Understanding helical magnetic dynamo spectra with a non-linear four-scale theory

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ABSTRACT
Recent magnetohydrodynamic (MHD) dynamo simulations for magnetic Prandtl number \(P > 1\) demonstrate that when MHD turbulence is forced with sufficient kinetic helicity, the saturated magnetic energy spectrum evolves from having a single peak below the forcing scale to become doubly peaked with one peak at the system (= largest) scale and one at the forcing scale. The system scale field growth is well modelled by a recent non-linear two-scale helical dynamo theory in which the system and forcing scales carry magnetic helicity of opposite sign. However, a two-scale theory cannot model the shift of the small-scale peak toward the forcing scale. Here I develop a four-scale helical dynamo theory which shows that the small-scale helical magnetic energy first saturates at very small scales, but then successively saturates at larger values at larger scales, eventually becoming dominated by the forcing scale. The transfer of the small-scale peak to the forcing scale is completed by the end of the kinematic growth regime of the large-scale field, and does not depend on magnetic Reynolds number \(R_M\). The four- and two-scale theories subsequently evolve almost identically, and both show significant field growth on the system and forcing scales that is independent of \(R_M\). Implications for fractionally helical turbulence are discussed.

Key words: MHD – turbulence – methods: numerical – stars: magnetic fields – ISM: magnetic fields – galaxies: magnetic fields.

1 INTRODUCTION
Most all-ionized astrophysical rotators are turbulent and carry magnetic fields. For astrophysical objects such as the Sun, helical dynamo theory was originally proposed to model the large-scale field generation on scales larger than that of the driving turbulence (e.g. Parker 1955; Steenbeck, Krause & Rädler 1966; Moffatt 1978; Parker 1979). Also of interest is to understand the overall shape and amplitude of the magnetic energy spectrum from helical or non-helical dynamo action on both large and small scales (e.g. Pouquet, Frisch & Leorat 1976; Kulsrud & Anderson 1983; Brandenburg et al. 1995; Hawley, Gammie & Balbus 1995a,b; Brandenburg 2001; Schekochihin et al. 2002a,b; Maron & Cowley 2002; Maron & Blackman 2002). When the effect of the growing magnetic field on the turbulent velocity is ignored, the dynamo model is kinematic. We know that kinematic theory is incomplete since the dynamo must represent a solution of the full non-linear magnetohydrodynamic (MHD) equations. In this context, much of theoretical dynamo research can be divided into two categories: (i) that which focuses on the fundamental principles, with the goal of determining a correct and useful non-linear theory, and (ii) that which parametrizes the backreaction of the growing magnetic field on the turbulent velocities and uses it in the dynamo equations to model the fields of specific objects in detail. This paper falls into the former category.

Turbulence that produces astrophysical dynamos can arise either from external forcing or be self-driven from shear. Examples of the former are supernova driving in the Galaxy (e.g. Ruzmaikin, Shukurov & Sokoloff 1988) or thermal convection driving in the Sun (e.g. Parker 1979), and an example of the latter is magneto-rotational instability (MRI) (Balbus & Hawley 1991, 1998). In either forced or self-driven dynamos, there exist both helical or non-helical varieties. Non-helical versus helical refers to the situation in which turbulence has either zero versus finite pseudoscalar quantities such as the kinetic helicity \(\mathbf{v} \cdot \nabla \times \mathbf{v}\), where \(\mathbf{v}\) is the turbulent velocity, and the overbar represents some volume average (within a given hemisphere for a rotator). Such a correlation can be derived from the statistical tendency for rising (falling) eddies to twist in the opposite (same) sense as the underlying rotation as they expand (contract) and conserve angular momentum in a stratified rotator. When kinetic helicity is present, the turbulence can generate a global large-scale field (e.g. Moffatt 1978; Parker 1979) the associated sign of flux of

