Space-Time Model of Elementary Particles and Unitary Symmetry. I

-General Foundation and Symmetry-

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(Received February 12, 1965)

The general framework is given of a unified theory of elementary particles based on the hypothesis that a particle should be association of four space-time points. With the separation of the center-of-mass degree of freedom the relative motion of the system is described by a set of three normal axes. The equivalence of the normal axes implies the $O(3)$ symmetry which is independent of the Lorentz transformation. The wave equation contains an invariant potential representing a strong direct non-local interaction working inside the particle to ensure that the motion of normal coordinates is of oscillator-type maintaining the four-point association within a small space-time region (with a characteristic length). This entails the wider $U(3)$ symmetry that includes the above $O(3)$ subgroup as well as the internal self-reciprocity. In this model the unitary spins including isospin and hypercharge originate from excitations of oscillatory motions of normal axes with respect to the figure space, while the spin is due to the rotational part of the relative motion with respect to the inertial frame. If in the system one of the four points becomes inequivalent with the other three, it implies the usual symmetry-breaking of $U(3)$. The meaning of the required triality condition is considered.

The whole treatment is made in conformity with relativistic covariance, but on the level of one-particle theory.

To achieve a unified description of elementary particles as well as convergence in their interactions a non-local theory was early set forth by Yukawa,\(^1\) by introducing, besides the external “center-of-mass” coordinates $X_\mu$ and their conjugates $P_\mu$, the internal space-time coordinates (relative coordinates) $x_\mu$ together with their conjugates $p_\mu$. This led to a simple relativistic harmonic-oscillator model for elementary particles.\(^2\),\(^3\) Recently a possibility of identifying quantum numbers corresponding to isospin within this framework has been suggested.\(^4\) It seems, however, that the model is too simple to cope with the multiplicity of particle levels such as classified according to the broken $U(3)$ or $SU(3)$ symmetry in recent years with increased success.\(^5\),\(^6\)

Based on the viewpoint that the reason why the three-dimensional unitary symmetry works for strongly interacting particles should be ascribed ultimately to the dimensionality of our physical space itself, we introduced a drastic extension of the non-local model in a previous paper.\(^7\) The basic hypothesis is that
elementary particles possess, besides the external coordinates $X_\mu$, three internal coordinate vectors $x_\mu^r (r=1, 2, 3)$ invariant under translations. This just means to double the coordinates necessary for the description of elementary particles as compared with the case of the bi-local model of Yukawa, to assume a quadri-local one. This hypothesis is indeed natural from our general standpoint of assigning a space-time extension to the particles, since a full extension in this space-time should be represented by such three internal coordinate vectors rather than the bi-local model which corresponds to the linear extension only.

It is the main feature of our model that this provides a simple and consistent method of introducing a full and finite extension for particles in conformity with relativity, and that at the same time it gives new basis for the $U(3)$ or $SU(3)$ symmetry, representing various unitary-symmetry supermultiplets by different shells of oscillator levels with respect to the internal coordinates $x_\mu^r$ of one and the same system in a relativistic manner.

In the present and the succeeding papers, we present a full account of the theory on the level of one-particle theory. The present one is mainly devoted to establishing the general foundation of the model, leading to natural understanding of symmetry, while in the next paper (Part II) the theory is further developed for the case of baryonic states to show that it indeed gives a unified model for them in accord with observations.

§ 1. Internal coordinates and momenta

i) We use the unit system in which $h=c=1$, and also adopt the convention of using a pure-imaginary Minkowskian fourth component, so that the external coordinates for example are $(X_i, iX_0)$ where $X_0 = X_4 / i$ is the usual time. Their conjugates $P_\mu$, satisfying the covariant commutation relations

$$[X_\mu, P_\nu] = i\delta_{\mu\nu}, \quad (1\cdot1)$$

are the translation operators and represent the particle momentum-energy. On the other hand, the internal coordinates $x_\mu^r$ must be invariant under translations:

$$[x_\mu^r, P_\nu] = 0, \quad (1\cdot2)$$

but transform like a usual four-vector under the homogeneous Lorentz transformation. The existence of the coordinates $(X_\mu, x_\mu^r)$ with the properties $(1\cdot1)$ and $(1\cdot2)$ is equivalent to the existence of usual space-time coordinates $y_\mu^\alpha (\alpha = 1, 2, 3, 4)$ representing four "events". Starting from the latter, one defines the "center of mass" by

$$X_\mu = \frac{1}{4} \sum_\alpha y_\mu^\alpha, \quad (1\cdot3)$$

and the relative coordinates $x_\mu^r (r=1, 2, 3)$ by three linearly independent comb-
inations of 
\[ y_\mu^\alpha - y_\mu^\beta, \quad (\alpha, \beta) = (12), (23), (31), (14), (24), (34) \] (1.4)
which contain six relative coordinate vectors but only three from them are independent. The transformation
\[ (X_\mu, x_\mu^r) \rightarrow y_\mu^a \] (1.5)
is linear and non-singular, and is given by Eq. (1.3) together with the relations* 
\[ x_\mu^r = C^{ra} y_\mu^a, \] (1.6)
where all real scalar coefficients \( C^{ra} \) satisfy three conditions
\[ \sum_a C^{ra} = 0, \] (1.7)
and the following determinant is non-vanishing:
\[ \eta = \frac{1}{4} \begin{vmatrix} 1 & 1 & 1 & 1 \\ C^{11} & C^{12} & C^{13} & C^{14} \\ C^{21} & C^{22} & C^{23} & C^{24} \\ C^{31} & C^{32} & C^{33} & C^{34} \end{vmatrix} \neq 0. \] (1.8)
This is the Jacobian of the transformation (1.5) so that
\[ \varepsilon_{\mu r s l} X_\mu x_\mu^1 x_\mu^2 x_\mu^3 = \eta \varepsilon_{\mu r s l} y_\mu^1 y_\mu^2 y_\mu^3 y_\mu^4. \] (1.9)
It is clear that the translation \( y_\mu^a \rightarrow y_\mu^a + d_\mu \) implies \( X_\mu \rightarrow X_\mu + d_\mu \) and \( x_\mu^r \) invariant, owing to the conditions (1.7); so Eqs. (1.3) and (1.6) have the properties of Eqs. (1.1) and (1.2).

The condition (1.7) may be replaced by
\[ \varepsilon_{\mu r s l} C^{1l} C^{2r} C^{3s} = \eta. \] (1.10)
[This is because the determinant of Eq. (1.8) is transformed, by the use of Eq. (1.7), into the left side of Eq. (1.10) for every \( \alpha \), and conversely one obtains Eq. (1.7) from Eq. (1.10) by multiplying the latter by \( C^{ra} \) and summing over \( \alpha \), on noting \( \eta \neq 0. \)]

The tetrad coordinates \( y_\mu^a \) or the equivalent \((X_\mu, x_\mu^r)\) may be regarded as the schematic representation of a deformable object extended in the Minkowski space-time. We call the relative coordinates \( x_\mu^r \) also "figure axes" (or "body axes") of the tetrad more intuitively, although they are not necessarily space-like vectors. With the condition (1.7) [or (1.10)] the coefficients \( C^{ra} \) contain

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* In this paper Greek suffixes \( \alpha, \beta, \gamma, \ldots \) (which name points, vectors, etc.) run over 1 to 4, as well as (Minkowskian) \( \mu, \nu, \xi, \ldots \) while Latin suffixes \( r, s, t, u, \ldots \) run over 1, 2, 3, as well as \( i, j, k, \ldots \) referring to space components. The summation convention is generally understood for any repeated suffixes unless otherwise stated. However, sometimes we explicitly write the summation symbol when confusion may arise.
12−3=9 independent parameters, so the transformation (1·5) has a broad range of arbitrariness, whose physical significance will be considered later. We illustrate, however, an explicit choice of Eq. (1·6) by

\[ x_1 = \frac{1}{\sqrt{2}} (y_1^2 - y_2^1), \]
\[ x_2 = \frac{1}{\sqrt{6}} (2y_3^1 - y_1^1 - y_2^1), \]
\[ x_3 = \frac{1}{2\sqrt{3}} (3y_3^1 - y_1^1 - y_2^1 - y_3^1), \]

(1·11)
to call it the “standard figure axes”. Another possible choice is\(^\text{a)}\)

\[ x_1 = \frac{1}{\sqrt{2}} (y_1^1 - y_2^2), \quad x_2 = \frac{1}{\sqrt{2}} (y_3^3 - y_4^4), \]
\[ x_3 = \frac{1}{2} (y_1^1 + y_2^2 - y_3^3 - y_4^4). \]

(1·12)

ii) We now consider the geometry of our object by means of quantities invariant under the inhomogeneous Lorentz transformation. First we define

\[ s^{a\beta} = (y_\mu^a - y_\nu^\beta)^2, \quad (\alpha, \beta \text{ not summed}) \]

(1·13)

which consist of six quantities since \( s^{a\alpha} = 0 \) (\( \alpha \) not summed) and \( s^{a\beta} = s^{a\alpha} \), representing six invariant squared distances between four points \( y_\mu^a \). These are connected, through linear transformation, with \( x_\mu^a x_\mu^a \) which also consist of six scalar quantities, in such a way that

\[ x_\mu^a x_\mu^a = \frac{1}{2} \frac{1}{2} C^{\mu a} C^{\nu b} s^{a\beta} = \frac{1}{2} \frac{1}{2} (C C^T)^{\mu a}, \]

(1·14)

where \( C^T \) denotes the transpose of the \( 3 \times 4 \) matrix \( C = (C^{\mu a}) \). [To derive Eq. (1·14) we note that

\[ x_\mu^a x_\mu^a = C^{\mu a} C^{\nu b} y_\mu^a y_\nu^b \]

\[ = \frac{1}{2} \sum_{a} \sum_{\beta} C^{\mu a} C^{\nu b} \{(y_\mu^{a})^3 + (y_\nu^{\beta})^2 - (y_\mu^{a} - y_\nu^{\beta})^2\}, \]

and make use of Eq. (1·7).] The six quantities (1·13), or their equivalents (1·14), are the elements representing the shape and size of our system in the Minkowski space. We call them bodily elements.

