Smoothed particle hydrodynamics with radiation transfer

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ABSTRACT

A method for adding radiation transfer to smoothed particle hydrodynamics is described. One dynamic time-step of an SPH calculation is run in the normal way but with an additional time-step constraint that the change in the internal energy of any particle is limited to the greater of 10 per cent of its initial value or the equivalent of a 2 K temperature increase. The internal energy of the SPH particles is then transferred to the elements of a tree structure (‘leafs’) that become the radiating elements. The radiation is simulated by streams of superphotons, called ‘phots’ that are followed through the system until either they are absorbed by another leaf or they leave the system. When the radiation transfer is complete the internal energy of the leafs is transferred back to the SPH particles ready for the next dynamical step. The method has been validated by comparison with theoretical results in a number of situations. The method is particularly useful where the modelled system contains subsystems with a variety of opacity regimes that may also change with time.

Key words: radiative transfer – methods: N-body simulations.

1 INTRODUCTION

Smoothed particle hydrodynamics (SPH), first introduced by Gingold & Monaghan (1977) and Lucy (1977), has been very successfully applied to the investigation of a wide range of astronomical processes. It has been greatly improved since its first introduction – for example, the time-consuming gravitational force calculations were reduced from \( n^2 \) dependence to \( n \log n \) dependence by dividing space into an hierarchical tree structure (Barnes & Hut 1986). Other modifications, some designed to deal with special situations, have been implemented from time to time (e.g. Randles & Libersky 1996).

There are many astronomical processes in which the transfer of radiation plays an important role in determining the behaviour of the system. It is possible simply to deal with this problem in two extreme situations that depend on the relationship between the dynamical time-scale and that for the transfer of energy by radiation. If the system is highly opaque then rates of change of thermal energy due to radiation are much slower than those due to dynamic processes. In such a case radiation transport can be ignored. At the other extreme, in a transparent regime, radiation transport will be so rapid that a body of gas will always take on the temperature of its environment. When the dynamical and radiation-transfer time-scales are comparable then other schemes have been used, based on an expected relationship between temperature and density within various ranges of density (e.g. Turner et al. 1995; Bate 1998). The successful application of these approximations depends on an a priori knowledge of the general form of the behaviour of the system under investigation.

Unfortunately it not possible to deal with radiation transfer in the most general case by the simple inclusion of an extra equation into the usual set of SPH coupled differential equations. When the opacity of the system is not too small then the diffusion approximation to radiation transfer is valid and the form of energy transfer is represented by a conduction-like equation. This allows the addition of an extra equation into the SPH system and a way of doing this was first described by Lucy (1977). The method requires the estimation of double derivatives from the positions of the SPH particles and is therefore highly sensitive to particle disorder. An alternative and more stable way of including the equation, based on a Taylor series expansion, has been given by Brookshaw (1995).

We have investigated interactions between a star and a very diffuse protostar in which the protostar was disrupted to form smaller bodies that subsequently collapsed. The relationship between the dynamical and radiation time-scales was different in different parts of the system and changed significantly with time. In addition, all the bodies were being continuously irradiated by the star, the heating effect of which had a large influence on the behaviour of the system. For this reason we developed a general radiation transfer scheme that would apply to all optical depth and time-scale regimes. It should be said that the final scheme was the product of several unsuccessful ideas, the failures of which pointed us towards a practical approach.

In the description of radiation transfer we shall assume that the reader is familiar with the general concepts of SPH and the Barnes & Hut (1986) tree-structure approach so that detailed descriptions of basic principles are not necessary.

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2 GENERAL IDEAS

Every element in a radiating system can be thought of as a source of photons that stream out in all directions until they are absorbed. Where they are absorbed depends on the linear absorption coefficient (opacity \( \times \) density) of the material through which they move and probability laws. The numbers of photons is much too large to simulate individually but, just as in the computational simulation of plasmas where superparticles are used, here we can use superphotons, referred to as \( \text{photos} \), each of which represents a large number of photons. The basic idea of dividing the computational domain into cells that both emit and absorb photons has been used in different ways by Oxley (1999), Lucy (1999) and Bjorkman & Wood (2001). Lucy’s formalism is particularly suited to lower-opacity situations in that the total path lengths of all the photon packets that traverse a cell are summed to give a radiation density within the cell and hence a temperature. Since the emission of a cell depends on the temperature an iteration procedure is required to give temperatures and radiation densities of cells that correspond to self-consistency. The Bjorkman & Wood formalism involves the absorption of photon packets in cells which are then re-emitted from the cell with changed energy corresponding to the revised cell temperature. A particular packet is scattered, absorbed and re-emitted until it leaves the system. Neither the Lucy nor Bjorkman & Wood approaches are directly related to the study of dynamical systems using SPH. Another approach by Kessel-Deynet & Burkert (2000), which is integrated into SPH, deals with the heating due to the action of ionizing radiation and involves a ray tracing process to determine the ionization field within different regions.

The Oxley procedure has some affinity with that of Bjorkman & Wood and the version described here is a direct development from the Oxley model.

