K-Matrix Formalism for Three-Body Scattering 
and Bound-State Scattering

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(Received April 12, 1965)

The $K$-matrix formalism corresponding to the $T$-matrix formalism given by Faddeev is described for the three-body and bound-state scatterings.

i) $K$-matrix equations for three-body scattering

In this paper we introduce $K$ matrix for three-body scattering and develop the $K$-matrix formalism corresponding to Faddeev's $T$-matrix formalism. In a previous paper (which is referred to, hereinafter, as A), brief but extensive explanations have been given for Faddeev's $T$-matrix formalism. The $K$-matrix formalism will be described in parallel with it. The symbols in A are also used here without explanation.

We start from the Lippman-Schwinger equation for the three-body scattering matrix $T(z)$,

$$T(z) = V - V G_0(z) T(z),$$

where $V = \sum V_\alpha$, ($\alpha = 1, 2, 3$), $V_\alpha$ being the two-body potentials ($V_1 = V_{23}$, etc.). The three-body Green function $G_0(z)$ can be divided, as usual, into two parts;

$$G_0(z) = (H_0 - z - i\varepsilon)^{-1} = P \frac{1}{H_0 - z} + i\pi\delta(H_0 - z)$$

$\equiv G_0^p(z) + i\pi\delta(H_0 - z).$ (2)

Here we define the three-body $K$ matrix as the solution of the equation which is obtained from Eq. (1) by substituting $G_0^p(z)$ for $G_0(z)$, namely,

$$K(z) = V - V G_0^p(z) K(z).$$

In the same way as above, the matrices $K_\alpha(z)$ and $K^{(a)}(z)$ which correspond to the scattering matrices $T_\alpha(z)$ and $T^{(a)}(z)$ in A, respectively, are defined by the following equations:

$$K_\alpha(z) = V_\alpha - V_\alpha G_0^p(z) K_\alpha(z),$$

and

$$K(z) = \sum_\alpha K^{(a)}(z),$$

$$K^{(a)}(z) = V_\alpha - V_\alpha G_0^p(z) K(z).$$

(5)
Then, we can easily obtain the equations in the $K$-matrix formalism, which entirely correspond to the Faddeev equations, and have parallel forms with them. This is done by the similar method to that used by Faddeev in obtaining the Faddeev equations from Eq. (1). These equations are

$$K^{(a)}(z) = K_a(z) - K_a(z) G_0^p(z) \sum_{\gamma=1}^{\infty} K^{(\gamma)}(z), \quad (a, \gamma = 1, 2, 3).$$

(6)

If we introduce another matrix $K_{(a)}(z)$ corresponding to the scattering matrix $T_{(a)}(z)$ by

$$K(z) = \sum_a K_{(a)}(z),$$

$$K_{(a)}(z) = V_a - K(z) G_0^p(z) V_a,$$

(7)

then we have, in the same way, another equation instead of Eq. (6),

$$K_{(a)}(z) = K_a(z) - \sum_{\gamma=1}^{\infty} K_{(\gamma)}(z) G_0^p(z) K_a(z).$$

(8)

Let us proceed further by introducing a third matrix $K_{a\beta}(z)$, which corresponds to $T_{a\beta}(z)$ in A. The matrix $K_{a\beta}(z)$ is defined by

$$K_{a\beta}(z) = V_a \delta_{a\beta} - V_a G_0^p(z) K_{(\beta)}(z) = V_a \delta_{a\beta} - K^{(a)}(z) G_0^p(z) V_{\beta}.$$

(9)

If a function $G^p(z)$ is defined as

$$G^p(z) = G_0^p(z) - G_{a}^p(z) K(z) G_0^p(z),$$

(10)

the matrix $K_{a\beta}(z)$ can also be written in a symmetric form,

$$K_{a\beta}(z) = V_a \delta_{a\beta} - V_a G^p(z) V_{\beta}.$$ 

(11)

The equation for $K_{a\beta}(z)$ can be easily deduced by applying Eq. (6) (or (8)) to $K^{(a)}(z)$ (or $K_{(\beta)}(z)$) in Eq. (9). This is

$$K_{a\beta}(z) = K_a(z) \delta_{a\beta} - K_a(z) G_0^p(z) \sum_{\gamma=1}^{\infty} K_{\gamma\beta}(z) = K_{\beta}(z) \delta_{a\beta} - \sum_{\gamma=1}^{\infty} K_{a\beta}(z) G_0^p(z) K_{\beta}(z),$$

(12)

which corresponds to Eq. (13) for $T_{a\beta}(z)$ given in A.

Thus, all the equations in the $K$-matrix formalism, as we can see from the above, can be written in similar forms to those in the $T$-matrix formalism, where $G_0^p(z)$ is substituted for $G_0(z)$ in the latter.