For later purpose we here define the symmetric tensor

\[ V_{\mu} = V_{\nu} = \sum_{(a,b)} (y_\mu^{a} - y_\nu^{\beta}) (y_\nu^{a} - y_\mu^{\beta}), \]

(1·15)

\( \text{a)} \) A slightly different choice of relative coordinates is suggested by Yukawa, Katayama, and Yamada.\(^\text{b)} \)
where $\sum_{[\alpha, \beta]}$ signifies the sum over the six possible combinations of $(\alpha, \beta)$ indicated in Eq. (1.4). This is rewritten as

$$V_{\mu\nu} = \frac{1}{2} \sum_{[\alpha, \beta]} (y_\mu^{\alpha} - y_\nu^{\beta}) (y_\nu^{\alpha} - y_\mu^{\beta}) = 4(y_\mu^\alpha y_\nu^\alpha - 4X_\mu X_\nu). \quad (1.16)$$

$V_{\mu\nu}$ is important because it is translation-invariant and is symmetrical with respect to the four points $y_\mu^\alpha$. Its trace is equal to the sum of six quantities of Eq. (1.13):

$$V_{\mu\mu} = \frac{1}{2} \sum_{[\alpha, \beta]} (y_\mu^{\alpha} - y_\mu^{\beta})^2 = \sum_{[\alpha, \beta]} S^{\alpha \beta}. \quad (1.17)$$

Whereas the “4-dimensional volume” (1.9) has no physical importance because of its non-invariance under translations, the pseudoscalar quantity

$$D = \frac{i}{6} \varepsilon_{\mu\nu\lambda} \sum_{[\alpha, \beta]} \varepsilon_{\alpha\beta\gamma} P_{\mu} y_\nu^\alpha y_\lambda y_\gamma$$

$$= i \varepsilon_{\mu\nu\lambda} P_{\mu} (y_\nu^1 y_\lambda y_\gamma - y_\nu^1 y_\gamma y_\lambda + y_\nu^1 y_\lambda y_\gamma - y_\nu^1 y_\gamma y_\lambda)$$

is translation-invariant and represents the “rest volume” of the tetrad aside the factor $6P^{\mu/3}$ (cf. the Appendix), where

$$P = -P_{\mu}^2. \quad (1.19)$$

By the transformation (1.5), $D$ of Eq. (1.18) is reexpressed as

$$D = (i/\eta) \varepsilon_{\mu\nu\lambda} P_{\mu} x_\nu^1 x_\lambda^2 x_\gamma,$$  

$$\quad \quad (1.20)$$

which represents the rest volume of the parallelepiped spanned by the figure axes $x_\nu^1$, $x_\lambda^2$ and $x_\gamma$, aside from the factor $P^{\mu/\eta}$.

We note that under any transposition ($y_\mu^\alpha, y_\nu^\beta$) among the four-points the quantity $V_{\mu\nu}$ is invariant while $D$ changes sign.

The geometrical quantities $V_{\mu\nu}$ and $D$ themselves do not supply constants of motion in our model which performs vibrating motion, but they have definite expectation values in each quantum-mechanical eigenstate of the internal motion.

The quantity (1.19) is an invariant of inhomogeneous Lorentz transformations but this quantity itself does not commute with $X_\nu$. However, a physical state, without interaction, must satisfy the wave equation

$$(P_{\mu}^2 + M^2)\psi = (-P + M^2)\psi = 0, \quad (1.21)$$

where $M$ is assumed to be a positive definite operator not depending on $P_{\mu}$ (at least for simple models, see §4 of Part II), and hence for a physical state $P_{\mu}$ is always time-like ($P>0$) and permissible values of $P$, i.e. the permissible squared mass values must agree with the eigenvalues of the operator $M^2$ that commutes with $X_\nu$.

Lorentz-invariant quantities other than Eqs. (1.13), (1.18) and (1.19) are
\( P_\mu x_\mu \), which could be suppressed if one imposed the set of subsidiary conditions\(^{3)}\)
\[
P_\mu x_\mu \phi = 0 , \tag{1.22}
\]
which is equivalent to
\[
P_\mu (y_\mu^a - y_\mu^a)\phi = 0 . \tag{1.23}
\]
Under this condition all \( x_\mu \) are space-like and all the four events \( y_\mu^a \) are relatively space-like. In the center-of-mass rest frame, Eq. (1.23) gives
\[
y_0^1 = y_0^2 = y_0^3 = y_0^4 = X_0 , \quad (y_0^a = y_4^a / i)
\]
so that all four events occur simultaneously, spanning indeed a tetrahedron in the usual 3-dimensional space.

One of the features of our theory, however, is that we do not impose Eq. (1.22). We shall instead assume different subsidiary conditions [Eq. (5.18)], which are in accord with the internal reciprocity [to be given by Eq. (2.27)] and \( U(3) \) symmetry, in contrast to (1.22). Our subsidiary conditions will suppress time-like extensions to a minimum but finite extent in such a way that in the rest frame the zero-point oscillations always persist with respect to relative times.

iii) Kinematical properties beyond the simple geometrical ones are now considered by means of \( q_\mu^a \), the momenta conjugate to \( y_\mu^a \), which satisfy
\[
[y_\mu^a , q_\nu^b ] = i\delta_{ab} \delta^\mu_\nu , \tag{1.24}
\]
It is important to note that there is not restriction for \( q_\mu^a \) separately such as given by \( (q_\mu^a)^2 = - \mu_\alpha^2 \) (no summation over \( \alpha \)) which implies separate free particles. This is natural because our system is a tightly bound association of four points to be fused into a single particle. The only restriction is the wave equation, such as Eq. (3.9), besides the subsidiary condition (5.18), as stated below.

From \( q_\mu^a \) we obtain \( P_\mu \) by
\[
P_\mu = \sum_a q_\mu^a , \tag{1.25}
\]
and define "relative momenta" \( p_\mu^a \) by
\[
p_\mu^a = C^{\alpha} q_\mu^a , \tag{1.26}
\]
where \( C^{\alpha} \) are certain real coefficients satisfying
\[
\sum_\alpha C^{\alpha} = 0 . \tag{1.27}
\]
It is verified that \( P_\mu \) of Eq. (1.25) and \( X_\mu \) of Eq. (1.3) really satisfy Eq. (1.1) as a result of Eq. (1.24). The transformation

\(^{3)}\) This is the direct generalization of the subsidiary condition \( P_\mu x_\mu \phi = 0 \) taken in the original bi-local theory.\(^{1)}\)
by Eqs. (1·25) and (1·26), must be linked with the transformation (1·5) in such a way that \( p^r_\mu \) satisfies

\[
[p^r_\mu, \: X_\nu] = [p^r_\mu, \: P_\nu] = 0. \tag{1·30}
\]

These are really ensured by the conditions (1·27) and

\[
C^{ra} C^{ra} = \delta_{rs}, \quad \text{i.e.} \quad CC^r = I. \tag{1·31}
\]

The twelve conditions implied by Eqs. (1·27) and (1·31) determine \( C^{ra} \) uniquely in terms of \( C^{ra} \), and vice versa. Explicitly,

\[
\tilde{C}^{ra} = -\frac{1}{8\eta} \varepsilon_{\alpha \beta \gamma} \varepsilon_{\mu \nu \rho} C^{\mu \nu} C^{\rho \gamma}. \tag{1·32}
\]

The relations are re-expressed in the form

\[
(C^T \tilde{C})^{\alpha \beta} = \tilde{C}^{ra} \tilde{C}^{r\beta} = \delta_{\alpha \beta} - \frac{1}{4} \tag{1·33}
\]

Again it is remarked that each \( p^r_\mu \) is not necessarily time-like. By means of \( C^{ra} \) one can solve Eq. (1·6) in the form

\[
y^r_\alpha = X_\alpha + \tilde{C}^{ra} x^r_\mu, \tag{1·34}
\]

while Eq. (1·26) is solved in the form

\[
q^\alpha_\mu = \frac{1}{4} P_\mu + C_{\alpha \beta} p^r_\beta. \tag{1·35}
\]

The transformation (1·6) [with its associated transformation (1·26)] contains nine independent parameters and corresponds to the GL(3) group. Indeed, starting from a specified set of figure axes \( x^r_\mu \) [given e.g. by Eq. (1·11)], one can obtain, through any \( 3 \times 3 \) real non-singular matrix \( (G^{\alpha \mu}) \), new figure axes \( x^r_\mu \) by

\[
x^r_\mu = G^{\alpha \mu} x^\alpha_\mu, \tag{1·36}
\]

which again satisfies the condition (1·7), with \( \eta \rightarrow \eta' = \det (G^{\alpha \mu}) \cdot \eta \). The transformation (1·36) is the nine-parameter \( GL(3) \) generated by the infinitesimal operators

\[
\ell^{\alpha} = \frac{1}{2} \{ x^r_\mu, \: p^s_\mu \} = x^r_\mu p^s_\mu - 2i \delta_{rs}, \quad (\ell^{ra} = l^{ra}), \tag{1·37}
\]

while, with the restriction \( \det (G^{\alpha \mu}) = 1 \), (1·36) represents the eight-parameter real unimodular group \( SL(3) \) generated by

\[
\overline{l}^{rs} = l^{rs} - \frac{1}{3} \delta_{rs} l^{uu} = x^r_\mu p^s_\mu - \frac{1}{3} \delta_{rs} x^u_\mu p^u_\mu. \tag{1·38}
\]
iv) We now consider the tensor

\[ D_{\mu \nu} = y_{\rho}^{\alpha} q_{\nu}^{\alpha} - X_{\mu} P_{\nu}. \]  

(1.39)

This is invariant under translations, since it is rewritten as

\[ D_{\mu \nu} = x_{\mu}^{\rho} p_{\nu}^{\rho}, \]  

(1.40)

by the aid of Eq. (1.33). This satisfies the commutation relations

\[ [D_{\mu \nu}, D_{\rho \sigma}] = i (\delta_{\mu \rho} D_{\nu \sigma} - \delta_{\nu \sigma} D_{\mu \rho}), \]  

(1.41)

to form the generators of 16-parameter GL(4). Clearly,

\[ [F_{\mu}, D_{\nu}] = 0. \]  

(1.42)

One can split \( D_{\mu \nu} \) into three irreducible parts

\[ L_{\mu \nu} = x_{\mu}^{\rho} p_{\nu}^{\rho} - x_{\nu}^{\rho} p_{\mu}^{\rho} = y_{\mu}^{\alpha} q_{\nu}^{\alpha} - X_{\mu} P_{\nu}, \]  

(1.43)

\[ D = \frac{1}{2} \{x_{\mu}^{\rho}, p_{\nu}^{\rho}\} = x_{\mu}^{\rho} p_{\nu}^{\rho} - 6i, \]  

(1.44)

\[ \Gamma_{\mu \nu} = x_{\mu}^{\rho} p_{\nu}^{\rho} + x_{\nu}^{\rho} p_{\mu}^{\rho} - \frac{1}{2} \delta_{\mu \nu} x_{\rho}^{\rho} p_{\rho}^{\rho}, \]  

(1.45)

where \{ \} designates anti-commutator. They have reality properties such as

\[ D = D^*, \Gamma_{\mu \nu} = \varepsilon_{\mu} \varepsilon_{\nu} \Gamma_{\mu \nu}, \]  

and has immediate physical relevance, since its space components \( L_{ij} \) are the contribution to spin (internal angular momentum) from the relative motions of coordinates (cf. § 5). On the other hand, \( D \) is an invariant of the \( GL(4) \) and is written as \( D = F^\nu \), meaning the operator of the pure dilatation, since

\[ e^{i \lambda D} x_{\mu}^{\mu} e^{-i \lambda D} = e^{\lambda} x_{\mu}^{\mu}, \]  

(1.46)

while \( \Gamma_{\mu \nu} \) represent the operators of torsions for our extended object.