The absorbing and emitting elements that are used here are leafs,1 individual cubic regions as produced by the Barnes & Hut tree code. However for the purposes of radiation transfer the resolution does not have to be as great as is usually employed for the determination of gravitational forces for which the smallest occupied leafs contain just one SPH point. Trials have shown that, for radiation transfer, in any region it is sufficient for the side of a leaf to be between one-third and two-thirds of the local smoothing length. We define the ‘level’ of a leaf in terms of the number of bisections of the dimension of the computational domain that are required to produce it. Thus a level 1 leaf occupies one eighth of the domain and a level 2 leaf one sixty-fourth. If for gravitational-force calculations the leaf for a particular SPH point has level \( m \) then we might estimate from its smoothing length that for radiation transfer only level \( n(<m) \) is required. There are various ways of representing a leaf but one is by a string of integers, e.g. 1758 316. The first digit, 1, indicates that the leaf is within the first-level octant labelled ‘1’ that could be defined as being within the region defined by \( 0 < x < 1/4 D, 0 < y < 1/4 D, 0 < z < 1/4 D \), where \( D \) is the dimension of the cubic computational domain. The second digit, 7, then represents the octant that the leaf occupies within the first level octant, and so on. If, for radiation transfer, the required level is 5 then the above-defined dynamical leaf is contained within the radiation leaf 17583.

The whole of the cubic computational domain is divided into radiation leafs which can be singly or multiply occupied by SPH points or empty. Fig. 1 gives a two-dimensional representation of a domain divided into leafs for radiation transfer.

\[ \rho_j = C_m \sum_i m_i W(r_{ij}) \]

\[ u^i_j = C_u \sum_i m_i u_i W(r_{ij})/\rho_j, \]

where \( m_i \) is the mass of particle \( i, u_i \) its intrinsic internal energy, \( \rho_j \) and \( u^i_j \) are the density and intrinsic internal energy at the centre of leaf \( j \) and \( r_{ij} \) the distance between the SPH point \( i \) and leaf centre \( j \). The constants \( C_m \) and \( C_u \) have values that ensure that mass and internal energy are conserved in the process of distributing these properties from the points to the leafs. It is a measure of the effectiveness of the distribution process that these constants are normally in the range 0.99 to 1.01. The temperature of a leaf, \( T_j \), can be obtained from \( u^i_j \) and the combination of density and temperature gives an estimate of opacity, again assumed uniform throughout the leaf.

The introduction of radiation transfer is based on the assumption that a superposition principle is applicable to energy changes. This means that in a particular time interval energy changes in the system can be taken as the sum of the energy changes arising from dynamical processes, without radiation transfer, and the radiation-transfer changes, without dynamical processes. This assumption will be valid if the energy changes owing to each kind of process are small in the time interval. For this reason, apart from the usual constraints to do with precision, the SPH dynamical time-step is further constrained by the condition that within the time step the specific internal energy of any point should not change by more than 10 per cent, or the equivalent of 2 K, whichever is the larger.

Once a time-step is chosen an SPH step is implemented in the usual way. Then, starting with the thermal state at the end of the SPH step, the radiation transfer process is implemented over the same time interval.

\[ W(r, h) = \frac{1}{8\pi h^3} \begin{cases} 1 - \frac{1}{2} \left( \frac{r}{h} \right)^2 + \frac{1}{2} \left( \frac{r}{h} \right)^3, & 0 \leq \frac{r}{h} \leq 1, \\ \frac{1}{8} \left( 2 - \frac{3}{2} \right)^2, & 1 \leq \frac{r}{h} \leq 2, \\ 0, & \frac{r}{h} > 2, \end{cases} \]

where \( r \) is the distance from the SPH point and \( h \) its smoothing length. With this kernel the density and the intrinsic internal energy are found at each leaf centre by placing the spline kernel proposed by Monaghan & Lattanzio (1985):
time interval. Radiation depends on the fourth power of temperature and it has been found that, to ensure precision and stability in the calculation of radiation transfer, the dynamical time-step must be broken up into a number of sub-steps – typically about 100. These sub-steps are referred to as radiation time-steps.

The actual process of radiation by the elements is a continuous one but is treated computationally as though at the beginning of a radiation time-step all the energy for the interval is emitted as a burst of photons from each leaf. Formally all the photons should have been absorbed or have left the system before the next burst at the beginning of the next radiation time-step but in practice, for the applications made so far, this condition is easily satisfied; in transparent regions the photons travel at the speed of light and in opaque regions, where they travel slowly by a diffusion process, they are quickly absorbed.

Although it does not affect the radiation transfer algorithm, the description of which is the main purpose here, for completeness we briefly describe the equations of state and opacities we have used.

3 THE EQUATION OF STATE

Although, in principle, the method for radiation transfer to be described could apply over a wide range of situations and materials the only applications have been for a 70:30 hydrogen/helium mixture, of the kind expected in solar-nebula theory, up to a temperature of 2000 K. Moving outside this range of conditions may introduce problems that have not been considered. In earlier work some allowance was made for a solid component of the material but this made so little difference that it could reasonably be excluded.

The specific heat capacity as a function of temperature has been derived from a number of sources, weighted by their reliability as judged by Touloukian & Ho (1970). The essential requirement for the present purpose is to be able to relate both temperature and the mean molecular weight of the material to the specific internal energy, which is derived from the basic SPH equations. Full details of how this was done have been given by Oxley (1999) but the outcome is summarized in Fig. 2.

The dissociation of molecular hydrogen is pressure dependent and under the conditions of the calculations that have been done, where pressures are very low by laboratory standards, it is appropriate to assume that the experimental results at very low (tending to zero) pressure are applicable. Molecular hydrogen begins to dissociate at about 2000 K at which point an effective energy sink is available so that a considerable increase in specific internal energy gives no increase in temperature. Once all the molecular hydrogen has dissociated then once again temperature increases with increasing internal energy. The effect of the dissociation of hydrogen is also seen in the mean molecular weight that falls until the dissociation is complete, after which it remains constant. There are some approximations in these relationships but, in the context of this application, they are quite satisfactory.