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which lasts many overturn times for the largest turbulent eddy. As discussed in Blackman (2003a) and Blackman & Tan (2003), such a global field is not to be confused with fields produced in non-helical simulations for which the largest turbulent scale is the box scale. In this latter case, magnetic energy will arise on the scale of the box, but the sign of its flux will change every overturn time-scale. For example, non-stratiﬁed MRI simulations (Hawley et al. 1995a) for which no kinetic helicity is present, produce magnetic energy with maximum azimuthal scale of the order of the box height, but the turbulence also extends to this scale, and the field does not maintain a coherent flux longer than a large eddy turnover time (~ an orbit time). In contrast, helical MRI simulations in a non-periodic box (Brandenburg et al. 1995) were shown to incur a helical dynamo that sustains a large-scale ﬂux over many orbit times. This latter type of global ﬁeld is extremely helpful in producing jets and coronae, since it can survive the buoyant rise without shredding on its way up (Blackman 2003b). (Note that Stone et al. 1996 claimed that no helicity was observed in stratified periodic box disc simulations. However, they averaged over their entire box to test for helicity, and since the northern and southern hemispheres must produce opposite signs, the box-averaged helicity would be expected to vanish.)

As the amplification of magnetic ﬁelds and the associated MHD turbulence represent highly non-linear problems, understanding the backreaction of the magnetic ﬁeld on the ﬁeld growth itself has been an evolving topic of study (e.g. Piddington 1981; Kleeorin & Ruzmaikin 1982; Kulsrud & Anderson 1992; Vainshtein & Cattaneo 1992; Cattaneo & Hughes 1996; Field, Blackman & Chou 1999; Blackman & Field 2000; Brandenburg 2001; Field & Blackman 2002; Blackman 2003a). Some recent developments (Brandenburg 2001; Blackman & Brandenburg 2002; Blackman & Field 2002; Brandenburg, Dobler & Subramanian 2002; Field & Blackman 2002; Maron & Blackman 2002) have emerged from focusing numerically and analytically on the simplest dynamos for which the underlying principles can be identiﬁed. Experiments have been performed in which a turbulent velocity spectrum is ﬁrst established in the absence of magnetic ﬁelds in a periodic box and then a weak seed spectrum of magnetic energy is input (e.g. Meneguzzi, Frisch & Pouquet 1981; Kida, Yanase & Mizushima 1991; Brandenburg 2001; Maron & Cowley 2002; Maron & Blackman 2002). The non-linear evolution the magnetic energy spectrum to saturation can then be studied as the system is driven. (Note that these simulations are distinct from those in which an initial uniform background ﬁeld is also assumed e.g. Maron & Goldreich 2001; Cho, Larfusan & Vishniac 2002)

The shape of the resulting spectrum depends on the fractional helicity. Deﬁne the fractional kinetic helicity \( f_h \equiv |(v_i \cdot \nabla) v_i|/v^2 \), where \( v_i \) is the turbulent velocity on the forcing scale and \( k_i \) is the forcing wavenumber. When forced such that \( f_h = 0 \), the magnetic spectrum saturates with the magnetic energy seemingly piling up close to the resistive scale for magnetic Prandtl number (the ratio of viscosity to magnetic diffusivity \( Pr_m \equiv v/\lambda \) > 1, not near the forcing scale (Kida et al. 1991; Maron & Cowley 2002; Schekochihin et al. 2002a,b). This contradicts, for example, observations of the Galactic magnetic ﬁeld, which does not seem to have a peak on the resistive scale (Beck et al. 1996). On the other hand, recent simulations of Haugen, Brandenburg & Dobler (2003) suggest that the peak may not be at the resistive scale even for \( Pr > 1 \). Determining the location of the peak for the non-helical dynamo for \( Pr > 1 \) is presently an active area of research. All studies seem to at least agree that the peak is below the forcing scale.

In contrast, simulations that force with \( f_h \sim 1 \) show the production of large-scale ﬁelds at the box scale and a peak at the forcing scale (Brandenburg 2001). Maron & Blackman (2002) studied what happens to the spectrum both above and below the forcing scale as a function of \( f_h \). By starting with an initial spectrum that represented the saturated state of an \( f_h = 0 \) simulation, they found that for \( f_h \) above a critical value (\( = k_1/k_2 \), where \( k_1 \) is the box scale) a peak at \( k_1 \) emerged, and as \( f_h \) was further increased towards 1, the peak originally at the subforcing scale increasingly depleted. A peak at the forcing scale emerged. The adjusted shape of the spectrum at and below the forcing scale established itself on a kinematic growth time-scale of the \( k_1 \) field.