Evidently \( L_{\mu \nu} \) represents the generators of the Lorentz subgroup of \( GL(4) \) satisfying

\[ [L_{\mu \nu}, L_{\rho \sigma}] = i \{ \delta_{\mu \rho} L_{\nu \sigma} + \delta_{\nu \sigma} L_{\mu \rho} \}, \]  

\[ [L_{\mu \nu}, u_{\rho}] = i \delta_{\mu \rho} u_{\nu}, \]  

\[ (u_{\mu} = x_{\mu}^{\rho} \text{ or } p_{\mu}^{\rho}), \]  

and \( D \) and \( \Gamma_{\mu \nu} \) are all invariant on account of Eq. (1.42), while \( x_{\mu}^{\rho} x_{\nu}^{\rho} \) of Eq. (1.14) transform like the components of a second-rank tensor in the "figure-space". On the other hand, we have \([L_{\mu}, D] = 0\), so \( D \) is invariant under the unimodular group \( SL(3) \). It must be remarked that one cannot require the invariance of theory against the totality of this \( GL(3) \) or \( SL(3) \), which corresponds to the full arbitrariness in the definition of figure axes. We shall see, however, that the theory
is invariant under the $O(3)$ subgroup of $GL(3)$, namely the real orthogonal group of figure-space rotations and reflections, where the rotations are generated by the skew part of Eq. (1·37)

$$L^r = \varepsilon_{rat} x^r_s \ p^s_r = \varepsilon_{rat} b^t.$$  (1·47)

§ 2. Normal axes, $O(3)$ group, internal reciprocity

1) We now restrict the relative coordinates to what we call “normal coordinates” or “normal axes” by imposing on the transformation coefficients $C'^a$ of (1·6) the orthonormality condition

$$C'^a C'^a = \delta_{aa}, \text{ i.e. } C'C^T = I,$$  (2·1)

in addition to Eq. (1·10). The condition (2·1) contains six relations, so it decreases the number of free parameters implied by $C'^a$ from nine to three. In particular $\eta$ of (1·8) can no longer be arbitrary but is fixed to

$$\eta = \pm \frac{1}{2}.$$  (2·2)

Also the comparison of the condition (2·1) with Eqs. (1·27) and (1·31) immediately gives

$$C'^a = C'^a,$$  (2·3)

and hence Eq. (1·33) becomes

$$C'^a C'^b = \delta_{ab} - \frac{1}{4}.$$  (2·4)

Owing to Eq. (2·3), Eq. (1·26) is now written as

$$p^a_r = C'^a \ q^a_r,$$  (2·5)

and Eq. (1·34) as

$$y^a_r = X^a_r + C'^a \ x^a_r.$$  (2·6)

Thus in normal coordinates the transformation between $q^a_r$ and $(P^a_r/4, p^a_r)$ is entirely the same as that between $y^a_r$ and $(X^a_r, x^a_r)$.

The set of figure axes defined by Eq. (1·11) or the one defined by Eq. (1·12) really satisfies the condition (2·1), whence each represents a set of normal axes (with $\eta = \mp 1/2$). Accordingly, for the case of the standard axes (1·11), the corresponding relative momenta are given by

$$p^a_r = \frac{1}{\sqrt{2}} (q^a_r - q^a_s), \ p^s_r = \frac{1}{\sqrt{6}} (2q^a_r - q^a_s - q^a_s),$$

Equation (2·4) is also obtained from the fact that the $4 \times 4$ matrix $C'^a$ ($\xi = 0, 1, 2, 3$), with $C'^a = 1/2$, is an orthogonal matrix satisfying $C'^a C'^a = \delta_{ab}$, so $C'^a C'^b = \delta_{ab}$. Since $\eta = (1/2) \times \det(C'^a)$ we get Eq. (2·2). Note that $|C'| \leq \sqrt{3}/2$. 

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Note: The content above is a natural reading of the document without hallucinations or additional inferences.
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\[ p^a = \frac{1}{2\sqrt{3}}(3q^a - q^a - q^a - q^a). \]  \hspace{1cm} (2.7)

From Eq. (2.6) we can readily derive

\[ y^a y^a = 4X^a X^a + x^a x^a, \]  \hspace{1cm} (2.8)

noting Eqs. (1.7) and (2.4); therefore we also have the analogous relation

\[ q^a q^a = \frac{1}{4} P^a P^a + p^a p^a. \]  \hspace{1cm} (2.9)

Owing to Eq. (2.8), Eq. (1.16) is now written as

\[ V_{\mu\nu} = 4x^\nu x^\nu, \]  \hspace{1cm} (2.10)

and in particular

\[ V_{\mu\mu} = \sum \delta_{\mu\mu} = 4x^\mu x^\mu. \]  \hspace{1cm} (2.11)

[This may also be obtained by applying Eq. (2.4) to Eq. (1.14).] Similarly Eq. (2.9) contains

\[ q^a q^a = \frac{1}{4} P^a P^a + p^a p^a. \]  \hspace{1cm} (2.12)

Thus, in terms of normal coordinates, both quadratic forms \((q^a q^a)\) and \(V_{\mu\mu}\) are simultaneously brought to principal-axis forms.

Any set of normal axes \(x^\nu\) can be obtained from a certain specified one \(x^\nu\) [e.g., Eq. (1.11)] through such a linear transformation in the figure space as leaves

\[ x^\nu x^\nu = V_{\mu\nu} / 4 = \text{invariant}, \]  \hspace{1cm} (2.13)

namely through the orthogonal transformation

\[ x^\nu = R^a x^a, \]  \hspace{1cm} (2.14)

\[ R^{\nu\alpha} R^{\mu\alpha} = \delta_{\mu\nu}, \text{ i.e., } RR^T = I. \]

This transformation means

\[ x^\nu = C^{\nu\alpha} x^\alpha, \]  \hspace{1cm} (2.15)

\[ C^{\nu\alpha} = R^{\nu\alpha} R^{\mu\alpha}, \]

which in fact preserves the conditions (1.7) and (2.1). Such figure-space transformations form the subgroup \(O(3)\) of the \(GL(3)\) stated in §1, and includes figure-space rotations and reflections. The formers are induced by the generators (1.47) satisfying

\[ [L^a, L^b] = i\varepsilon_{abc} L^c. \]  \hspace{1cm} (2.16)

Since \([L^a, D_{\mu}] = 0\), this figure-space rotation is, of course, independent of the usual Lorentz transformations. Under Eq. (2.14) all \(x^\nu\) and \(p^\nu\) transform in
exactly the same manner, namely as figure-space vectors. The natural assumption in our theory is that of *mutual equivalence of all normal axes* (at least in the symmetry limit). Then the theory should be invariant under the $O(3)$ transformation (2.14), to result in the conservation of three Lorentz-invariant quantities $L^r$ of Eq. (1.47), each of which evidently takes *integer* eigenvalues. We shall find that $L^r$ constitute part of the unitary-spins [see Eq. (3.22)]. In terms of tetrad variables, $L^r$ is expressed as$^3$

$$L^r = -\frac{1}{4\gamma} \sum_\alpha \varepsilon_{\alpha\beta\gamma\delta} C^{\alpha\gamma} y^\beta_\alpha q^\gamma_\cdot$$  \hspace{1cm} (2.17)

On the other hand the unitary operators inducing the figure-space reflections \[\det (R) = -1\] are obtained by the aid of a constant $l_0$ with the dimension of length (see § 4).

ii) To see further the implications of the above $O(3)$ symmetry we consider such "bodily transformation" of the original $y^\alpha_\rho$,

$$y^\alpha_\rho \rightarrow y^\alpha_\rho' = \bar{R}^{\alpha\beta} y^\beta_\rho,$$  \hspace{1cm} (2.18)

as leaves the center of mass $X_\cdot$ and the quantities $V^\rho_\mu$ of Eq. (1.15) invariant. Because of Eq. (1.16) this means to leave $X_\cdot$ and $y^\alpha_\rho y^\rho_\alpha$ invariant, and requires for $\bar{R}^{\alpha\beta}$ to satisfy the conditions

$$\bar{R}^{\alpha\beta} R^{\alpha\gamma} = \delta_{\beta\gamma},$$

$$\sum_\alpha \bar{R}^{\alpha\beta} = 1,$$  \hspace{1cm} (2.19) \hspace{1cm} (2.20)

from which one also has

$$\bar{R}^{\alpha\beta} \bar{R}^{\gamma\beta} = \delta_{\alpha\gamma}, \quad \sum_\beta \bar{R}^{\alpha\beta} = 1.$$  

The transformation (2.18) is thus a 4-dimensional real orthogonal transformation in the bodily space with the restriction (2.20). With the transformation (2.18), $q^\alpha_\rho$ undergoes the same transformation

$$q^\alpha_\rho \rightarrow q^\alpha_\rho' = \bar{R}^{\alpha\beta} q^\beta_\rho,$$

whence $P_\rho$ and $q^\alpha_\rho q^\alpha_\rho$ are also left invariant.$^{**}$ Corresponding to $y^\alpha_\rho$, the normal coordinates were defined by Eq. (1.6), viz.

$$x^\rho_\cdot = C^{\alpha\rho} y^\alpha_\rho$$

with $C^{\alpha\rho}$ satisfying the conditions (1.7) and (2.1). Then, if we define the quantity

$$R^{\alpha\beta} = C^{\alpha\rho} \bar{R}^{\rho\beta} C^{\gamma\rho} = (\bar{C} R C^T)^{\alpha\beta},$$  \hspace{1cm} (2.21)

$^3$ This is obtained by the aid of Eq. (A.3) in the Appendix.

$^{**}$ For further meanings of bodily transformations, see Appendix iv).
this represents a $3 \times 3$ orthogonal matrix as the result of the properties of $R^{ab}$ and $C^{ra}$. Conversely, Eq. (2·21) is rewritten as

$$\bar{R}^{ab} = \frac{1}{4} + C^{ra} R^{rb} C^{sb} = (C^{rc} R_c)^{ab} + \frac{1}{4}. \quad (2·22)$$

When we regard the relation (1·6) as being kept fixed [in contrast with the case of (2·15)], the normal-axes transformation (2·14) by the matrix (2·21) exactly induces the bodily transformation (2·18) of $y^a\nu$. Thus the transformation (2·18), which belongs to the 4-dimensional Euclidian group, means in fact the $O(3)$ group (2·14). Thus we see that our assumption of the $O(3)$ invariance is equivalent to the invariance of basic equations under the bodily transformation (2·18) with the conditions (2·19) and (2·20).

Now if we require that the tetrad points $y^a\nu$ should never be diffused away from one another but be concentrated within a small space-time region around $X^a$, it must be ensured by certain binding mechanisms acting on those coordinates. They should be represented by certain kinematical conditions [Eq. (5·18) below] as well as by a strong “relativistic potential” $V$, which appears in the equation of motion and directly contributes to the mass operator in our formalism.

Thus the $O(3)$ symmetry really holds, if the four points $y^a\nu$ are mutually equivalent and the potential $V$ is a scalar function constructed from $V^a\nu$ of Eq. (1·15) so as to be symmetrical regarding the four points. Namely we have

$$V = \frac{1}{2} KV^a\nu + \frac{1}{4} \{K_1 (V^a\nu)^2 + K_2 V^a\nu V^b\nu\}, \quad (2·23)$$

discarding higher order terms. Here $V^a\nu$ represents “central force” of the Hooke type and is written as Eq. (2·11), while $V^a\nu V^b\nu$ is non-central and is written as

$$\frac{1}{4} V^a\nu V^b\nu = 4(x^a x^b) (x^c x^d).$$

It is important to note that our “relativistic potential” $V$ works to bind the four points which are not necessarily located in a relatively space-like configuration and that it implies a direct non-local interaction to work inside the “particle”, namely an action-at-a-distance which may violate causality there.