4 OPACITY

At low temperature opacity is dominated by solid grains. Above about 150 K volatile ices have evaporated but residual refractory grains still provide the bulk of the opacity. Once grains have completely evaporated, at about 1500 K, hydrogen molecules and H$^-$ are the major contributors to opacity.

For temperatures up to 630 K opacity depends only on temperature and can be parametrized in the form

$$\kappa = aT^b$$

(Henning & Stognienko 1996) where the values of $a$ and $b$ are given in Table 1.

For higher temperatures, opacity is mainly due to the gas component and is dependent both on density and temperature. Most of the entries in Table 2, covering the temperature range 630–12 500 K,

![Figure 2](https://academic.oup.com/mnras/article-abstract/343/3/900/1123713/123713)
Table 2. Opacity table for the temperature range 630–12 500 K.

<table>
<thead>
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<th>( \log T )</th>
<th>-4.0</th>
<th>-3.5</th>
<th>-3.0</th>
<th>-2.5</th>
<th>-2.0</th>
<th>-1.5</th>
<th>-1.0</th>
<th>-0.5</th>
<th>0.0</th>
<th>log ( R )</th>
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<td>-1.479</td>
<td>-1.439</td>
<td>-1.342</td>
<td>-1.147</td>
<td>-0.840</td>
<td>-0.430</td>
<td>0.023</td>
<td>0.476</td>
<td>0.887</td>
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<tr>
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<td>-1.285</td>
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<td>-0.729</td>
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<td>-0.538</td>
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<td>-0.590</td>
<td>-0.590</td>
<td>-0.590</td>
</tr>
</tbody>
</table>

SPH with radiation transfer
have been given by Alexander & Ferguson (1994). To provide a table for better interpolation a variable \( R \) is introduced where

\[
R = \frac{\rho}{T_b^3} \tag{5}
\]

and \( T_b \) is the temperature in units of \(10^6\) K. Here, in contrast with some other workers, we use SI units so that density is in units kg m\(^{-3}\). The last three columns and the bottom row are an extension to the data given by Alexander & Ferguson. The column extension is to enhance the range of densities. For \( \log T < 3.5 \) it is reasonable to accept the values for \( R = 4.0 \) since the rate of change with \( R \) is so low. For \( T > 3.5 \) values are estimated from other sources of opacity data for similar mixes of materials. The row for \( \log T = 2.8 \) has been taken from (4).

The general behaviour pattern of systems is not usually greatly affected by modest changes of opacity although, of course, the detailed numerical results will change.

5 RADIATION BY THE LEAFS

The individual leafs that are radiating are cubical elements initially taken to have a uniform opacity, \( \kappa \), density \( \rho \) and temperature \( T \). If the side of the leaf cube is \( d \) then the luminosity of the leaf is

\[
L = 6d^2 \sigma T^4 [1 - \exp(-\tau)], \tag{6}
\]

where \( \sigma \) is Stefan's constant and \( \tau \) is the optical thickness of the leaf, \( \kappa \rho d \).

Within a period of time \( t \) each face of the leaf is taken as a source of \( n \) phots, each carrying energy \( Lt/\sigma n \). These phots emanate from the centre of the face in pseudo-random directions that satisfy rules imposed by the program user. For example, for most of the applications made so far each leaf has emitted 24 phots (\( n = 4 \)) in each radiation time-step. The rule we have imposed is such that for the face \(+x\), with an outward normal in the positive \( x \) direction, direction cosines, \((l, m, n)\) are chosen randomly but with the constraint that \( l \) is positive and the four phots have \((m, n)\) with sign combinations \((+, +), (+, -), (-, +), \text{and} (-, -)\). In addition we have imposed an additional constraint that the probability of a particular direction is proportional to \( l \), which leads to the statistical satisfaction of Lambert's cosine law that ensures that the face is perceived with the same brightness from any angle of view. For different numbers of phots per leaf other statistical patterns can be imposed on the directions of the phots. Detailed procedures for generating distributions satisfying particular conditions have been given by Woolfson & Pert (1999).

6 THE ABSORPTION OF PHOTS

When a phot leaves the face of one leaf it immediately enters another, unless the emitting leaf is at the boundary of the domain in which case the energy carried by the phot is lost from the system. We illustrate in Fig. 3, in two dimensions, the passage of a phot through a neighbouring leaf.

The optical thickness of the path within the leaf is \( \tau_s = \kappa \rho s \), where \( s \) is the total path within the leaf. The probability that the phot will be absorbed in the leaf is given by

\[
P_a = 1 - \exp(-\tau_s) \tag{7}
\]

and the decision of whether or not the phot has been absorbed is made by comparing \( P_a \) with \( r \), a uniform deviate in the range zero to one generated by a standard pseudo-random number generator. If \( P_a \) is greater than \( r \) then the phot is absorbed, otherwise not. If the phot is not absorbed then its path in the next leaf it traverses is calculated, starting from point B, and once again a decision is made as to whether or not it is absorbed. Eventually the phot is either absorbed in a leaf or it leaves the computational domain and represents energy radiated out of the system.

If the phot is absorbed then there are three possible ways of considering its absorption.

(i) The energy from the phot is uniformly spread over the whole leaf so that the absorbed energy is centred on the leaf centre.

(ii) The average position of absorption along the line AB is found and taken account of in some way in considering subsequent radiation.

(iii) A Monte Carlo selection of the absorption position is made using the known probability distribution along AB.