The growth rate and saturation value of the large-scale ﬁeld at \( k_1 \) is now well understood by a non-linear two-scale dynamo theory based on magnetic helicity conservation and exchange between the small scale (assumed to be the forcing scale) and the box (large) scale (Blackman & Field 2002; Field & Blackman 2002): growth of the large-scale ﬁeld corresponds to a growth of large-scale magnetic helicity. Magnetic helicity conservation dictates that a small-scale helical ﬁeld must then also grow with the opposite sign. Because the growth driver for the large-scale helicity also depends on the small-scale magnetic helicity, the growth of the latter ends up quenching the large-scale dynamo. However, there is an important limitation of the two-scale theory. Why should the growth of the small-scale compensating magnetic helicity occur at the forcing scale and not at some much smaller scale? Answering this can help to understand why the small-scale magnetic energy peak initially at subforcing scales in simulations by Maron & Blackman (2002) migrates to the forcing scale when sufﬁcient kinetic helicity is input.

To address these questions, I develop a simple four-scale approach which predicts that the small-scale magnetic helicity and current helicity grow fastest on the smallest scale on which there is both a ﬁnite turbulent velocity and magnetic ﬁeld. (This would be the viscous scale for \( Pr_m \geq 1 \) and the resistive scale for \( Pr_m \leq 1 \).) The magnetic energy and current helicities saturate fastest at these small scales. Eventually, however, larger values are reached at larger scales until the small-scale growth is dominated by the forcing scale, essentially justifying a two-scale approach by the time the kinematic growth phase for the \( k_1 \) ﬁeld ends. It is important to emphasize that the mean-ﬁeld formalism herein describes the dynamical evolution of both the large- and small-scale magnetic and current helicities. The backreaction of the large-scale current helicity on the kinetic helicity, and thus on the growth of the large-scale ﬁeld is treated dynamically. We will see that signiﬁcant growth of the large-scale ﬁeld proceeds unimpeded at early times, but at late times, the backreaction from the small-scale ﬁeld slows the growth to a resistively limited pace. It is important to emphasize that the dynamical mean-ﬁeld theory studied herein, albeit a simpliﬁed theory, goes beyond the kinematic mean-ﬁeld theory of standard textbooks, which does not include the backreaction.

In Section 2, the basic role of helicity conservation in large-scale dynamo theory is discussed and the four-scale set of equations (three of which are coupled) are derived. In Section 3, the equations are solved. The solutions and their interpretation are discussed in Section 4. Section 5 gives our conclusions.

2 THE FOUR-SCALE NON-LINEAR DYNAMO

The mean-ﬁeld dynamo (MFD) theory has been a useful framework for modelling the in situ origin of large-scale magnetic ﬁeld growth in planets, stars and galaxies (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin & Sokoloff 1983) and has also been invoked to explain the sustenance of ﬁelds in fusion devices (see Ortolani & Schnack 1993; Bellan 2000; Ji & Prager 2002,
where the overbar indicates averages that vary on scales of wavenumber $k_1^{-1}$, the $(\ )^2$ indicates averages that vary on scales $\geq k_2^{-1}$. Equation (10) follows because $v_2 = 0$, as I have assumed $k_4$ is taken at the resistive scale and that mixed scale correlations vanish.

In what follows I will solve for the non-linear time evolution of the magnetic helicity at each of the four scales. Blackman & Field (2002) showed that a proper non-linear theory must technically include the time evolution of $\mathbf{E}_2$ and by generalization here, also the time evolution for $\mathcal{E}_3$ into the theory. However, they also showed that for the specific case for which triple correlations in $\mathcal{E}_2$ are treated as having time terms with a damping time $\tau_2 = 1/k_2v_2$, neglecting the time evolution of the turbulent electromotive force does not qualitatively influence the solution. Adopting this specific case, the implication of their result for the present theory is that

$$\mathbf{E}_2 = \frac{\tau_2}{3}(j_3 \cdot \mathbf{b}_2 - v_2 \cdot \nabla \times v_2)\mathbf{B} - \bar{\beta}_2 \mathbf{J},$$

where $\beta_2$ is a diffusivity computed from velocities at the $k_2$ scale. (The form of the diffusivity is not essential for the present discussion, and I will later scale it to its kinetic value.) Similarly, we also have