We note a little that the relation similar to that in the $T$-matrix formalism holds owing to the definition of $G^p(z)$ by Eq. (10). That is

$$K(z) G_0^p(z) = VG^p(z).$$

(13)

Similarly, if we define $G_{a}^p(z)$ by

$$G_{a}^p(z) = G_0^p(z) - G_{a}^p(z) K_a(z) G_0^p(z),$$

(14)
the relation
\[ K_a(z) G_0^a = V_a G_0^a (z) \]  (15)
is also found.

**ii) Relations between \( K \) matrices and \( T \) matrices**

The matrices \( K(z), K_a(z), K^{(a)}(z), K_{(a)}(z) \) and \( K_{a\beta}(z) \) are related to the matrices \( T(z), T_a(z), T^{(a)}(z), T_{(a)}(z) \) and \( T_{a\beta}(z) \) as follows, respectively:

First, similarly to the case of two-body scattering,
\[
T(z) = K(z) - i\pi K(z) \delta(H_0 - z) T(z) = K(z) - i\pi T(z) \delta(H_0 - z) K(z). \]  (16)

Combining Eq. (16) with Eq. (5) and the similar equation for \( T^{(a)}(z) \), we have the following relation:
\[
T^{(a)}(z) = K^{(a)}(z) - i\pi K^{(a)}(z) \delta(H_0 - z) T(z) = K^{(a)}(z) - i\pi T^{(a)}(z) \delta(H_0 - z) K(z). \]  (17)

In the same way, we have
\[
T_{(a)}(z) = K_{(a)}(z) - i\pi T(z) \delta(H_0 - z) K_{(a)}(z) = K_{(a)}(z) - i\pi K(z) \delta(H_0 - z) T_{(a)}(z). \]  (18)

Further, applying Eq. (9) and the similar equation for \( T_{a\beta}(z) \) to the above-described relations, we can obtain
\[
T_{a\beta}(z) = K_{a\beta}(z) - i\pi \sum_{\gamma} \sum_{\overline{\gamma}} T_{\gamma\overline{\gamma}}(z) \delta(H_0 - z) K_{\beta\gamma}(z) = K_{a\beta}(z) - i\pi \sum_{\gamma} \sum_{\overline{\gamma}} K_{\gamma\overline{\gamma}}(z) \delta(H_0 - z) T_{\beta\gamma}(z). \]  (19)

**iii) Wave functions for bound-state scattering**

Lastly, we define the wave functions for bound-state scattering in the \( T \-) and \( K \)-matrix formalism, and deduce the equations satisfied by them. Though the wave function and its equation in the \( T \)-matrix formalism have been described by Faddeev, we will give in a somewhat different way a brief discussion which we need later.

First, we define the wave function \( \Phi_a \) and \( \psi_a \), which describe the free state and scattering state, respectively, with a particle \( a \) and a bound state of \( a \) channel. The function \( \Phi_a \) is the solution of the homogeneous equation;
\[
\Phi_a = -G_0(z) V_a \Phi_a, \]  (20)
and \( \psi_a \) is the solution of the equation;
\[
\psi_a = \Phi_a - G_a(z) \sum_{\gamma \neq a} V_{\gamma} \psi_{\gamma}, \]  (21)
where \( G_a(z) = (H_0 + V_a - z)^{-1} \). Then, it was shown by Faddeev that the equation for \( \phi_a \) can be reduced to the following form:

\[
\psi_a = \sum_b \psi_a^{(b)},
\]

\[
\psi_a^{(b)} = \Phi_a \delta_{ab} - G_a(z) T_{\beta}(z) \sum_{\gamma \neq \beta} \phi_a^{(\gamma)}.
\]  

The reduction from Eq. (21) to Eq. (22) is simply performed by the following way: By using the relation for the total Green function \( G(z) \),

\[
G(z) = G_0(z) - G_0(z) VG(z)
\]  

and

\[
T_{a\beta}(z) = V_a \delta_{a\beta} - V_a G(z) V_\beta,
\]  

Eq. (21) can be written as

\[
\psi_a = \Phi_a - G(z) \sum_{\gamma \neq a} V_\gamma \Phi_a
\]

\[
= \sum_b [\Phi_b \delta_{a\beta} - G_0(z) \sum_{\gamma \neq \beta} T_{\beta\gamma}(z) \Phi_a].
\]  