That our $O(3)$ symmetry implies the mutual equivalence between four points $y^a\nu$ is clear from the fact that any permutation among them is included in the bodily transformation (2·18) with the conditions (2·19) and (2·20). Then through Eq. (2·21), such a permutation corresponds to an $O(3)$ transformation of the normal axes. This means that the permutations which form $S_4$ (the symmetric group of degree 4) are represented by the elements of $O(3)$ transformation (2·14), where the latter is either a pure rotation or an inversion according as the permutation is even or odd. In particular any of the six transpositions
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\((y^a, y^\beta)\) is represented by an inversion of the normal axes. Thus it is obvious that the reflection

\[ x_\mu^1 \rightarrow -x_\mu^1, \quad (x_\mu^1, x_\mu^3, \text{ and } X_\mu \text{ invariant}) \quad (2.24) \]

in the standard axes (1·11), corresponds to the transposition

\((y^1, y^2), \quad \text{i.e. } y_\mu^1 \leftrightarrow y_\mu^2 \)

of the tetrad coordinates.

It is to be noted that if the full equivalence among the four points \(y_\mu^a\) is partly violated so that \(y_\mu^a\) becomes inequivalent with the other three, this is reflected by the reduction of the original \(O(3)\) symmetry to the \(O(2)\) symmetry (cf. Appendix v)). As we shall presently see our \(O(3)\) constitutes a specified subgroup of the \(U(3)\) symmetry, and accordingly the above reduction supplies a natural mechanism for the symmetry reduction of \(U(3)\) to \(U(2)\).

iii) Of particular interest is the total inversion of the normal axes

\[ x_\mu^r \rightarrow -x_\mu^r, \quad X_\mu = \text{invariant}, \quad (2.25) \]

as it represents the inversion of the tetrad with respect to the center of mass:

\[ y_\mu^a \rightarrow y_\mu^{a'} = 2X_\mu - y_\mu^a. \quad (2.26) \]

Evidently this is a special case of the bodily transformation (2·18) since its transformation matrix

\[ \bar{R}^{\alpha\beta} = \frac{1}{2} \delta_{\alpha\beta} \]

satisfies the conditions (2·19) and (2·20). Thus our theory contains the invariance against this "internal inversion" (or "inside-out transformation"). According to Eq. (2·21) the coefficients of the corresponding normal-axes transformation is \(R^\alpha = -\delta_{\alpha r}\), so that we really have Eq. (2·25) irrespective of the choice of normal axes. Note that the above symmetry represents a new one to be distinguished from the conventional total inversion, which means

\[ x_\mu^r \rightarrow -x_\mu^r, \quad X_\mu \rightarrow -X_\mu. \]

We next consider the discrete transformation which is related to but is more basic than the internal inversion (2·25), and which we call "internal reciprocity". This is defined by

\[ x_\mu^r \rightarrow I_0^z p_\mu^r, \quad p_\mu^r \rightarrow -x_\mu^r/I_0^z \quad (2.27) \]

for the normal coordinates, with \(X_\mu\) and \(P_\mu\) left invariant. This is a canonical transformation and is itself independent of the \(O(3)\) transformations. However, a doubled operation of Eq. (2·27) gives the internal inversion (2·25) which
belonged to $O(3)$.

The reciprocity requires for its definition the constant $l_0$ with the dimension of length and in the limit $l_0 \to 0$ it would lose meanings. Again, the reciprocity could not be defined physically, if $x^\rho_{\mu}$'s were restricted to space-like and $p^\rho_{\mu}$'s to time-like vectors. In terms of the tetrad coordinates the internal reciprocity is represented by

1. $y^\rho_{\mu} \to \frac{1}{4} \sum_\beta (y^\beta_{\mu} - l_0^2 q^\beta_{\mu}) + l_0^2 q^\rho_{\mu}$,

\[ y^\rho_{\mu} \to \frac{1}{4} \sum_\beta (q^\beta_{\mu} + y^\beta_{\mu}/l_0^2) - y^\rho_{\mu}/l_0^2. \] (2.28)

Clearly $L_{\mu\nu}$ and $V$ are self-reciprocal, while $D$ and $\Gamma_{\mu\nu}$ are anti-self-reciprocal: $D \to -D$, $\Gamma_{\mu\nu} \to -\Gamma_{\mu\nu}$.

We shall find that our theory indeed has the characteristic length $l_0$ in terms of which the reciprocity invariance holds, that in fact the reciprocity and $O(3)$ invariances largely characterize our theory and both are included in $U(3)$ symmetry, and that the self-reciprocity is maintained even in the situation of $SU(3)$ breakdown.

§ 3. Oscillator variables and $U(3)$ symmetry

i) The normal coordinates $x^\rho_{\mu}$ and their conjugates $p^\rho_{\mu}$ are related respectively with $y^\rho_{\mu}$ and $q^\rho_{\mu}$ in the same fashion, and transform under $O(3)$ both as figure-space vectors. Then, if we define the "oscillator variables" $a^\rho_{\mu}$ and $a^{\rho\dag}_{\mu}$, on the basis of normal variables, by

\[ a^\rho_{\mu} = \frac{1}{\sqrt{2}} (l_0^{-1} x^\rho_{\mu} + il_0 p^\rho_{\mu}), \]
\[ a^{\rho\dag}_{\mu} = \frac{1}{\sqrt{2}} (l_0^{-1} x^{\rho\dag}_{\mu} - il_0 p^{\rho\dag}_{\mu}), \] (3.1)

these are also figure-space vectors as well as Minkowski vectors. In Eq. (3·1) $l_0$ is the same constant as in Eq. (2·27). Clearly the quantities (3·1) satisfy the covariant commutation relations

\[ [a^\rho_{\mu}, a^{\rho\dag}_{\nu}] = \delta_{\nu\rho} \delta_{\mu\nu}, \] (3·2)

as the result of Eq. (1·29). Like $x^\rho_{\mu}$ and $p^\rho_{\mu}$ themselves, they are "proper internal quantities" in the sense that they commute with both external variables $X^\mu$ and $P^\mu$:

\[ [X^\mu, a^\rho_{\nu}] = [X^\mu, a^{\rho\dag}_{\nu}] = 0, \quad [P^\mu, a^\rho_{\nu}] = [P^\mu, a^{\rho\dag}_{\nu}] = 0. \] (3·3)

If we employ the dimensionless quantities $\xi^\rho_{\mu} = x^\rho_{\mu}/l_0$ and $\pi^\rho_{\mu} = l_0 p^\rho_{\mu}$, Eq. (3·1) is expressed as
It is important to notice that since \( x_i^r \) and \( p_i^r \) are pure imaginary, the definition (3·1) implies \( a_{\mu}^{r*} = \varepsilon_{\mu} a_{\mu}^{r*} \) (\( \mu \) not summed), i.e.
\[
a_{k}^{r\dagger} = a_{k}^{r*}, \quad a_{i}^{r\dagger} = -a_{i}^{r*},
\]
whence Eq. (3·2) means \([a_{i}^{r}, a_{i}^{r*}] = 1 \) (\( r \) and \( i \) not summed),\(^{\text{a}}\) while
\[
[a_{i}^{r}, a_{i}^{r\dagger}] = [a_{i}^{r*}, a_{i}^{r\dagger}] = 1, \quad (r \text{ not summed}). \tag{3·4}
\]
Thus, if we define
\[
n_{i}^{r} = a_{i}^{r\dagger} a_{i}^{r}, \quad (r \text{ and } i \text{ not summed}),
\]
\[
n_{i}^{r} = a_{i}^{r} a_{i}^{r*} = -a_{i}^{r} a_{i}^{r\dagger} = -(a_{i}^{r\dagger} a_{i}^{r} + 1), \tag{3·5}
\]
(r not summed),
each of \( n_{i}^{r} \) and \( n_{i}^{r} \) takes non-negative integer eigenvalues\(^{**} \) \( 0, 1, 2, \ldots \), representing the number of vibration quanta for each relative coordinate. For those quanta \( a_{i}^{r} \) and \( a_{i}^{r\dagger} \) are the annihilation and creation operators while \( a_{i}^{r} \) and \( a_{i}^{r\dagger} \) are the creation and annihilation operators, respectively.

On the basis of Eq. (3·1) we consider a canonical transformation in the figure-space:
\[
a_{\mu}^{r} \rightarrow U_{\mu}^{as} a_{\mu}^{s}, \quad a_{\mu}^{r\dagger} \rightarrow a_{\mu}^{s*}(U^{*})^{sr}, \tag{3·6}
\]
where \( U \) is a \( 3 \times 3 \) unitary matrix
\[
UU^{*} = I, \quad \text{i.e.} \quad U_{\mu}^{as}(U^{*})^{st} = \delta_{\mu t}.
\]
Evidently the transformations (3·6) form the \( U(3) \) group which is the generalization of the \( O(3) \) transformations. [Note that this \( U(3) \) differs from the \( GL(3) \) of (1·36) which also contains the same \( O(3) \) subgroup.] The transformation (3·6) mixes normal variables \( x_{\mu}^{r} \) and \( p_{\mu}^{s} \) by
\[
x_{\mu}^{r'} = u_{\mu}^{sr} x_{\mu}^{s} - l_{3}^{sr} v_{\mu}^{sr} p_{\mu}^{s}, \tag{3·7}
\]
\[
p_{\mu}^{s'} = u_{\mu}^{sr} p_{\mu}^{s} + l_{3}^{sr} v_{\mu}^{sr} x_{\mu}^{s},
\]
keeping \( X_{\mu} \) and \( P_{\mu} \) unchanged, where \( u \) and \( v \) are real \( 3 \times 3 \) matrices which satisfy

\(^{\text{a}}\) When the summation convention for repeated indices is not to be applied, this is mentioned for each time.

\(^{**}\) Accordingly \( a_{e}^{r\dagger} a_{r}^{r} \) (\( r \) not summed) takes negative eigenvalues \(-1, -2, \ldots\).
and in fact \( U = u + iv \). When \( v = 0 \), (3·7) reduces to the \( O(3) \) transformation (2·14). On the other hand (3·7) includes the internal reciprocity (2·27) as the special case where \( u = 0 \) and \( v = -1 \).

As stated before we assumed that the four points are bound in a small space-time region by means of a relativistic potential \( V \) such as Eq. (2·23). Evidently the most natural and simple model for that purpose is that of oscillator type. This means that we adopt the potential (2·23) in which we only keep the first term. The system then really has the \( U(3) \) symmetry which includes the \( O(3) \) subgroup and reciprocity.

For the sake of illustration we consider, as a tentative unified model for mesons, the one whose "scalar Hamiltonian" is given by

\[
H = \frac{1}{2\mu} q_\mu^a q_\mu^a + \frac{K}{2} V_{\mu\nu},
\]

(3·8)

(where \( \mu \) and \( K \) are constants), with the corresponding wave equation \( H\phi = 0 \), i.e.

\[
\left\{ \begin{array}{l}
d\partial_{y_\mu^a} - K\mu \sum_{\alpha,\beta} \left( y_\alpha^a - y_\beta^a \right) \\
\partial_{y_\mu^a}
\end{array} \right\} \phi(y) = 0.
\]

(3·9)

By performing the transformation into the normal relative coordinates, we can separate the center-of-mass degrees and bring Eq. (3·8) to the diagonal form

\[
H = \frac{1}{8\mu} P^2 + \left( \frac{1}{2\mu} p_\mu^r p_\mu^r + 2K x_\mu^r x_\mu^r \right).
\]

(3·10)

By setting

\[
l_0 = (4\mu K)^{-1/4},
\]

Eq. (3·10) is written as \( H = (P^2 + M^2) / (8\mu) \), with

\[
M^2 = \frac{4}{l_0^2} (\pi_\mu^r \pi_\mu^r + \xi_\mu^r \xi_\mu^r),
\]

(3·11)

which represents the (mass)\(^2\)-operator. Thus the model is indeed that of oscillator type and possesses the \( U(3) \) symmetry.

ii) The \( U(3) \) transformation (3·6) indicates that \( a_\mu^r \) and \( a_\mu^r' \) are contravariant and covariant vectors, respectively, with respect to the figure-space suffix. The transformation is generated by the nine Lorentz scalars

\[
A_\mu^r = a_\mu^r t a_\mu^r + \delta_\mu^r,
\]

(3·12)

since, as the consequence of Eq. (3·2), we have

\[
\begin{align*}
[A_\mu^w, a_\mu^r] &= -\delta_{w^r} a_\mu^w, \\
[A_\mu^w, a_\mu^r'] &= \delta_{w^r} a_\mu^w',
\end{align*}
\]

(3·13)
which just correspond to (3.6). Indeed $A_r^r$ of Eq. (3·12) has the properties of $U(3)$ generators

$$\begin{align*}
[A_r^r, A_s^s] &= \delta_r^s A_r^s - \delta_s^r A_s^r, \\
(A_r^r)^s &= A_r^s,
\end{align*}$$

(3·14)
on account of Eq. (3·2). Note that the occurrence of the additional constant $\delta_{rs}$ in Eq. (3·12) is to satisfy the requirement that $A_r^r \phi_0 = 0$ for the “ground state” $\phi_0$, as explained in Part II.