The simplest of the above choices is (i) but there is a strong objection to using it. In the semi-opaque and transparent regime the limiting rate at which radiation can move through the system during a time-step is the speed of light. For the simulations which have been done the radiation transfer time-steps are of order 0.1 yr, during which time light can travel about 6000 au. This is considerably larger than the largest bodies through which radiation passes in the simulation and is even larger than the size of the computational domain so we need not be concerned about the actual speed of passage. However, for opaque regimes the transmission of radiation is a diffusion process and the effective speed of passage through material is much slower. If we imagine a phot moving in the \(+x\) direction being absorbed in a very opaque leaf then the mean point of absorption will be close to the entry surface in the leaf. Assuming that the phot has been absorbed uniformly over the whole leaf is tantamount to speeding up its motion through the material.

A process that slows down the passage of phots in the opaque regime is now described using absorption assumption (ii) and where, for simplicity of explanation, it is assumed that the phots are moving in the \(+x\) direction. For a phot entering a leaf the probability that it will be absorbed at a distance between \( x \) and \( x + dx \) from the entry point is

\[
p(x) \, dx = \kappa \rho \exp(-\kappa \rho x) \, dx.
\]

If the cubical leaf has side \( d \) then the probability that it is absorbed somewhere in the leaf is, from (7),

\[
\int_0^d p(x) \, dx = P_{\text{leaf}} = 1 - \exp(-\kappa \rho d) = 1 - \exp(-\tau),
\]

the result given in (7).
For absorption method (ii) the mean distance, $\delta$, travelled by the photon from entering the leaf to absorption is given by

$$\delta = \frac{\int_0^d sp(x)dx}{\int_0^d p(x)dx},$$

which gives

$$\frac{\delta}{d} = 1 - \frac{\exp(-\tau)}{1 - \exp(-\tau)}$$

(8)

and the photon's energy can be considered as deposited at that point. For absorption method (iii) the deposition point is chosen by a Monte Carlo process according to the probability as a function of distance from the entry point. A random number generator is used to give a uniform deviate $r$ in the range 0 to 1. The distance from the entry point at which the photon is absorbed is then given by

$$s = \frac{1}{\kappa \rho} \ln \left( \frac{1}{1 - r} \right)$$

(9)

to give the required distribution. If $s > d$ then the photon is not absorbed in the leaf.

In practice the absorption method (ii) has been found to be simpler than (iii) and to give satisfactory results. The same process, as described above, is applied to a photon moving in a general direction; the length of its path through a leaf is calculated, the probability that it is absorbed is found from (7) and, if it is absorbed, then the point of absorption is found from (8). What is recorded for each absorbed photon is its leaf of origin, its energy per unit time and per unit temperature of the origin leaf ($s_{fr}$), the absorbing leaf and the position of absorption within the absorbing leaf.

7 NON-UNIFORM LEAVES

Immediately after an SPH dynamical step the leaves are assumed to be of uniform temperature so that all the photons emitted from it have equal energy. When the pattern of emitted and absorbed photons has been established then it is possible to find the loss and gain of internal energy from each leaf for the first radiation step. The energy of each photon is found by multiplying $s_{fr}$ by the time-step and $T_f^i$, where $T_f$ is the temperature of the emitting leaf. The loss of energy from each leaf is just the total energy of the photons that leave it during the time-step. The gain of energy of each leaf is just the sums of the energies of the absorbed photons but since the positions of absorption of the photons have been recorded it is also possible to work out the centre of energy in the leaf at the end of the time-step. If the leaf, with dimension $d$, density $\rho$ and centre at $(x_c, y_c, z_c)$, has initial intrinsic internal energy $\mu_l$ then the initial internal energy associated with the leaf is

$$iU_l = \mu_l \rho d^3.$$  

(10)

If $N$ photons are absorbed, the $ith$ of which has energy $e_i$ and absorption position $(x_i, y_i, z_i)$, and the $n$ photons that are emitted have total energy $E_i$ then the final internal energy of the leaf is

$$jU_f = iU_l - E_i + \sum_{i=1}^{N} e_i.$$  

(11a)

with the $x$ coordinate of the final centre of energy

$$x'_f = \frac{1}{jU_f}\left\{ (iU_l - E_i)x_c + \sum_{i=1}^{N} e_i x_i \right\}.$$  

(11b)

The $y$ and $z$ coordinates of the final centre of energy are found similarly.

Once the centre of energy is displaced from the centre of the leaf then it is clear that the radiation from the leaf in the next radiation time-step should be asymmetrical. The average temperature in the leaf, $T$, may be found from the average intrinsic internal energy

$$jU_f = \frac{iU_f}{\rho d^3}$$

as indicated in Fig. 2. However, we take the temperature of the radiating faces of the leaves as different to reflect the asymmetry in the distribution of internal energy.

We have already indicated that the simple assumption that the internal energy associated with absorbed photons is uniformly distributed over a leaf is unsatisfactory. In the method we have elected to use the mean positions of the absorbed photons in a leaf are found and lead to an asymmetric distribution of energy in the leaf. For estimating the emission from the leaf faces we now take the simplest possible assumption that gives asymmetry, that $\nabla u$ is constant over the leaf. In all practical cases the addition to the internal energy of leaves within a dynamic time-step is small so that $\nabla u$ is not only constant but also small so that the centre of energy of the leaf at the end of each dynamical time-step is displaced from the centre of the leaf by, at most, one tenth of a leaf edge and mostly much less than that. Another consequence of having a small $\nabla u$ is that the effective specific heat of the material will be virtually constant so that $\nabla T$ is small and constant over the leaf.

