This book, Volume 31 of the familiar *Mathematics in Science and Engineering* series of Academic Press, is a collection of ten separate contributions to the field of optimization of dynamical systems. It follows by four years the publication in the series and by the same editor of a similar volume entitled *Optimization Techniques With Application to Aerospace Systems*.

The purpose of this book is to assemble under one cover contributions which, because of length and style of exposition, are not particularly appropriate for technical journals, and yet are very important to researchers and students in a field which is still evolving. Some of the chapters are devoted to particular problems in optimization arising from applications; others are essentially mathematical. However, even the theoretical chapters contain illustrative problems. The underlying principle in this book seems to be to bring to the attention of as wide an audience as possible some of the results and methods in optimization which have been obtained in the near past.

The material of this volume is divided into two parts, each containing five chapters; the first part is devoted to variational techniques and extensions and applications of the calculus of variations are presented there. Part 2 is primarily of a geometric nature, and the contributions there are to optimal control theory and some of its extensions and applications. Let us briefly describe these contents.

Chapter 1 and Chapter 2, written by B. Garfinkel, deal with inequalities and discontinuities in a variational problem. The inequalities studied naturally arise from problems with bounded state and control variable, which fit in the problem of Lagrange; the necessary and sufficient conditions for the calculus of variations as well as corner conditions and singularities are treated and numerical examples are carried out. The second chapter deals with the mathematical problem which commonly arises in the study of optimization problems with discontinuous control variables and which leads to the subject of discontinuous solutions in the calculus of variations; the chapter presents the necessary and sufficient conditions for an extremal, the corner conditions for the construction of a refracted continuation of an extremal, as well as its existence and uniqueness; again, numerical computations of examples are presented.

Chapter 3, entitled "Singular Externals," by H. J. Kelley, R. E. Kopp, and H. G. Meyer, recapitulates and relates some of the work by the authors on this difficult problem; the presentation is limited to the case where the control variable appears linearly. The examples presented, which include Goddard's Problem and Lawden's Spiral, are very interesting to those concerned with applications.

The last two chapters in Part 1 are in the nature of applications of the calculus of variations. Chapter 4, "Thrust Programming in a Central Gravitational Field," by A. I. Lurie, discusses the problem of prescribing the optimal thrust operation for applications arising in rocketry. K. A. Lurie, in Chapter 5, discusses "The Meyer-Belz Problem for Multiple Integrals: Some Optimum Problems for Elliptic Differential Equations arising in Magnetohydrodynamics."

Part 2 opens with a contribution by H. Halkin on "Mathematical Foundations of System Optimization" which is a simple, elegant, and eminently readable complete proof of the maximum principle which demands a minimum of mathematical background; two appendices give some of the limited background needed from the theory of ordinary differential equations and from the geometry of convex sets. This presentation of the maximum principle should be most welcome by both teachers and practitioners of optimal control theory.

In Chapter 7, A. Blaquiere and G. Leitmann present an investigation "On the Geometry of Optimal Processes," in which the geometry of the limiting surfaces in the state space of a dynamical system which behaves in an optimal fashion is described. As a consequence of the global and local properties of limiting surfaces the maximum principle is obtained and the relation between the maximum principle and dynamic programming is established from a geometric point of view. This chapter closes with some illustrative examples.

Chapter 8, by S. P. Diliberto, is entitled "The Pontryagin Maximum Principle." There, a reformulation of the proof of the maximum principle is presented and the methods used yield a proof of the bang-bang principle for nonlinear systems. In Chapter 9, entitled "Synthesis of Optimal Controls," B. Paiewonsky presents an exposition of Neustadt's synthesis method, describing in detail the time optimal control problem and touching on the extension to systems with effort constraints; computational considerations are discussed and two examples are given in some detail.

The final chapter, by E. K. Blum, is entitled "Optimal Problems for Elliptic Differential Equations arising in Optimal Control." The central idea is to show the relationship between the problem of Meyer and optimal control problems and how to attack these problems through an abstract Lagrange multiplier rule. A convergent gradient procedure for the numerical evaluation of optimal trajectories is obtained from the abstract multiplier rule.

As the above contents indicate, a considerable amount of material is presented in this volume. The presentations are rather clear, indeed the authors have obviously made a considerable effort to be as pedagogical as possible, given the highly mathematical nature of the subject. The many examples shown are well selected and meaningful. This is not a book for the uninitiated in this subject, but rather a reference work for rather new material whose importance is evident.

This volume is a very welcome addition to the body of literature on optimization theory and its applications.

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**REVIEWED BY BERNARDO RETCHKIMAN**

This book is written primarily for an introductory course in linear systems for advanced seniors or first-year graduate students. It claims to emphasize the fundamental principles, underlying different mathematical methods of analysis and synthesis, using the state-space approach. The prerequisites in mathematics are not clearly stated nor those of circuit analysis or conventional linear control systems; although whenever needed, the author refers to the appropriate bibliography.

The subject matter is generally well organized and presented. The first chapter presents a brief discussion of the state-space general equation, and flow graph representation, together with associated concepts and properties. At the end of this chapter as well as in all other remaining eight chapters that form the book, the author has included useful sections named "Supplementary Notes and References," where he refers the reader to