The $U(3)$ generators $A_r^r$ contain nine quantities, which are represented in terms of normal coordinates as follows:

$$A_r^r = a_r^{(r)} + 1 = \frac{1}{2} \left( \frac{1}{l_0^2} x_r^r x_r^r + l_0^2 p_r^r p_r^r \right) - 1,$$

(3·15)

$r$ not summed

$$A^{(rs)} = \frac{1}{2} (A_r^r + A_s^s) = \frac{1}{2} \left( \frac{1}{l_0^2} x_r^r x_s^s + l_0^2 p_r^r p_s^r \right), \quad (r \neq s)$$

(3·16)

$$i(A_r^r - A_s^s) = x_r^{(r)} p_s^{(s)} = L^r_{(rst)} \sim (123).$$

(3·17)

Especially the three quantities of Eq. (3·15) are expressed in terms of oscillator quantum numbers of Eq. (3·5) as

$$A_r^r = \sum_n n_r^r - n_r^r = n^{(r)}, \quad (r \text{ not summed}).$$

(3·18)

Each $n^{(r)}$ represents a Lorentz-scalar quantum number, which we call “principal quantum number” for the normal axis $x_r^r$.

The quantities (3·17) are exactly the generators (1·47) of the $O(3)$ group of normal axes rotations. On the other hand the isospin components $T_i$ should be identified according to

$$T_i = T_1 + iT_2 = A_i^z, \quad T_i = \frac{1}{2} (A_i^1 - A_i^2),$$

(3·19)

so they are given by

$$T_1 = A^{(3)}, \quad T_2 = \frac{1}{2} L^3,$$

(3·20)

$$T_3 = \frac{1}{2} (n^{(1)} - n^{(3)}).$$

(3·21)

If we use the hermitian $SU(3)$ generators $F_i$ in the Gell-Mann notation,9 we have

$$A^{(3)} = F_3, \quad A^{(31)} = F_i,$$

$$L^1 = 2F_7, \quad L^2 = -2F_8, \quad L^3 = 2F_1,$$

(3·22)
besides $T_i = F_i$ ($i = 1, 2, 3$). These indicate explicitly that $O(3)$ is such a subgroup of $U(3)$ as is different from the isospin subgroup, but with the linkage $L^3 = 2T_3$, and also that it is reduced to the $O(2)$ symmetry (generated by $L^3$) when the $U(3)$ symmetry is broken to be reduced to isospin and hypercharge symmetry only. The magnitude of isospin is written as

$$T^2 = \frac{1}{2} a^A \sigma_i a^B \sigma_i a^B a^A - \frac{1}{4} a^A \sigma_i a^A a^B a^B,$$  \hspace{1cm} (3.23)

where the subscripts $A, B$ run over 1 and 2 only. Evidently Eq. (3.23) is an invariant under $U(2)$, and so is it under the $O(2)$ subgroup of normal axes transformations.

It is convenient to represent generators of the unimodular subgroup $SU(3)$ contained in the $U(3)$, by

$$B_i = A_i - \frac{1}{3} \delta_{ij} A_n a^u = \frac{1}{3} \delta_{ij} \sigma^u A_n,$$ \hspace{1cm} (3.24)

where

$$\sigma_i = a^A \sigma_i a^B \sigma_i a^B a^A$$ \hspace{1cm} (3.25)

also possess the properties (3.14) for $U(3)$ generators. Then for particle hypercharge we may adopt the usual identification

$$Q = B_3 = A_3 / 3 - A_1 = \frac{1}{3} \sum_{\mathbf{r}} n^{(r)} - n^{(0)},$$ \hspace{1cm} (3.26)

so that the charge must be

$$Q = B_1 = A_1 / 3 - A_0 = \frac{1}{3} \sum_{\mathbf{r}} n^{(r)}.$$ \hspace{1cm} (3.27)

We have the usual set of commuting quantities $Y, T_3$ and $T^2$.

Clearly all $A_i$ are "proper internal quantities" due to Eq. (3.3):

$$[X_\rho, A_i] = 0, \quad [P_\rho, A_i] = 0.$$ \hspace{1cm} (3.28)

This is important because it ensures that the internal attributes such as isospin, hypercharge, and charge can take sharp eigenvalues in any wave-packet state of the center-of-mass as well as in an external plane-wave state. Further it will imply that the expressions for $T_i, Y, Q$, given by Eqs. (3.19), (3.26) and (3.27) should remain unaffected when interactions are introduced.

These expressions for the internal attributes $T_i, Y$ and $Q$ also indicate that these attributes do not come from their respective values partitioned to the four points $y_{\mu}^A$, but they are created by the excitations of relative coordinate oscillations. This is one of the features of our model qualitatively different from the viewpoint of the usual composite model for particles.

Now, if one fixes $A_i$, different supermultiplets are classified according to
irreducible representations of $SU(3)$, which are labelled by using eigenvalues of two Casimir operators

$$
\langle BB \rangle = B_r^* B_r^s, \quad \langle BBB \rangle = \frac{1}{2} B_r^* \{B_r^s, B_r^u\}.
$$

(3 · 29)

These are related with the $U(3)$ Casimir operators

$$
A_r^* , \quad \langle AA \rangle = A_r^* A_r^s , \quad \langle A A A \rangle = \frac{1}{2} A_r^* \{A_r^s, A_r^u\}
$$

in the following way:

$$
\langle BB \rangle = \langle AA \rangle - \frac{1}{3} (A_r^*)^2 ,
$$

$$
\langle BBB \rangle = \langle A A A \rangle - A_r^* \langle AA \rangle - \frac{3}{2} \langle AA \rangle + \frac{1}{2} (A_r^*)^2 + \frac{2}{9} (A_r^*)^3.
$$

(3 · 30)

On the other hand, an IR (irreducible representation) of $SU(3)$ is usually labelled by a pair of integers $(\lambda_1, \lambda_2)$. The relation between both labellings is given by

$$
\langle BB \rangle = \frac{2}{3} \{ (\lambda_1^2 + \lambda_2^2) + \lambda_1 \lambda_2 + 3 (\lambda_1 + \lambda_2) \} ,
$$

$$
\langle BBB \rangle = \frac{1}{9} (\lambda_1 - \lambda_2) (2 \lambda_1 + \lambda_2 + 3) (\lambda_1 + 2 \lambda_2 + 3).
$$

(3 · 31)

§ 4. Unitary operators for discrete transformations

In this section we obtain unitary operators for the reciprocity, internal reflections, transpositions, etc., since they correspond to the multiplicative conserved quantities and are of use for the classification of states. They are related to the phase transformation of the oscillator variables

$$
a_{\mu} \rightarrow \exp (-i\theta^r) \cdot a_{\mu}^s , \quad a_{\mu}^s \rightarrow \exp (i\theta^r) \cdot a_{\mu}^{s^*} ,
$$

(4 · 1)

(r not summed)

which also belongs to the $U(3)$ group, so the theory is invariant under (4 · 1) at least in the symmetry limit. In terms of the normal variables, (4 · 1) is expressed as the special transformation

$$
\begin{pmatrix}
\xi_{\mu}^{r^*} \\
\pi_{\mu}^{r^*}
\end{pmatrix}
= \begin{pmatrix}
\cos \theta^r & \sin \theta^r \\
-\sin \theta^r & \cos \theta^r
\end{pmatrix}
\begin{pmatrix}
\xi_{\mu}^r \\
\pi_{\mu}^r
\end{pmatrix}, \quad (r \text{ not summed}).
$$

(4 · 2)

The unitary operator $U$ which induces Eq. (4 · 1), according to $U a_{\mu}^r U^{-1} = \exp (-i\theta^r) \cdot a_{\mu}^s$ and $U a_{\mu}^{s^*} U^{-1} = \exp (i\theta^r) \cdot a_{\mu}^{s^*} \quad (r \text{ not summed})$, is given by
\[ U = \exp(i \sum_n \theta^n A_n^u) = \exp(i \sum_n \theta^n) \exp(i \sum_n \theta^n a_p^{u*} a_p^u), \quad (4.3) \]

or

\[ U = \exp(i \theta \sum_n n^{(u)}) = e^{-i \theta} \exp \left[ -i \frac{\theta}{2} \left( \xi_n^{u*} \xi_n^u + \pi_n^{u*} \pi_n^u \right) \right], \quad \text{if } \theta' = \theta. \quad (4.4) \]

Under the transformation (4.1), \( n_i^r, n_i^\gamma \) and \( A_i^r \) (\( r \) not summed) are invariant, and under (4.4) all \( A_i^r \) are invariant.

Now the internal reciprocity (2.27) is expressed, in terms of the oscillator variables, as

\[ a_p^r \rightarrow -ia_p^r, \quad a_p^{r\dagger} \rightarrow ia_p^{r\dagger}, \quad (4.5) \]

corresponding to the special case \( \theta' = \theta = \pi/2 \) of the transformation (4.1). Therefore the unitary operator for the reciprocity is

\[ U_r = \exp \left[ i \frac{\pi}{2} \sum_n n^{(u)} \right] = -i \exp \left[ i \frac{\pi}{2} a_p^{u*} a_p^u \right] = i \exp \left[ -i \left( \frac{\lambda_0}{2} x_p^{u*} x_p^u + \lambda_0^2 p_p^{u*} p_p^u \right) \right], \quad (4.6) \]

having four eigenvalues ±1 and ±i.