We now consider the emission from the $+x$ and $-x$ faces of a leaf on the basis that the variation of temperature is restricted to the $x$ direction; the result of doing the analysis with a general $\nabla T$ gives the same result. If the temperature at the centre of the leaf, taken as coordinate origin, is $T_0$, then the temperature at a point $(x, y, z)$ is

$$T_i = T_c + \frac{\Delta T}{dx} x.$$  

(13)

The centre of energy within the leaf will be at a distance from the leaf centre given by

$$\bar x = \frac{\int_{-d/2}^{d/2} T_i x \, dx}{\int_{-d/2}^{d/2} T_i \, dx} = \frac{d^2 \Delta T}{12 T_0} x,$$

(14)

where $d$ is the length of the leaf edge. Combining (13) and (14) enables the temperature at any point of the leaf to be expressed in terms of the central temperature and the position of the centre of energy as

$$T_i = T_c \left( 1 + \frac{12 \bar x}{d^2} \right).$$  

(15)

A thin slab of the leaf between $x$ and $x + d$ will emit energy towards the $+x$ face at a rate

$$\sigma T_i^4 \kappa \rho dx$$

per unit area where $\sigma$ is Stefan’s constant, and be absorbed while passing through a distance $d/2 - x$ before it crosses the face. Summing the contributions of slabs for the whole leaf the rate of emission of energy for the $+x$ face per unit area is

$$F_i = \sigma \kappa \rho T_i^4 \exp(-\kappa \rho d/2) \int_{-d/2}^{d/2} \left( 1 + \frac{12 \bar X}{d^2} \right)^4 \exp(\kappa \rho x) \, dx.$$  

(16)

Changing variables to $X = x/d$ and $\tau = \kappa \rho d$ gives

$$F_i = \sigma T_i^4 \tau \exp(-\tau/2) \int_{-1/2}^{1/2} (1 + 12 \bar X X)^4 \exp(X \tau) \, dX.$$  

(17)
The assumption of a uniform leaf gives an emission rate per unit area of
\[ G_\lambda = \sigma T^4_\lambda \{1 - \exp(-\tau)\} \] (18)
and the ratio \( Q(x, \tau) = F_\lambda / G_\lambda \) gives the factor of difference in assuming a non-uniform leaf. It is possible to evaluate \( Q(x, \tau) \) analytically but it is more efficient to tabulate it over the possible range of the variables. The partial results displayed in Table 3 illustrate the range of values of \( Q(X, \tau) \) that occur.

In most applications the optical thickness of the individual leafs is small as is the displacement of the centre of energy from the leaf centre so that the value of \( Q \) is usually between 1.00 and 1.05. A characteristic of the results that requires explanation is that the face remote from the displacement of the centre of energy has a greater emission than if the leaf was uniform. This is because the \( T^4 \) dependence on radiation leads to enhanced total radiation when the temperature is not uniform – that is illustrated by the relationship \( 3^4 + 1^4 \approx 2 \times 2^4 \). The increase in emission is counterbalanced by the greater extra path to the remote face but this gives an overall value of \( Q \) less than unity only for higher optical depths.

After each radiation time-step the new centre of energy can be found from (11b) where, now, \( x_\tau \) is taken as the previous centre of energy. It is implied that the energy lost by emission is centred on the previous centre of energy, which is consistent with the relative temperatures of the different faces. Although this algorithm deals satisfactorily with the emission and absorption from a single leaf it does introduce an inconsistency into the overall pattern since a face shared by two leafs is associated with a different temperature for each leaf. This turns out not to be a problem in practice since what is of interest is the final temperature distribution of the system as indicated by the specific internal energies of individual SPH points.

Table 3. A selection of tabulated values of \( Q(X, \tau) \).

<table>
<thead>
<tr>
<th>( X \times 10^{-3} )</th>
<th>( 2^{-7} )</th>
<th>( 2^{-5} )</th>
<th>( 2^{-3} )</th>
<th>( 2^{-1} )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.742</td>
<td>1.731</td>
<td>1.686</td>
<td>1.510</td>
<td>0.922</td>
</tr>
<tr>
<td>0.08</td>
<td>1.469</td>
<td>1.460</td>
<td>1.426</td>
<td>1.294</td>
<td>0.844</td>
</tr>
<tr>
<td>0.06</td>
<td>1.261</td>
<td>1.255</td>
<td>1.230</td>
<td>1.136</td>
<td>0.808</td>
</tr>
<tr>
<td>0.04</td>
<td>1.115</td>
<td>1.111</td>
<td>1.095</td>
<td>1.034</td>
<td>0.819</td>
</tr>
<tr>
<td>0.02</td>
<td>1.028</td>
<td>1.026</td>
<td>1.019</td>
<td>0.989</td>
<td>0.881</td>
</tr>
<tr>
<td>0.00</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.02</td>
<td>1.030</td>
<td>1.031</td>
<td>1.039</td>
<td>1.069</td>
<td>1.184</td>
</tr>
<tr>
<td>0.04</td>
<td>1.117</td>
<td>1.121</td>
<td>1.137</td>
<td>1.199</td>
<td>1.441</td>
</tr>
<tr>
<td>0.06</td>
<td>1.265</td>
<td>1.271</td>
<td>1.295</td>
<td>1.394</td>
<td>1.781</td>
</tr>
<tr>
<td>0.08</td>
<td>1.474</td>
<td>1.483</td>
<td>1.517</td>
<td>1.657</td>
<td>2.215</td>
</tr>
<tr>
<td>0.10</td>
<td>1.750</td>
<td>1.761</td>
<td>1.807</td>
<td>1.994</td>
<td>2.755</td>
</tr>
</tbody>
</table>

