Substituting equations (63), (64), and (66) into equation (62) gives
\[ \frac{4e^{2}}{M^{2}} \left( \frac{2k}{M^{3}} + \frac{q^{2}}{M^{3}} \right) > \frac{4e^{2}}{M^{4}} + \frac{4e}{M^{4}} \left( \frac{k^{2}}{M^{4}} + \frac{q^{2}}{M^{4}} \right) \]
\[ c > \frac{k}{M} \text{ or } c > q \left( \frac{M}{k} \right)^{1/4} \]  
(67)

Combining equations (52) and (67) show that for stability
\[ c > \frac{63,000 \text{ (hp)}}{D_{p} H \text{ (rpm)}} \left( \frac{M}{k} \right)^{1/4} \]  
(68)

The Q-factor of the system, which is a convenient measure of the freedom from damping, is given by
\[ Q = \frac{f_{0 \text{ (resonance)}}}{f_{0 \text{ (static)}}} = \frac{p_{e}}{p_{e}} = \frac{k}{c} \]  
(69)

and
\[ \omega_{n} = \left( \frac{k}{M} \right)^{1/4} \]

Therefore
\[ Q = \frac{k}{c} \left( \frac{M}{k} \right)^{1/4} = \left( \frac{kM}{c} \right)^{1/4} \]  
(70)

Combining equations (68) and (70) gives
\[ \left( \frac{kM}{c} \right)^{1/4} > \frac{63,000 \text{ (hp)}}{D_{p} H \text{ (rpm)}} \left( \frac{M}{k} \right)^{1/4} \]

or
\[ k > \frac{63,000 \text{ (hp)}}{D_{p} H \text{ (rpm)}} \]  
(71)

**DISCUSSION**

W. K. Bodger

The author is to be commended on introducing a new quantum of understanding and a new field of investigation into the dynamics of turbomachinery. The results of this pioneering effort offer a new set of clues to the unraveling of some very frustrating mysteries in overhung rotors.

In the Appendix, assumption 4 seems questionable. An alternate assumption would be that circumferential flow does occur and that this flow is resisted by friction, as is normal in closed ducts. This will introduce an additional relation:

\[ \omega_{s} = \frac{V}{r} \left( \frac{4}{\pi} \frac{D_{h}}{D_{s}^{2}} \frac{dp}{d\theta} \right)^{1/4} \]

where \( \omega_{s} \) is the one-side circumferential flow, \( D_{h} \) is the hydraulic diameter in the annular space between the seal teeth, and \( f \) is a friction factor. Due to the nonlinearity of the \( \omega_{s} \) relation, this complicates the solution for \( p \); however, the result should be more realistic than that given by equations (18), (19), and (25). Other extensions of the labyrinth seal theory would be the case of more than two seal teeth and/or the case of subcritical flow past the nozzle teeth. It may be that the subcritical flow case is covered by appropriate values of \( N_{1} \) and \( N_{2} \). However, the author gave no real clue as to the meaning of these coefficients.

F. F. Ehrlich

The author is to be congratulated for the identification focus of attention on two dynamic aeroelastic phenomena that should be of great interest to those concerned with the vibration of high-speed turbomachinery rotors.

But the discusser would like to indicate certain possible discrepancies in the detailed analysis of the phenomena:

1. For the case of dynamic eccentricity of rotor inducing a whirling torque, the criterion of stability is, as derived from the Alford formulation (equation (68)), a requirement that damping be sufficient to overcome destabilizing torque

\[ c > \frac{\text{(const.)} \rho \beta}{\omega_{n} D_{h} H} \]

Some benefit is ascribable to system stiffening if it results in an increase in the natural (whirling) frequency of the system \( \omega_{n} \). The increase of whirling frequency results in an increase in energy dissipation (for constant damping coefficient) which, in turn, implies greater stability. But, this effect is much smaller and of less significance (half order) than the first-order effect of stiffness implied by equation (71) and the related parameter listed in Table 4 and calculated in Table 3. The inaccuracy is introduced by ignoring the magnification factor \( Q \) in equation (71). In the case of linear damping, \( Q \) is proportional to the square root of rotor stiffness, and its variance cannot be neglected in assessing stiffness effects on the stability parameter. In fact, for the alternative assumption that system damping coefficient is inversely proportional to vibration frequency (as is often assumed in structural damping problem), the stability of the system appears to be independent of rotor stiffness.

2. For the case of aeromechanical forces associated with labyrinth seals, it appears that the analysis of the Appendix is not consistent with the conclusions stated in the text. The possible negative value of the coefficient \( A_{1} \) in equation (25) implies a possible negative real root (by Descartes rule). This implies an instability trend, not toward whirling motion, but toward a “softening” of the rotor stiffness and reduction of natural frequency toward zero. It would appear from examination of the coefficient \( A_{1} \) that the natural damping is actually enhanced by the aeroelastic forces, and one might expect the aeromechanical forces to result in a reduction in any whirl tendency.