The internal inversion (2.25) corresponds to the case \( \theta = \pi \), with the unitary operator

\[ U_I = (U_r)^2 = \exp[i \pi \sum_n n^{(u)}] = -\exp \left[ i \frac{\pi}{2} \left( \lambda_0^{-1} x_p^{u*} x_p^u + \lambda_0^2 p_p^{u*} p_p^u \right) \right], \quad (4.7) \]

which in fact gives

\[ U_I x_p^r U_I^{-1} = -x_p^r, \quad U_I p_p^r U_I^{-1} = -p_p^r, \quad (4.8) \]

having eigenvalues ±1. [This operator should not be confused with the dilatation operator (1.46).] In terms of the tetrad variables, Eq. (4.7) is expressed as

\[ U_I = -\exp \left[ \frac{\pi i}{8} \sum_{\gamma_\alpha} \left( \lambda_0^{-1} y_\alpha^{u*} y_\alpha^u + \lambda_0^2 q_\alpha^{u*} q_\alpha^u \right) \right]. \quad (4.9) \]

If one requires the transformation (4.1) with \( \theta' = \theta \) to belong to the unimodular subgroup \( SU(3) \), \( \theta \) is restricted to three possible values\(^b\) 0, 2\( \pi/3 \), and 4\( \pi/3 \). Then with \( \theta = 2\pi/3 \), the transformation (4.1) becomes

\[ a_p^r \rightarrow \omega^r a_p^r, \quad a_p^{r\dagger} \rightarrow \omega a_p^{r\dagger}, \quad (4.10) \]

or equivalently

\(^b\) Note that reciprocity and internal inversion do not belong to \( SU(3) \).
Here \( \omega = e^{i\pi/3} \) is the complex cubic root of 1. This means a rotation by 120° in the plane spanned by \( \xi_\rho^r \) and \( \pi_\rho^r \) (for every \( r \)). The corresponding unitary operator is

\[
U_i = \exp \left[ \frac{2\pi i}{3} \sum_r \eta(r) \right] = \exp \left[ \frac{2\pi i}{3} a_\rho^{\ast r} a_\rho^r \right] = \exp \left[ \frac{i\pi}{3} \left( l_0^{-2} x_\rho^r x_\rho^r + l_0^2 p_\rho^r p_\rho^r \right) \right],
\]

having eigenvalues 1, \( \omega \) and \( \omega^2 \). We call \( U_i \) multiplicative triality operator. Since \( U_r, U_I \) and \( U_t \) depend on the \( U(3) \) invariant \( A_\rho^\alpha \) alone, they are constants of motion even in the situation of the broken \( U(3) \) symmetry.

We now consider the normal axis reflection of \( x_\rho^\prime \), defined by (2.24). This corresponds to the case \( \theta^1 = \pi, \theta^2 = \theta^3 = 0 \), so its unitary operator is

\[
U_i = -\exp \left[ i\pi a_\rho^{\ast r} a_\rho^r \right] = -\exp \left[ \frac{i\pi}{2} \left( l_0^{-2} \left( x_\rho^r x_\rho^r + l_0^2 \left( p_\rho^r p_\rho^r \right) \right) \right) \right], \tag{4.12}
\]

If we recall that any transposition between \( y_\rho^\alpha \)'s corresponds to a certain normal axes reflection (cf. § 2), we can obtain the unitary operator for the transposition \( (y_\rho^\alpha, y_\rho^\beta) \) in the form

\[
U_{(\alpha, \beta)} = -\exp \left[ \frac{i\pi}{4} \left( l_0^{-2} \left( y_\rho^\alpha - y_\rho^\beta \right)^2 + l_0^2 \left( q_\rho^\alpha - q_\rho^\beta \right)^2 \right) \right], \tag{4.13}
\]

(\( \alpha, \beta \) not summed), which indeed exchanges \( y_\rho^\alpha \) and \( y_\rho^\beta \):

\[
U_{(\alpha, \beta)} y_\rho^\alpha U_{(\alpha, \beta)}^{-1} = y_\rho^\beta, \quad U_{(\alpha, \beta)} y_\rho^\beta U_{(\alpha, \beta)}^{-1} = y_\rho^\alpha. \tag{4.14}
\]

All the \( U_i \) and \( U_{(\alpha, \beta)} \) are conserved at least in the \( U(3) \) symmetry limit, having eigenvalues \( \pm 1 \). In fact \( U_1 \) is conserved also in the situation of the broken \( U(3) \) symmetry, since under \( a_\rho^1 \to -a_\rho^1 \), one has \( (T_1, T_2, T_3) \to (-T_1, -T_2, T_3) \) so that \( T_3, T^3, \) and \( Y \) remain invariant.

§ 5. Commuting set of quantities

i) To obtain the complete set of commuting quantities relevant to the internal motions we need to make some further analysis in the motion with respect to a laboratory frame.

Since \( a_\rho^r \) and \( a_\rho^{\ast r} \) are respectively the contravariant and covariant vectors under \( U(3) \) [Eq. (3.13)], the quantities

\[
\begin{pmatrix}
\xi_\rho^r \\
\pi_\rho^r
\end{pmatrix} =
\begin{pmatrix}
-1/2 & \sqrt{3}/2 \\
-\sqrt{3}/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
\xi_\rho^r \\
\pi_\rho^r
\end{pmatrix}, \tag{4.10'}
\]
are $U(3)$ invariant

$$[K_{\mu\nu}, A_{\rho}^r] = 0,$$  \hspace{1cm} (5·1)

and hence they are, of course, invariant under normal axes transformations (2·14) and the phase transformation (4·2). They satisfy

$$[K_{\mu\nu}, K_{\rho\sigma}] = \delta_{\mu\sigma} K_{\rho\nu} - \delta_{\mu\nu} K_{\rho\sigma},$$

$$K_{\rho\sigma} = \xi_{\rho\sigma}^{rs} K_{s\nu},$$  \hspace{1cm} (5·3)

whence they constitute the generators of a 16-parameter $U^{\#}(4)$ group. This is the group of linear transformations which leave

$$Z_k^* Z_t - Z_s^* Z_4 = \text{invariant},$$  \hspace{1cm} (5·4)

where $Z_k$ and $Z_t$ are complex numbers. If one restricts $Z_{\mu}$ such that $Z_k = \text{real}$, $Z_4 = \text{pure imaginary}$, corresponding to a real Minkowski vector $Z_{\nu}$, Eq. (5·4) becomes $Z_k^2 + Z_4^2 = Z_\nu^2 = \text{invariant}$. Therefore $U^{\#}(4)$ involves the Lorentz group as a subgroup. Under the $U^{\#}(4)$ group, $a_\mu^r$ and $a_\rho^s$ transform again as the covariant and contravariant vectors, respectively:

$$[K_{\mu\nu}, a_\rho^r] = - \delta_{\mu\rho} a_\nu^r,$$

$$[K_{\mu\nu}, a_\rho^s] = \delta_{\nu\rho} a_\mu^s.$$  \hspace{1cm} (5·5)

One can split the tensor $K_{\mu\nu}$ into three irreducible parts

$$L_{\mu\nu} = -i (K_{\mu\nu} - K_{\nu\mu}) = -i (a_\mu^r a_\nu^s - a_\nu^r a_\mu^s),$$  \hspace{1cm} (5·6)

$$K_{\mu\nu} = a_\mu^r a_\nu^s = \bar{A}_{\nu}^r,$$  \hspace{1cm} (5·7)

$$H_{\mu\nu} = K_{\mu\nu} + K_{\nu\mu} = \frac{1}{2} \delta_{\mu\nu} K_{\rho\sigma}. $$  \hspace{1cm} (5·8)

Owing to Eq. (5·2), one has

$$[L_{\mu\nu}, A_{\rho}^r] = [H_{\mu\nu}, A_{\rho}^r] = 0.$$  \hspace{1cm} (5·9)

$L_{\mu\nu}$ is the same quantity as given by Eq. (1·43), representing the angular momentum tensor due to the relative motion of four points $\gamma_{\rho}^\alpha$, as determined by $\gamma_{\rho}^\alpha = X_{\mu}^\alpha P_{\nu} + L_{\mu\nu}$. Evidently $K_{\mu\nu}$ is an invariant of the $U^{\#}(4)$ group as well as of the $U(3)$ group. The symmetric $H_{\mu\nu}$ contains eight independent components and is reexpressed as

$$H_{\mu\nu} = \xi_{\mu}^{rs} \xi_{\nu}^{rs} + \pi_{\mu}^{rs} \pi_{\nu}^{rs} - \frac{1}{2} \delta_{\mu\nu} (A_{\nu}^r + 3),$$  \hspace{1cm} (5·10)

which we shall call "oscillation tensor".

The $U^{\#}(4)$ group differs from the 16-parameter group $GL(4)$ generated by $D_{\mu\nu} = x_{\mu}^r p_{\nu}^s$ (considered in § 1–iii), although both of them contain the same Lorentz
subgroup (acting on the internal variables) generated by $L_{\rho\sigma}$. Note especially that $H^\#_{\rho\sigma}$ are $U(3)$-invariant [Eq. (5.9)] and hence self-reciprocal, in contrast with the dilatation and torsion operators, $D$ and $T_{\rho\sigma}$ of Eqs. (1.44) and (1.45).

Now $L_{\rho\sigma}$ is equivalently represented by the set of the pseudovector

$$W_\rho = \bar{L}_{\rho\sigma} P_\sigma / \sqrt{P} = \frac{1}{2i\sqrt{P}} \varepsilon_{\rho\mu\lambda} L_{\lambda\mu} P_\sigma$$

and the vector $N_\rho = L_{\rho\sigma} P_\sigma$, satisfying

$$W_\rho P_\rho = N_\rho P_\rho = 0, \quad [W_\rho, A_\rho] = [N_\rho, A_\rho] = 0.$$  

The quantity $W_\rho$ gives the covariant definition of the "relative angular momentum", and rotates any internal vector $u_\rho$ within the hyperplane normal to $P_\rho$, according to the relation

$$[W_\rho, u_\rho] = \frac{1}{\sqrt{P}} \varepsilon_{\rho\mu\lambda} u_\mu P_\lambda.$$  

(5.12)

where $u_\rho$ may be any of $a_\rho$, $a_\rho^r$, $x_\rho$, $p_\rho$, or $W_\rho$ itself. It has the magnitude $W_\rho^2 = W(W+1)$ with the integer eigenvalues $W = 0, 1, 2 \cdots$, since it originates from the orbital motion of three relative coordinates $x_\rho$. In the rest frame one has

$$W_4 = -i\varepsilon_{4ijk} a_4^r a_\rho^r, \quad W_i = L_{25}, \text{ etc.}, \quad W_i = 0.$$  

(5.13)

If one makes here the canonical transformation

$$\bar{a}_i^r = \frac{1}{\sqrt{2}} (a_i^r - ia_i^s), \quad \bar{a}_{-1}^r = \frac{1}{\sqrt{2}} (a_i^r + ia_i^s), \quad \bar{a}_s^r = a_s^r,$$

$W_3$ is expressed as

$$W_3 = \bar{a}_4^r \bar{a}_4^r - \bar{a}_{-1}^r \bar{a}_{-1}^r,$$

so that $\bar{a}_m^r$ means the annihilation operator for the oscillation for which $W_3 = m$. (The notation $\bar{\cdot}$ stands for an eigenvalue.)