This effect is particularly important when the boundary leafs are opaque because all the radiation from the system is taking place from those leafs. The extent of the excess radiation from boundary leafs depends on the detailed way that their outer surfaces relate to the true boundary of the system. In tests of radiation transfer, described later, various factors were applied to the outward radiation from boundary leafs to see what effect they had in making numerical results consistent with those from theory. Different situations gave different optimum factors by which the radiation from boundary surfaces should be reduced. These varied between about 0.3 and 0.8. Since in a general application it is not possible to choose a best value for that particular application, a mean value of 0.5 for reducing the outward radiation from boundary leafs has been found to be a reasonable compromise. This is straightforward if the fact that a leaf is at a boundary can be established. For a system containing a single object there is no problem. If the photon leaves the computational domain having traversed a region of zero optical thickness then the emitting leaf was at a boundary. It is more complicated when the system contains many objects for then the photon may not reach the edge of the computational domain before it encounters another object. A rule-of-thumb which gives satisfactory results is to accept that the emitting leaf is at a boundary if the photon traverses two empty leafs after emission.

9 RETURNING ENERGY TO SPH POINTS

Once the radiation transfer is complete the internal energy from the leafs may be transferred back to the points, making the system ready for the next dynamical step. This is done by using the kernel (1) with smoothing lengths associated with the centres of energy of the leafs. The intrinsic internal energy of the \( i \)th SPH point is then found from
\[ u_i = \frac{1}{m_i} \sum_j C_j W(r_{ij}, h_j) u_j \rho_i d_j^3 \] (19)
where \( C_j \) is a normalization constant that gives conservation of energy by making the internal energy received by the SPH points equal to that contained in leaf \( j \). All the asymmetry in the energy content of the leafs is now implicit in the distribution of internal energy amongst the SPH points. When the next allocation is made to radiation leafs the internal energy of each leaf can be taken as uniform within it.
10 A SUMMARY OF THE RADIATION PROCESS

The process begins at the end of a dynamical time-step.

(1) Break up dynamical time-step into a number (~100) of radiation time-steps.

(2) Set up a system of leaves for radiation. The level of each leaf containing an SPH point is reduced so that its dimension is some fraction of the smoothing length of the point. The unique set of radiation leaves containing SPH points is found, some of which will be multiply occupied. Empty leaves are then found to complete the occupation of the computational domain.

(3) The internal energy of the SPH points is distributed to the radiation leaves.

(4) Photons are emitted from the radiation leaf surfaces taking account of the position of the centre of energy within the leaf. If the photons are not lost from the system then their locations in the absorbing leaves are found. If the surface is identified as a boundary then the photons emitted from it are reduced in energy by a factor of 2.

(5) The loss and gain of energy of each leaf are used to update its internal energy and to determine a new centre of energy.

(6) If radiation has taken place over the whole dynamical time-step then redistribute internal energy to the SPH points and move on to the next dynamical step. Otherwise return to step 4.

11 TESTING RADIATION TRANSFER

As has already been stated the radiation transfer mechanism was developed to deal with a particular astrophysical process and testing it has been confined to the conditions that would exist within that process. It might be possible to adapt it to situations involving stellar temperatures but that is not claimed for it on the basis of present experience.

There are five basic characteristics that have been tested:

(a) the luminosity of an isolated sphere which checks the emission properties of leaves,

(b) the total absorption of a uniform sphere illuminated by a point source, which checks the absorption properties of leaves,

(c) the distribution of temperature with time within a uniform infinite slab illuminated by a parallel beam of radiation falling normally on its surface,

(d) the distribution of temperature with time within a non-uniform layered infinite slab illuminated by a parallel beam of radiation falling normally on its surface, and

(e) the diffusion of radiation through an opaque infinite slab.

These five tests are now described.

11.1 Luminosity of an isolated sphere

This test is concerned with the rate at which energy is lost from the surface of a confined uniform-density, uniform-temperature sphere of gas. The analytical solution is known in this case as

\[ L = 4\pi \sigma R^2 T^4 \{1 - \exp(-\tau)\} \]  \hspace{1cm} (20)

where \( \sigma \) is the Stefan constant, \( R \) the radius of the sphere, \( T \) its temperature and \( \tau \) the optical depth across its diameter given by

\[ \tau = 2R_k\rho. \]  \hspace{1cm} (21)

The model radiator consists of 5946 particles arranged on a hexagonal-close-packed grid (HCP) that is a standard three-dimensional packing arrangement, a description of which will be found in textbooks of solid-state physics) fitted into a sphere of radius 1 au. The effective radius, \( R_e \), for the purpose of comparing calculated results with theory was taken as the furthest distance of a radiation leaf centre from the centre of the sphere and this turned out to be 1.072 au. The model is completely scalable in temperature but for the purpose of the calculations the temperature was taken as 100 K. The calculated luminosity was estimated from the energy associated with all the photons that escaped from the sphere in a single time-step divided by the time-step. The theoretical luminosity was calculated from (20) with the radius \( R_e \). The ratio of the luminosity from the model to that from theory is shown in Fig. 5 over a wide range of optical thicknesses of the sphere.

The calculated luminosity is about 16 per cent low at very small optical depths and about 5 per cent high at the other extreme of very high optical depths. There is a peak, rising to about 28 per cent higher than theory for optical depths between about 10 and 100. This peak also appears in the results given by Oxley (1999) with a very different process for moving photons. In judging the quality of the agreement between modelling and theory it should be borne in mind that, over the range of optical thickness considered, the absolute values of luminosity vary over four orders of magnitude.