If we define the wave function \( \psi_a^{(b)} \) by

\[
\psi_a^{(b)} = \Phi_b \delta_{a\beta} - G_0(z) \sum_{\gamma \neq \beta} T_{\beta\gamma}(z) \Phi_a,
\]  

and use the equation in A,

\[
T_{\beta\gamma}(z) = T_\beta(z) \delta_{\beta\gamma} - T_\beta(z) G_0(z) \sum_{\delta \neq \beta} T_{\delta\gamma}(z),
\]  

then, we can easily obtain Eq. (22). Moreover, we will make a little mention of the bound state scattering amplitude, which has been described in reference 3) and in A. The amplitude for a bound state scattering

\[
M_{a\beta}(z) = (\Phi_a, \sum_{\gamma \neq a} V_\gamma \Phi_\beta)
\]  

is expressed by the scattering matrix \( T_{\gamma\beta}(z) \) as

\[
M_{a\beta}(z) = (\Phi_a, V_\beta (1 - \delta_{a\beta}) \Phi_\beta) + (\Phi_a, \sum_{\gamma \neq a} \sum_{\delta \neq \beta} T_{\gamma\delta}(z) \Phi_\beta).
\]  

This expression is easily obtained by using Eqs. (25), (23) and (24). (This is further rewritten by Eq. (20) as

\[
M_{a\beta}(z) = - (\Phi_a, V_a G_0(z) V_\beta \Phi_\beta) (1 - \delta_{a\beta})
\]

\[
+ (\Phi_a, V_a G_0(z) \sum_{\gamma \neq a} \sum_{\delta \neq \beta} T_{\gamma\delta}(z) G_0(z) V_\beta \Phi_\beta).
\]  

If we put, as given by Lovelace, \( \Phi_a V_a = \phi_a \) (the form factor \( \phi_a \) was defined in connection with the residue of the bound state pole in A), we have

\[
M_{a\beta}(z) = - \phi_a G_0(z) \phi_\beta (1 - \delta_{a\beta}) + \phi_a G_0(z) \sum_{\gamma \neq a} \sum_{\delta \neq \beta} T_{\gamma\delta}(z) G_0(z) \phi_\beta.
\]  

\[
M_{a\beta}(z) = - \phi_a G_0(z) \phi_\beta (1 - \delta_{a\beta}) + \phi_a G_0(z) \sum_{\gamma \neq a} \sum_{\delta \neq \beta} T_{\gamma\delta}(z) G_0(z) \phi_\beta.
\]
This is equivalent to the amplitude defined by the residue of poles in $A$.)

Now, we proceed to discuss the bound-state scattering in the $K$-matrix formalism. We introduce, as usual, a “standing wave” solution $\phi_\alpha^0$ which satisfies the equation

$$\phi_\alpha^0 = \Phi_\alpha - \sum_{\gamma \in \mathcal{A}} G^\gamma_\alpha (z) V_\gamma \phi_\gamma^0. \quad (32)$$

By the similar way to that given for $\phi_\alpha$ in the above, we can obtain the following relations for $\phi_\alpha^0$: Similarly to Eq. (26), the wave function $\phi_\alpha^{0(\beta)}$ is defined by

$$\phi_\alpha^{0(\beta)} = \Phi_\alpha - \delta_{\alpha\beta} G_\alpha^0 (z) \sum_{\gamma \in \mathcal{A}} K_{\beta\gamma} (z) \Phi_\gamma. \quad (33)$$

Applying Eq. (12) to Eq. (33), we obtain the equation

$$\phi_\alpha^{0(\beta)} = \Phi_\alpha - \delta_{\alpha\beta} G_\alpha^0 (z) K_{\beta\gamma} (z) \sum_{\gamma \in \mathcal{A}} \phi_\gamma^{0(\gamma)}, \quad (34)$$

which corresponds to the Faddeev equation (22). Furthermore, the element of the $K$ matrix for bound-state scattering is introduced by the definition

$$R_{\alpha\beta} (z) = (\Phi_\alpha, \sum_{\gamma \in \mathcal{A}} V_\gamma \phi_\gamma^0). \quad (35)$$

This is expressed by the matrix $K_{\gamma\beta} (z)$ as

$$R_{\alpha\beta} (z) = (\Phi_\alpha, V_\beta (1 - \delta_{\alpha\beta}) \Phi_\beta) + (\Phi_\alpha, \sum_{\gamma \in \mathcal{A}} \sum_{\delta \in \mathcal{B}} K_{\gamma\delta} (z) \Phi_\delta), \quad (36)$$

which entirely corresponds to Eq. (29).

This formulation will be extended to the composite-particle scattering (including resonance scattering) in the next work.

The authors would like to thank Dr. M. Monda and Dr. Y. Takahashi for their careful reading of the preprint. They also would like to thank Miss S. Ideura for kindly preparing this manuscript.

References

3) C. Lovelace, Phys. Rev. 135 (1964), B1225.