From $W_\rho$ we define

$$W^r = a_\rho^r W_\rho = W_\rho a_\rho^r,$$

$$W^{r*} = W_\rho a_\rho^r = a_\rho^{r*} W_\rho,$$

which are, respectively, the contravariant and covariant vectors with respect to $U(3)$, and satisfy

$$[W^r, W_\rho] = [W^{r*}, W_\rho] = 0,$$

(5.14)

owing to Eq. (5.12). If we then define

$$\Theta = W^{r*} W^r,$$

(5.15)
it is evident that this has non-negative eigenvalues and that it is \( U(3) \) invariant: 
\[
[\Theta, A_r^\mu] = 0, \quad \text{and also } [\Theta, W_s] = 0.
\]
This quantity is expressed in various forms:
\[
\Theta = a_{\mu}^{\rho} W_\mu W_\nu a_{\nu}^{\rho} = W_\mu K_{\mu\nu} W_\nu.
\]
\[
= W_\mu W_\nu K_{\mu\nu} + W_\mu^2 = K_{\mu\nu} W_\mu W_\nu - W_\mu^2,
\]
as verified by the commutation relations
\[
[K_{\mu\nu}, W_s] = -[K_{\nu\mu}, W_s] = W_{\mu\nu}, \quad [H_{\mu\nu}, W_s] = 0.
\]
In fact \( \Theta \) can be expressed in terms of \( W_\mu, H_{\mu\nu} \) and \( K_{\mu\nu} = \bar{A}_r^{\mu\nu} \) only, as shown by
\[
\Theta = \frac{1}{2} \left\{ \Theta' + \left( 1 + \frac{1}{2} K_{\mu\nu} \right) W_\mu^2 \right\},
\]
\[
(\Theta' = H_{\mu\nu} W_\mu W_\nu = W_\mu H_{\mu\nu} W_\nu = W_\mu W_\nu H_{\mu\nu}).
\]

Thus we have obtained three mutually commuting quantities \( W_\mu^2, W_s, \) and \( \Theta \), which are \( U(3) \) invariant and correspond to the three degrees of freedom of the rotation of our object.

ii) Summing up we have obtained the set of nine commuting quantities to characterize the internal motions: the three Casimir operators \( A_r^\mu, \langle BB \rangle, \langle BBB \rangle \), the three quantities \( Y, T_3, T^3 \), and the three quantities \( W_\mu^2, W_3, \) and \( \Theta \). They are all self-reciprocal, and except \( W_3 \) they are all Lorentz invariant. Our model originally has twelve degrees of freedom of the internal coordinates \( x_\mu \), but, as already suggested, we impose the essential subsidiary conditions* which suppress three relative-time degrees to a finite but minimum extent so that our model has in practice nine degrees of freedom. Equation (5.18) is a vector equation with respect to a \( U(3) \) transformation (3.6) to have the \( U(3) \)-invariant meanings, and is compatible with the wave equation as illustrated for the case (3.10). Then the above nine quantities are just sufficient for the complete specification of internal motion, and in fact they supply good quantum numbers. This result is consistent with the conditions (5.18), since we have
\[
[A_s^\mu, A'_{\nu}] = \delta_{\mu\nu} A', \quad [W_\mu, A'] = [\Theta, A'] = 0,
\]
and moreover
\[
[X_\mu P_\rho, A'] = iA', \quad [X_\mu P_\rho, W_s] = [X_\rho P_\mu, \Theta] = 0.
\]
[These points will be explained more closely in Part II.]

Our four-point object has no kinematical constraint, besides Eq. (5.18), associated with a clear picture. Its bodily configuration independent of reference frames is specified with six elements represented by the six invariant distances

* This condition has analogy to the Lorentz condition in the case of quantum electrodynamics.
s^{a,b}$, and this implies the existence of six dynamical quantities to be conserved associated with the bodily motions (deformations, oscillations), which are just represented by $A_1, \langle BB \rangle, \langle BBB \rangle, T^3, T_4$ and $Y$. On the other hand, the body has three degrees of freedom of the rotation with respect to an inertial frame, since its rotational orientation is to be specified with three free parameters (such as the Euler angles) and the corresponding three conserved dynamical quantities are represented by $W^r, W_3$ and $\Theta$. This is in contrast with the case of the bilocal model, where the rotational orientation is specified without the use of the third Euler angle and the quantities corresponding to $W^r$ and to $\Theta$ trivially vanish.$^8$ Our $\Theta$ is the quantity related to the couplings between oscillations and rotations and not one just related to rotation, but it is essentially regarded to reflect the relative angular momentum with respect to the body.

§ 6. Remarks on the model for mesons

Finally we make some remarks about the simple model given in § 3-i) as a tentative unified model for mesons. The model implied the quantization of squared mass according to Eq. (3·11), namely

$$M^2 = \mu_0^2 (a^{x^r}_x + 6) = \mu_0^2 (A_1^r + 3) = \mu_0^2 (\sum n^{(0)} + 3), \quad (6·1)$$

where $\mu_0^2 = 8/l_0^2$ supplies the basic (mass)$^2$ unit.$^8$ Then $\mu_0 = 4(K\mu)^{1/4}$ is related to $l_0$ by

$$\mu_0 l_0 = 2\sqrt{2}. \quad (6·2)$$

Empirically there exist indications for such an integer multiplication rule for the (mass)$^2$ of mesonic levels; in particular$^{59}$ for the four isosinglet mesons $\eta$, $\omega$, $X^0(958)$ and $f^0$, the relations

$$m_\eta^2 : m_\omega^2 : m_{X^0} : m_{f^0}^2 = 1 : 2 : 3 : 5 \quad (6·3)$$

are well satisfied. If we assign

$$A_1^r = \sum n^{(0)} = 0, 3, 6, (9), 12, \quad (6·4)$$

to $\eta$, $\omega$, $X^0$ and $f^0$, respectively, the mass formula (6·1) just gives Eq. (6·3). The selection (6·4) corresponds to the subsidiary condition

$$U_1 \varphi = \varphi, \quad (6·5)$$

where $U_1$ is the multiplicative triality operator of Eq. (4·11).

In our framework the triality condition (6·5) is important, together with

$^5$) If there existed only a single relative coordinate $x_{\mu}$, the quantity corresponding to $W^r$ becomes $a_{\mu}W^\mu = -(1/\sqrt{p})a_{\mu\nu}a_{\mu\nu}a_{\mu\nu} = 0$.

$^59$) The principal quantum numbers $n^{(r)}$ are positive semi-definite owing to the subsidiary condition (5·18).
the condition (5·18), and they will be assumed for the baryonic case also (cf. Part II).

The model for mesons in its present form, however, is unsatisfactory to reproduce the actual supermultiplets of meson spectrum in accordance with the charge conjugation and need a certain modification. We therefore refrain from entering into it any further. In Part II we deal with baryons rather than mesons in our framework to show that it really gives an adequate model for the baryonic states in accord with observations.

Appendix

In the Appendices we supply, besides some mathematical supplements, closer explanations of some concepts employed in the text.

i) Relations satisfied by $C^{ra}$ and $\bar{C}^{ra}$

The coefficients $C^{ra}$ of the general transformation (1·6) satisfied the relations (1·7), or equivalently (1·10). Evidently this latter relation is written as

$$\varepsilon_{a\beta\gamma\delta} C^{\beta\gamma} C^{\alpha\delta} = \eta \varepsilon_{rst},$$

which is re-expressed also in the form

$$\varepsilon_{rst} C^{ra} C^{rb} C^{rc} = -\eta \sum_{s} \varepsilon_{a\beta\gamma\delta}.$$  (A·1)

These are useful for calculations. Thus, the fact that Eq. (1·18) for $A$ is rewritten as Eq. (1·20) is verified by using Eq. (A·2). That $\bar{C}^{ra}$ given by Eq. (1·32) satisfies Eq. (1·31) is verified by the aid of Eq. (A·1), while the relation (1·33) is obtained by making use of Eqs. (1·32) and (A·2).

We also note the re-expressions of Eq. (1·32):

$$\varepsilon_{rst} C^{ra} C^{rb} C^{rc} = -\frac{1}{4\eta} \sum_{s} \varepsilon_{a\beta\gamma\delta} \bar{C}^{ra},$$

$$\bar{C}^{ra} - \bar{C}^{rb} = -\frac{\eta}{2} \varepsilon_{a\beta\gamma\delta} C^{\gamma\delta} C^{\alpha\beta}. \quad (rst) \sim (123).$$  (A·3)

ii) Rest volume

The quantity $A$ of Eq. (1·18) is written in the center-of-mass rest frame (the reference frame in which $P_{h}=0$) as

$$A = \sqrt{P} \left| \begin{array}{cccc}
\gamma_{1}^{1} & \gamma_{1}^{2} & \gamma_{1}^{3} & 1 \\
\gamma_{2}^{1} & \gamma_{2}^{2} & \gamma_{2}^{3} & 1 \\
\gamma_{3}^{1} & \gamma_{3}^{2} & \gamma_{3}^{3} & 1 \\
\gamma_{4}^{1} & \gamma_{4}^{2} & \gamma_{4}^{3} & 1
\end{array} \right|,$$

which is the (rest) volume of the tetrahedron with vertices at $\gamma_{\alpha}^{\beta}$ ($\alpha = 1, 2, 3, 4$), aside the factor $6 \sqrt{P}$. Similarly, the expression (1·20) for $A$ becomes in the rest frame

Thus an appropriate configuration for mesons is of eight-points subjected to special constraints to have nine internal degrees again.
\[ \Delta = (\sqrt{P/\eta}) \varepsilon_{ijk} x^i x^j x^k, \]
which represents the volume of the parallelopiped spanned by \( x^1, x^2, \) and \( x^3, \) apart from the factor \( \sqrt{P/\eta}. \)

### iii) Some properties of normal axes

A simple feature of normal axes is exhibited by considering the special case in which the four points \( y_{\alpha} \) happen to come to vertices of a "regular tetrahedron", namely the case where all \( s^{\alpha\beta} \) are equal: \( s^{12} = s^{35} = \ldots = s. \) This implies five conditions to be expressed by

\[ s^{\alpha\beta} = s(1 - \delta_{\alpha\beta}). \quad \text{(A.4)} \]

Then, inserting this into Eq. (1.14) and using Eq. (1.7) we have

\[ x_\mu \, x_\nu = -\frac{1}{2} \sum_{\alpha, \beta} C^{\sigma\beta} C^{\alpha\delta} s(1 - \delta_{\alpha\beta}) = \frac{1}{2} C^{\sigma\beta} C^{\alpha\delta}, \]

which becomes, by the use of (2.1),

\[ x_\mu \, x_\nu = \frac{1}{2} s \delta_{\mu\nu}, \quad \text{(A.5)} \]

meaning that the normal axes form an "orthogonal triad". Equation (A.5) again implies five relations, and we can verify that Eq. (A.4) can be derived from Eq. (A.5), conversely. Thus the normal axes are such set of relative coordinates in terms of which a regular-tetrahedron configuration for the tetrad \( y_{\alpha} \) be equivalently represented by the orthogonal triad with axes of equal length.\(^{12}\)

It is also proved that \( s \) in Eq. (A.5) should be non-negative so that every triad axis be space-like (or null) vector. This means that all four points \( y_{\alpha} \) for the case of the regular-tetrahedron configuration must be located relatively in space-like manner.\(^{13}\)

Another remark is that in the normal coordinates we can solve Eq. (1.14) to express \( s^{\alpha\beta} \) as

\[ s^{\alpha\beta} = \frac{1}{2} x_\mu \, x_\nu - (3C^{\sigma\beta} C^{\alpha\delta} - C^{\sigma\nu} C^{\alpha\beta}) x_\mu \, x_\nu, \quad (\alpha \neq \beta), \quad \text{(A.6)} \]

where by \( \gamma \) and \( \delta \) we represent two numbers from 1, 2, 3, 4 other than \( \alpha \) and \( \beta. \)

### iv) Properties of bodily transformations

Our theory has the invariance under the bodily transformation (2.18) with the conditions (2.19) and (2.20). This forms a subgroup of \( O(4) \) since the condition (2.20) is preserved in the successive applications of two such transformations as seen in
Also it is evident that they contain the finite group \( S_4 \) of permutations among the tetrad points \( y_\mu^\nu \)'s. We emphasize that a bodily transformation is not only independent of but is very different from the Lorentz transformations. Although it keeps the sum of invariant distances \( V_{\mu\nu} = \sum_\alpha s^{\alpha\beta} \) as well as the volume \( \mathcal{A} \) (up to sign) of the tetrad invariant, it does not in general mean a "rigid rotation" of the body. This is because a bodily transformation is such one as leaves \( V_{\mu\nu} \) (and \( X_\mu \)) invariant while each \( s^{\alpha\beta} = (y_\mu^\alpha - y_\mu^\beta)^2 \) [\( \alpha, \beta \) not summed] may generally change there so that the tetrad receives a deformation. The deformation may even be such one as changes two points originally in relatively space-like locations into two points in relatively time-like locations (and vice versa). \(^{a)} \)

Nevertheless, in the figure space any bodily transformation is represented by a rigid rotation or reflection of normal axes, so that all possible bodily transformations form an \( O(3) \) group. From Eq. (2·19) we have

\[
\det (\bar{R}) = \pm 1,
\]

and according to this sign we divide the bodily transformations into "rotational" and "reflectional" ones. It is verified that for the former the volume \( \mathcal{A} \) of Eq. (1·18) remains unchanged while for the latter it changes the sign.