11.2 Absorption of a uniform sphere

When the system contains a source of radiation, such as a star, then the other bodies of interest are constantly irradiated. The test designed here is to compare the radiation received from a point source by a uniform spherical body with a particular optical thickness to that derived from theory. If the spherical body has radius \( R \) and optical thickness \( \tau \) and has its centre at distance \( D \) from the point source such that \( K = D/R \) then, from theory, the radiation absorbed by the body per unit time is

\[ \frac{dE}{dt} = \frac{L}{2} \int_0^\infty \sin \theta [1 - \exp(-\tau \sqrt{1 - K^2 \sin^2 \theta})] \, d\theta, \]  \hspace{1cm} (22)

where \( L \) is the luminosity of the point source and \( \sin \theta_m = 1/K \) (Oxley 1999).

The model sphere was similar to that described for the luminosity test with respect to the number of points, their distribution and the definition of effective radius. There were 10^6 photons impinging on the sphere and absorption was found for \( K = 2 \) and \( K = 10 \) over a range of optical thickness from 0.000 01 to 10 000. The results of the test are shown in Fig. 6. For high optical thickness, as expected, all
the photons are absorbed and calculation and theory agree. For optical thickness between about 1 and 100 there is a small drop in the curve which then seems to pick up again at lower optical thickness. For the lowest range of optical thickness there is considerable fluctuation owing to the fact that few photons are being absorbed. Nevertheless there seems to be a trend for calculation to be below theory for very small optical thickness that may be a result of some effect from the inevitably poorly-defined boundary.

Figure 6. Absorption from a point source for $K = 2$ and $K = 10$.

Figure 7. The temperature distribution in the sphere after ten years.
11.3 Temperature distributions within illuminated bodies

The first test consisted of irradiating a spherical body by a point source and noting the distribution of temperature within it as a function of time. The body is uniform, is represented by 5946 points and has nominal radius 1 au. As previously noted, because of kernel boundary spread its effective radius was 1.072 au. Its optical thickness is 40 and its centre is at distance 2 au from a point source of luminosity $4 \times 10^{26}$ W. Initially the body is at a uniform temperature of 30 K. Fig. 7 shows the distribution of temperature within a central slice of the body after 10 yr where each SPH point is represented by a circle shaded to represent the temperature and the circles are large enough to give continuous coverage of the area. The appearance of the temperature contours more-or-less agrees with what intuition would suggest – although there are clear small-scale anomalies reflecting the Monte Carlo nature of the process. When the model is run for a longer time an equilibrium temperature profile is reached after about 60 yr. At this stage the average temperature of the ten points closest to the source was 319 K and for the ten points furthest away it was 49 K. The theoretical equilibrium temperature of a body at a distance of 1 au from the source is 281 K and this should apply to the parts of the body closest to the source. However, because of kernel boundary spreading the absorbing surfaces of the closest leaves receiving radiation were somewhat closer to the source.

Figure 8. A projected view of the SPH points and leaves for an infinite slab. Because of the periodic boundary conditions the photon moving along AB reappears moving along B'C.

Figure 9. A comparison of temperature profiles within an illuminated uniform slab for various times as found by the SPH program and from a precise finite-difference calculation. The optical thickness in each case is (a) 0.25, (b) 1.0, (c) 4.0 and (d) 20.0.
It is not easy to compare numerically the result of the calculation for a sphere with a theoretical result. For this reason a test was devised in which an infinite uniform slab of material was irradiated normally on one face. The actual simulated region consisted of a block of length $d$ in the $x$ direction, corresponding to the thickness of the slab, and of length $\frac{d}{2}$ in the $y$ and $z$ directions. The infinite area of the slab is then introduced by applying periodic boundary conditions in the $y$ and $z$ directions; this is a well-known device used in many areas of computational modelling. The $x$–$y$ projection of the block is shown in Fig. 8 together with SPH points within it, this time defined on a cubic lattice. There is one SPH point per leaf and the leaf outer surfaces in the $y$ and $z$ directions containing boundary SPH points coincide with the boundaries of the block. If a photon leaves a leaf containing a $y$ or $z$ boundary point (along path AB) then the periodic boundary conditions bring it back through the opposite boundary (along path B'C).

In the computational model we have used there were 32 points in the $x$ direction, defining the thickness of the slab, $10^6$ m, and 8 points in the $y$ and $z$ directions that, through the periodic structure, produced the infinite slab. The illuminating radiation was in the form of a parallel beam of intensity $1500$ W m$^{-2}$ falling normally on the left-hand face. The density of the material was $1000$ kg m$^{-3}$ and it had a constant specific heat corresponding to a gas with mean molecular mass $4 \times 10^{-27}$ kg.

The SPH results were compared with those from a finite-difference calculation that divided the thickness of the slab into $10,000$ equal slices. The radiation was absorbed by the slab in the usual declining exponential way as was the radiation from each strip by its neighbours in both directions. This was a very precise calculation that could be regarded as a ‘theoretical’ result against which to assess the SPH results.

Because the leaf surfaces coincide with slab surfaces in this application the radiation from surface leaves was not reduced by a factor of 2, as described in Section 9. The results for values of optical thickness -- 0.25, 1.0, 4.0 and 20.0 -- are shown in Fig. 9. Comparisons are made at different times between the theoretical result, shown as a line, and the SPH results, shown as points with a range shown as a vertical bar. The SPH temperatures are derived as averages of the temperatures of SPH points having the same $x$ values and the bars show the standard deviations of the 64 values contributing to the average. The best agreement seems to be for the slabs with optical thickness 1.0 and 4.0 although there is a tendency for the higher temperatures to dip at the receiving end of the slab. This is particularly marked for the slab of optical thickness 20.0 where the increase in temperature is about 20 per cent lower than theory.

