in numerical computing and has been termed as stacking and tracking. For any such window, the initial and final times may be taken respectively as \( t_0 + n \Delta t \) and \( t_0 + (n + 1) \Delta t \), where \( t_0 \) represents the starting of the data set and \( 0 \leq n \leq M - 1 \). The window under consideration is the \( n_0 \) window. Let

\[ I = \int_{t_0 + n \Delta t}^{t_0 + (n + 1) \Delta t} h(\tilde{t}) e^{-i2\pi\tilde{f} \tilde{t}} d\tilde{t} \]

\[ = \int_{0}^{\Delta t} h(t + t_0 + n \Delta t) e^{-i2\pi\tilde{f} \tilde{t}} d\tilde{t} \]

\[ \tilde{t} = t + t_0 + n \Delta t \]

Hence the FT for the data of the time-window under consideration is given via
Figure 3. FT of a FM signal of frequency, $f_o = 25$ Hz from a source located at $(\pi/9, \pi/4)$ with a resolution of $9.67 \times 10^{-8}$.

Figure 4. Power spectrum of the complete response of a Doppler modulated signal of frequency, $f_o = 25$ Hz from a source located at $(\pi/9, \pi/4)$ with a resolution of $9.67 \times 10^{-8}$. 

\[ [\hat{h}(f)]_s = \int_0^{\Delta t} \cos[\Phi(t + t_o + n\Delta t)]e^{-i2\pi f(t + t_o + n\Delta t)} dt \]  

(35)

Taking the initial time of the data set

\[ t_o = 0 \]  

(36)

and proceeding as in the previous section we obtain

\[ [\hat{h}(f)]_s \simeq \frac{1}{2\omega_{rot}} e^{i2\pi(f_o - f)\Delta t} - \mathcal{R} - \mathcal{Q} \]

\[ \times \left[ \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} e^{iAk}[C_\iota - iD_\iota]; \right. \]

\[ \left. \right. \]  

\[ \mathcal{A}_\iota = \frac{(k+m)\pi}{2} + 2\pi n\Delta t(f_o - f) - \mathcal{R} - \mathcal{Q} \]

\[ \mathcal{B} = \frac{\mathcal{I} \mathcal{L} \mathcal{A}(N)}{(k^2+m^2)(k+m)^2}, \]

\[ \mathcal{C}_\iota = \sin(\nu\tau)\cos(ak + mt + k\nu - \mu) \]

\[ - \frac{\nu}{m} \cos(\nu\tau)\sin(ak + mt) - \sin(k\nu + m\mu), \]

\[ \mathcal{D}_\iota = \cos(\nu\tau)\cos(ak + mt + k\nu - \mu) \]

\[ + \frac{\nu}{m} \sin(\nu\tau)\sin(ak + mt) - \cos(k\nu + m\mu), \]

\[ \lambda = \phi - an\tau, \quad \zeta = \delta - n\tau, \]

\[ \tau = \omega_{rot}\Delta t, \quad n = 0, 1, 2, 3, \ldots, M - 1. \]  

(39)

The FT of the complete response would now be given via

\[ [\hat{R}(f)]_s = \frac{1}{2} \left\{ e^{-i\nu f_0} [\hat{h}(f + 2f_{rot})]_s \right. \]

\[ \times \left[ h_{\nu s} \left( F_{1s} + iF_{2s} \right) + h_{\nu s} \left( F_{2s} - iF_{1s} \right) \right] \]

\[ + e^{i2\pi f_0} [\hat{h}(f - 2f_{rot})]_s \]

\[ \times \left[ h_{\nu s} \left( F_{1s} - iF_{2s} \right) - h_{\nu s} \left( F_{2s} + iF_{1s} \right) \right] \]

\[ + e^{-i\nu f_0} [\hat{h}(f + f_{rot})]_s \]

\[ \times \left[ h_{\nu s} \left( F_{1s} + iF_{2s} \right) + h_{\nu s} \left( F_{2s} - iF_{1s} \right) \right] \]

\[ + e^{i\nu f_0} [\hat{h}(f - f_{rot})]_s \]

\[ \times \left[ h_{\nu s} \left( F_{1s} - iF_{2s} \right) - h_{\nu s} \left( F_{2s} + iF_{1s} \right) \right] \]

\[ + 2[h(f)]_s h_{\nu s} \left[ F_{3s} - ih_{\nu s} F_{3s} \right]. \]

(40)

\section{N-Component Signal}

The FT in equations (27) and (31) are valid for a pulsar which emits GW signals at single frequency \( f_o \). However, there are known physical mechanisms which generate GW signals consisting of many components. An axially symmetric pulsar undergoing free precession, emits quadrupole GW at two frequencies, \( f_o \) and \( 2f_o \), where \( f_o \) is equal to the sum of the spin frequency and the precession frequency (Zimmermann & Szedenski 1979). The quadrupole GW from a triaxial ellipsoid rotating about one of its principal axes consists of one component only (Thorne 1987). In this case the signal has frequency twice the size of the spin frequency of the star. In general, if a star is non-axisymmetric and precesses, the GW signal consists of more than two components. Recently, new mechanisms e.g. \( r \)-mode instability of spinning neutron stars (Andersson 1998; Lindblom et al. 1998; Owen et al. 1998) and temperature asymmetry in the interior of the neutron star with misaligned spin axis (Bildsten 1998) have been discussed in the literature.

In view of the above discussion CGW signal may consist of frequencies which are multiple of some basic frequencies. An analysis of the GW data of N-component signal has been made recently by Jaranowski & Krolak (2000). We do not continue this analysis as the requisite formalism is analogous to what we have presented in Paper I and the earlier sections of this paper.

\section{Discussion and Conclusions}

The analysis and results obtained in Paper I regarding FT of the response of a Laser Interferometer have been generalized in this paper. In this context following points must be noted.

(i) For 120-d observation time data, the resolution provided by fast Fourier transform (FFT), which is equal to \( 1/T_o \), is sufficient to represent the structure of sidebands.

(ii) In every case discussed, it turned out that the maximum power lies in the frequency \( f_o + 2f_{rot} \). However, this is not established conclusively if this result is generic.

(iii) Throughout our analysis in Paper I and this paper we have employed following conditions: (1) the phase of the wave is zero at \( t = 0 \); and (2) the observation time of the data set is from \( t = 0 \) to \( t = T_o \).

(iv) By making trivial modifications, we could get the general results with non-zero initial time and phase.

It is pointed out that in choosing the parameters for the source and the detectors we have been constrained by the computer memory and efficiency. Accordingly the choice of the parameters has been a matter of convenience. The application of the results obtained in the present and the earlier paper to the problem of all-sky search for continuous gravitational wave sources will be made in our forthcoming article (Srivastava & Sahay 2002b).

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