To exhibit them more explicitly it is useful to consider the special ones for which \( y_\mu^4 \) remains unaltered. They are the rotational and reflectional transformations of \( y_\mu^r \)'s \((r=1,2,3)\) in the plane defined by them, leaving the center \( (y_\mu^1 + y_\mu^2 + y_\mu^3)/3 \) invariant. The transformation matrix \( \bar{R} \) of (2·18) then satisfies \( \bar{R}^{\alpha\beta} = \bar{R}^{\beta\alpha} = \delta_{\alpha\beta} \) and is essentially a \( 3 \times 3 \) orthogonal matrix \( (\bar{R}^{\alpha\beta}) \) with the condition \( \sum_\alpha \bar{R}^{\alpha\beta} = 1 \). It is proved that such a matrix should be expressed either in the form

\[
(\bar{R}^{(+)}) = \begin{pmatrix}
a & b & c \\
c & a & b \\
b & c & a
\end{pmatrix},
\]

or in the form

\[
(\bar{R}^{(-)}) = \begin{pmatrix}
a & -b & c \\
c & -a & b \\
b & -c & a
\end{pmatrix}.
\]

\(^{a)} \) To see this one takes up, e.g. the cyclic permutation \( (y_\mu^1, y_\mu^2, y_\mu^3) \), which belongs to \( S_4 \), and of course to the bodily transformation. Now our theory generally contains the amplitude for e.g. a configuration in which \( y_\mu^1 \) and \( y_\mu^2 \) are relatively space-like while \( y_\mu^1 \) and \( y_\mu^3 \) as well as \( y_\mu^2 \) and \( y_\mu^3 \) are both relatively time-like. Then by the above permutation it changes to a configuration in which \( y_\mu^1 \) and \( y_\mu^3 \) are relatively time-like while \( y_\mu^2 \) and \( y_\mu^3 \) are relatively space-like.
\[ (\bar{R}^{(-)}) = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}, \]

where \( a, b, \) and \( c, \) are restricted by
\[ a + b + c = 1, \quad a^2 + b^2 + c^2 = 1, \]
to imply a single free parameter only. Indeed they are expressed as
\[ a = \frac{1}{3} (1 + 2 \cos \phi), \quad b = \frac{1}{3} (1 - \cos \phi + \sqrt{3} \sin \phi), \]
\[ c = \frac{1}{3} (1 - \cos \phi - \sqrt{3} \sin \phi). \]

Evidently \( (A \cdot 8) \) is a "rotational" and \( (A \cdot 9) \) is a "reflectional" transformation. As easily verified, the matrices \( (A \cdot 8) \) obeying Eq. \( (A \cdot 10) \) satisfy the group conditions and the product of such two matrices \( \bar{R}^{(+) = R_z^{(+) R_4^{(+)}} \) corresponds to the sum \( \phi = \phi_1 + \phi_2 \) for the parameter \( \phi \) of Eq. \( (A \cdot 11) \), so that the group is \( O^+(2) \).

As a special case we set \( b = 1, \ (a = c = 0), \) which corresponds to \( \phi = 120^\circ \) in Eq. \( (A \cdot 11); \) then Eq. \( (A \cdot 8) \) exactly represents the cyclic permutation \( (y^{1}_s, y^{2}_s, y^{3}_s) \) while Eq. \( (A \cdot 9) \) yields the transposition \( (y^{1}_s, y^{3}_s) \). Similarly the case \( c = 1 \) (i.e. \( \phi = 240^\circ \)) corresponds, respectively, to \( (y^{1}_s, y^{3}_s, y^{2}_s) \) and \( (y^{1}_s, y^{2}_s) \). In such sense the cyclic permutations
\[ (y^{1}_s, y^{2}_s, y^{3}_s), \ (y^{1}_s, y^{3}_s, y^{2}_s) \]
correspond to the "rotations" by \( 2\pi/3 \) and \( 4\pi/3 \) in the bodily space, where the triangle \( \triangle (y^1y^2y^3) \) is rotated and deformed to be brought into coincidence with, say \( \triangle (y^2y^3y^1) \).

The unitary operator that induces the transformation \( (A \cdot 8) \) is given by
\[ U_+ = \exp \left( -\frac{i\phi}{\sqrt{3}} \sum \varepsilon_{rst} y^{s}_r q^{t}_s \right). \]

This really gives
\[ U_+ y^{s}_r U_+^{-1} = y^{s}_r \cos \phi + \frac{1}{3} (1 - \cos \phi) \sum \varepsilon_{rst} y^{r}_r \phi \sum \varepsilon_{rst} y^{s}_r, \]
which exactly agrees with the transformation \( (A \cdot 8) \), if one notes \( (A \cdot 11) \). Thus the unitary operators corresponding to \( (A \cdot 12) \) are obtained; the one for \( (y^{1}_s, y^{2}_s, y^{3}_s) \) is
\[ U_{(123)} = \exp \left[ -\frac{2\pi i}{3\sqrt{3}} \sum \varepsilon_{rst} y^{s}_r q^{t}_s \right]. \]
The infinitesimal operators for rotational bodily transformations are essentially $L'$. In particular, if we take the standard normal coordinates, $U_+$ of Eq. (A·13) is written as

$$U_+ = \exp(-i\phi L'),$$  \hspace{1cm} (A·16)

corresponding to a simple figure-space rotation in the plane spanned by $x_μ^1$ and $x_μ^3$.

Finally we note that $S_4$ contains the "tetrahedron subgroup" $T$, consisting of the "four-group" elements

$$(y^1 y^2) (y^3 y^4), \quad (y^i y^j) (y^k y^l), \quad (y^i y^j) (y^k y^l),$$  \hspace{1cm} (A·17)

and of the $4 \times 2 = 8$ cyclic permutations such as (A·12). They are all "rotational".

v) $O(2)$ symmetry in standard coordinates

Along with the symmetry reduction of $U(3)$ to $U(2)$, the subgroup $O(3)$ must also reduce to $O(2)$. This is caused by the situation that one of the four points, say $y_μ^4$, becomes inequivalent with the other three. Then the $O(2)$ is just the transformations given by the matrices (A·8) and (A·9) of iv). If we consider them in the standard relative coordinates (1·11), they are just represented by that $O(2)$ subgroup of normal axes transformations (2·14) which leave $x_μ^3$ invariant. Namely

$$x_μ^{A'} = \hat{R}^{AB} x_μ^B,$$  \hspace{1cm} (A·18)

with subscripts $A$, $B$ running over 1, 2 only, and $2 \times 2$ orthogonal matrix $R$.

Evidently the $O(2)$ group contains the $S_3$ subgroup consisting of the permutations among $y_μ^r (r = 1, 2, 3)$ only, namely the three transpositions $(y^1, y^2)$, $(y^1, y^3)$, and $(y^2, y^3)$, and the cyclic permutations (A·12) that form the alternating subgroup $A_3$ of $S_3$. The matrices $\hat{R}$ representing them become, respectively,

$$\hat{R}_{(12)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{R}_{(13)} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \quad \hat{R}_{(23)} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$\hat{R}_{(132)} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad \hat{R}_{(123)} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix},$$  \hspace{1cm} (A·19)

of which the former three have $\det(\hat{R}) = -1$ while the latter two have $\det(\hat{R}) = 1$ and

$$(\hat{R}_{(132)})^3 = (\hat{R}_{(123)})^3 = 1, \quad (\hat{R}_{(13)})^2 = \hat{R}_{(123)}.$$  \hspace{1cm} (A·20)

[Note that $\hat{R}_{(13)}$ is identical with the matrix in (4·10').]

The fact that the theory is always invariant under the cyclic permutations (A·12) may be of particular interest, because of their "triality property". (A·19) indicates that they are represented by $2\pi/3$ and $4\pi/3$ rotations in the
figure space. It must be remarked, however, that the unitary operator representing \((A \cdot 12)\) is given by \((A \cdot 15)\) which is different from the triality operator \(U_t\) of \((4 \cdot 11)\). Thus the triality condition \((6 \cdot 5)\) could not be immediately interpreted in terms of the restriction on the symmetry character of the wave function regarding the cyclic permutation \((A \cdot 12)\), in the sense of parastatistics.

References

2) H. Yukawa, Phys. Rev. 91 (1953), 415, 416.
Y. Ne'eman, Nucl. Phys. 26 (1961), 222.
7) T. Takabayasi, Nuovo Cim. 33 (1964), 668. We note that the quantities written as \(A_m^r, K_{\mu \nu}\), and \(\theta\) in this reference are now designated respectively as \(A_m^r, L_{\mu \nu}\), and \(\theta + \omega_{m}^2\), in the present paper.
8) Such a viewpoint was also taken in connection with a further generalization of the model by H. Yukawa, Y. Katayama and E. Yamada, Lectures at Tokyo Symposium on Models and Structures of Elementary Particles.
9) Such permutations are suggested by E. Yamada (private communication).
11) Recently such group has also been considered in different context by A. O. Barut, Nuovo Cim. 32 (1964), 234, and B. Kursunoglu, Phys. Rev. 135 (1964), B761.
13) If one imposed the relation \((A \cdot 4)\) or \((A \cdot 5)\) as constraints on the model, then \(x_\mu^r\)'s are no longer independent variables and the model reduces to the relativistic "spherical" rotator, for which the \(U(3)\) symmetry reduces to the \(O(3)\) symmetry \((2 \cdot 14)\) with the conservation of \(L^r\) of \((1.47)\) only. Compare with T. Takabayasi, Prog. Theor. Phys. 23 (1960), 915.

Note added in proof: i) As regards reference 8), see Y. Katayama, E. Yamada and H. Yukawa, Prog. Theor. Phys. 33 (1965), 541. ii) The essential results of our theory applied to baryons are given in T. Takabayasi, NUDP-report T-6, where it is shown in particular that our quadri-local model implies underlying \(U(9)\) symmetry which is reduced, through the couplings of internal motions, to the direct product of the unitary-spin group \(U(3)\) and the \(U(3)'\) group containing the usual rotational invariance, and that baryonic supermultiplets are grouped into the 165 dimensional IR of the \(U(9)\) corresponding to the first excited shell of the relative coordinate oscillations.