The total absorption from the SPH calculation also agreed well with theory. In the form (optical depth, SPH absorption, theoretical absorption) these were (0.25, 0.2205, 0.2212):(1.0, 0.6280, 0.6321):(4.0, 0.9760, 0.9817):(20.0, 1.0000, 1.0000).

### 11.4 Temperature distribution within layered slabs

A characteristic of the SPH process is that it does not handle abrupt changes of physical conditions too well. Thus a body with a sharp
boundary within surroundings of lower density will be represented as though its density continuously changed in some region spanning the actual boundary. This spread of physical conditions is due to the use of a kernel to represent continuously varying properties by the properties of a discrete number of points. Clearly the same behaviour will be true for radiation transfer if there are abrupt changes of opacity, or linear absorption coefficient ($\kappa \rho$), within the system. To illustrate this we again use an infinite slab model, with a slab of thickness $10^5$ m, this time layered so that the absorption abruptly changes at various depths. To avoid the confusion of varying too many properties, which gives the problem of disentangling different influences, the density of the slab is taken as constant, 1000 kg m$^{-3}$, but its opacity has a relative variation as follows:

- between depths 0 and $2.5 \times 10^5$ m 1 unit,
- between depths $2.5 \times 10^5$ m and $7.5 \times 10^5$ m 4 units,
- between depths $7.5 \times 10^5$ m and $8.75 \times 10^5$ m 2 units,
- between $8.75 \times 10^5$ m and $10^6$ m 5 units.
The unit was adjusted to give a designated overall optical thickness for the slab. For leaves adjacent to slab surfaces the opacity was taken as that of the neighbouring surface region. Fig. 10 shows a comparison of the SPH results with finite-difference results for optical thickness 0.25, 1.0 and 4.0.

The SPH temperature profiles follow the expected pattern quite well. The total absorption is similar to that from the uniform slab and agrees well with theory. There is some blurring where the opacity is discontinuous but this is no different from the blurring found for sharp changes of density. A necessary condition to obtain a reasonable result is that the leaves must be small enough to give the required resolution. However, this is a general requirement for simulating physical systems and radiation transfer is unlikely to give any extra constraints.

11.5 Rate of radiation transport in the opaque region

The slow penetration of energy into the slab for optical thickness 20.0, where the diffusion approximation would be valid, is evident from Fig. 9(d). The time for photons to move by diffusion a distance x is given by

\[ t_D = \frac{\kappa \rho x^2}{c} \]  \quad (23)

For the slab the value of \( \kappa \rho \) can be found from the physical thickness and optical thickness of the slab from

\[ \tau = \kappa \rho d \]. \quad (24)

Actually, (23) is only applicable where the radiation density in the body is much greater than the thermal energy per unit volume, which is not true in this case. The radiation density is given by

\[ U_{\text{rad}} = \frac{4\sigma T^4}{c} \]  \quad (25)

and the thermal energy density by

\[ U_{\text{therm}} = \frac{3k \rho T}{2\mu} \]. \quad (26)

When \( U_{\text{therm}} \gg U_{\text{rad}} \), which is true here, then the time in (23) must be multiplied by the ratio \( U_{\text{therm}}/U_{\text{rad}} \). Putting this factor into (23) and using (24) the revised time is \( t_D' \) and

\[ t_D' = \frac{3k \rho \tau}{8\sigma \mu T^3} \]. \quad (27)

It is not absolutely clear how to test (27) rigorously from the results given in Fig. 9(d). One approach is to note that at time 2000 yr the average value of \( T^3 \) is about 220 K and that in the next 2000 yr the distance traversed by the energy is of order \( 4 \times 10^4 \) m. With \( x = 4 \times 10^4 \) m and \( T = 220 \) K, and other values as previously given, the left-hand side of (27) has value 25.3 s m\(^{-2}\) and the right-hand side has value 42.9. While the agreement is not precise, because the model does not correspond to the conditions assumed by the equation apart from other reasons, it is good enough to confirm that the general form of behaviour for large opacities is followed.

12 CONCLUSION

The description of radiation transfer by the passage of photons, taking into account the expected locations of where they are absorbed, is close to the reality of the actual physical processes that occur. Four of the five validation tests have indicated quite good agreement between calculation and theory. The fifth – that concerned with the diffusion approximation for high opacities – gives a reasonable result that, at the very least, indicates a correct behaviour pattern.

By its very nature it is unlikely that in a general application it will give results with respect to radiation transfer that are correct at the few percent level. However, it will usually be far better than the various approximations that have usually been used. The process gives the correct general behaviour for both the transparent and opaque regimes and therefore may be used with confidence even when opacity regimes vary from place to place and from time to time in the model under investigation. There is a price to pay for this improvement. The time for a normal SPH calculation is increased by anywhere between one and two orders of magnitude, the time being less for very opaque systems where the photons are quickly absorbed and do not have to be followed for long distances.

The development of this radiation transfer process has been completely directed towards the needs of one particular model. Rather than regarding this specific formulation as a universal recipe for radiation transfer in general, it would be better to think of it as providing a set of principles that can be adapted to the special needs of the problem in hand.

The presentation of this work has been much improved by the comments and suggestions made by a referee of the initial version and we have pleasure in acknowledging this contribution.

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