3. For the case of aeromechanical forces associated with labyrinth seals, it would appear that the circumferential passage between seal teeth may form a first-order mechanism for inhibiting any significant circumferential pressure nonuniformities hypothesized by Alford. One may estimate the importance of the effect by comparing the area available for inflow of unbalancing flow (say, over 90 deg of the circumference) to the area available for circumferential flows to balance out pressure nonuniformities:

\[ \epsilon = \frac{(\pi/4)D_{1}B_{1}}{h_{L}} \]

Using Alford’s data from Table 2 indicates that this parameter is probably much less than unity, which indicates that circumferential flow area is, indeed, large enough to suppress circumferential pressure nonuniformities.

4. In the discusser’s experience, most experimentally observed instances of whirling motion synchronous with rpm have their onset at an rpm approximately twice the frequency of the induced whirling motion. The hypothesized mechanisms do not contain any special rationale or accommodation to this fact. Has the author any insight on this observation as related to the hypothesized aeroelastic stability criteria proposed?

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Footnotes:
E. W. Snow

The two different forms of self-excited whirl discussed in the paper and which have been the cause of serious trouble on certain designs of aircraft gas turbines built by the General Electric Co. will serve as a valuable warning to other engine designers of the potential dangers which can arise when attempts are made to build very lightweight engines characterized by having low natural fundamental whirling speeds—for example, by the omission of a center bearing. While such designs may be satisfactory as regards normal synchronous whirl excited by out-of-balance, they may run into difficulty from one or another or all of the various types of self-excited vibration which are possible. As far as Rolls-Royce engines are concerned, there is no record of either of the two forms discussed in the paper having been the cause of any serious trouble. Applying the torque deflection criterion to a 17-stage compressor of a large single-shaft engine gives a figure of 1.37, disregarding both stiffening from the disk spacers on the one hand and bearing-housing flexibility on the other. Roughness troubles with this engine have been attributed to disturbance of balance due to the built-up construction. As might be expected, the value of the criterion applied to the housepower compressor having only 12 stages of a modern two-shaft engine is very low—0.24.

The type of stationary damping referred to in the final paragraphs of the paper has been applied successfully to the L. P. turbine bearing of one design of turboprop engine. Extensive rig tests were carried out to establish the optimum combination of materials to give adequate wear rates for long life reliability. In more recent designs, effective amplitude reduction has been obtained by interposing a thin film of oil between the bearing and its housing, although in this case the mechanism is not simply a damping one.

Den Hartog (4th edition p. 319-320) mentions that self-excited whirl due to pressure changes associated with varying clearances in labyrinth seals occurred in the contrarotating Ljungstrom designs of radial-flow steam turbines.

Author's Closure

Several of the discussions point out that the circumferential flow in the narrow annular passage between seal teeth tends to significantly reduce the circumferential variation of static pressure within this passage. When a steady pressure is applied to the mass of air in the circumferential passage between seal teeth, tending to move it along in a circumferential direction, there exists a definite relation between the applied pressure and the flow produced. As pointed out by Mr. Bodger, this resistance is due to viscosity, expressed by the familiar friction factor. However, when the applied pressure is alternating, viscosity is not the only force opposing the movement of the gas; there is also a factor dependent on the frequency. The ratio between the applied pressure and the velocity produced by it is known as the acoustic impedance. For high frequencies, the acoustic impedance is the primary factor in controlling circumferential flow in the narrow annular passage between seal teeth. In fact, some of the fatigue failures of labyrinth seals, discussed in reference [1], show evidence of acoustic oscillations in the narrow annular passages. The most prominent pressure oscillations appear to propagate in the forward direction, i.e., the same direction as rotor rotation.

The mean air velocity in the narrow annular passage between seal teeth is typically from $1/4$ to $1/2$ the corresponding tangential speed of rotating seal component. This mean circumferential velocity of the air affects the acoustic oscillations of air in the narrow annular passage between seal teeth. The effect is discussed in a paper by Whitehead.

The wavelength $\lambda$ in the circumferential direction is $\frac{\pi D_a}{n}$ where $n$ is the number of circumferential waves. An approximate equation of the rotating pressure pattern is

$$p(\theta_1, t) = A_s \cos[n(\theta - \Omega t + \phi_1)]$$

$\Omega$ is the angular velocity of the rotating pressure pattern and is typically twice or more the angular speed of rotor.

Dr. F. F. Ehrich's contribution mentioned "a possible negative value of the coefficient $A_t$ in equation (35)." So far as is known, this particular coefficient is always positive.

In reply to paragraph 4 of Dr. Ehrich's contribution, one engine exhibited whirl at ratios of whirl speed to actual rotor rpm, varying from

$$\begin{align*}
51.8 & = 0.405 \\
128 & = 0.508
\end{align*}$$

It is true that the most severe whirls occurred at the lower ratio for the specific engine studied. The data available to the author suggest that the speed ratio may be related to the rotating pressure patterns in the narrow annular passages between seal teeth, particularly for $n = 1$.

Mr. Snow's discussion of the use of damping applied to bearing supports is very interesting and suggests that this damping makes it practicable to deliberately introduce radial flexibility in bearing supports so as to locate the engine system natural frequencies in advantageous speed ranges. Of particular interest is the generous disclosure that effective amplitude reduction has been obtained by interposing a thin film of oil between the bearing and its housing. In the writer's opinion, provision for adequate damping in the bearing housing will make a significant contribution toward protecting future turbomachinery from both synchronous and asynchronous vibration.

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