$$\mathbf{E}_3 = \frac{\tau_1}{3}(j_3 \cdot \mathbf{b}_3)\mathbf{B} - \bar{\beta}_3 \mathbf{J},$$

and

$$(v_3 \times \mathbf{b}_3)_2 = \frac{\tau_1}{3}(j_3 \cdot \mathbf{b}_3)\mathbf{B} - (\bar{\beta}_3)_2 j_2,$$

where $\tau_1 = 1/k_3v_3$, and I assume that there are no kinetic helicity contributions at the $k_3$ scale. This is reasonable because I am focusing on the case in which the kinetic helicity is peaked at the forcing scale, and kinetic helicity does not cascade efficiently. If the $k_1$ field is maximally helical, the current helicity, magnetic helicity and magnetic energy for the $k_3$ scale are simply related by powers of $k_1$. (However, the third term of 8 and the second term of 9 involve a subtlety that will be explained below.) Combining equations (6)–(13) we have

$$\bar{\Delta}_1 H_1 = \frac{2\tau_2}{3}(k_2^2 H_2 + f_2 k_3 v_2^2)k_1 H_1 + \frac{2\tau_2}{3}(k_1 H_1 k_3^2 H_3)$$

$$- 2 \left(\lambda + \bar{\beta}_3 + \bar{\beta}_3\right)k_1^2 H_1,$$

$$\bar{\Delta}_2 H_2 = -\frac{2\tau_2}{3}(k_2^2 H_2 + f_2 k_3 v_2^2)k_1 H_1 - \frac{2\tau_2}{3}(k_1 H_1 k_3^2 H_3 + f_2 k_3 v_2^2)k_2 H_2$$

$$+ 2\bar{\beta}_3 k_1^2 H_1 - 2g_2(\bar{\beta}_2)k_1^2 H_2 - 2k_2^2 H_2,$$

$$\bar{\Delta}_3 H_3 = \frac{\tau_1}{3}(k_1 H_1 k_3^2 H_3) - \frac{2\tau_1}{3}(k_1 H_1 k_3^2 H_3 + 2\bar{\beta}_3)k_2^2 H_2$$

$$+ 2g_2(\bar{\beta}_2)k_1^2 H_2 - 2k_2^2 H_2.$$
The last equation is decoupled from the others and represents decay. Thus in the present approximation scheme, where correlations between different scales vanish and \( k = k_s \geq k_3 \) there is no helicity exchange with the resistive scale. Equation (17) can then be subsequently ignored.

The third and fifth terms of (15) and the second and fifth terms of (16) involve the quantities \( f_u \) and \( g_u \). These are positive quantities and they come from the subtlety in dotting the second and third terms of (13) with \( b_2 \): dotting the third term of (13) with \( b_2 \) gives the term \( \langle (\beta_1) f_2 \cdot b_2 \rangle_{\text{vol}} = \langle (\beta_1) \rangle_{\text{vol}} \langle f_2 \cdot b_2 \rangle_{\text{vol}} = \langle (\beta_1) \rangle_{\text{vol}} \langle \beta_1 \rangle_{\text{vol}} \langle f_2 \rangle_{\text{vol}} G_0 k_2^2 H_3 \), where \( G_0 \) accounts for the deviation of \( \langle (\beta_1) \rangle_{\text{vol}} \) from unity, and I also assume that \( \langle \beta_1 \rangle_{\text{vol}} \) accounts for both the fractional helicity at the \( k_2 \) scale and the deviation of the ratio \( \langle (\beta_1) \rangle_{\text{vol}} / \langle (\beta_1) \rangle_{\text{vol}} \) from unity. Note that unlike \( b_2^2, b_3^2 \) does not enter these equations so that even if there is a large pile-up in magnetic energy at \( k_3 \) initially, only its force-free component couples into the above equations.

To write equations (14)–(16) in dimensionless form, I define \( h_1 = H_1(k_3/v_2), h_2 = H_2(k_3/v_2), h_3 = H_3(k_3/v_2), R_M = v_2/k_3, \tau = t v_2 k_3, \psi = \psi_0 + \psi_2 = -f_0 k_2^2 v_2 \), so that \( f_u \) accounts for both the fractional helicity at the \( k_3 \) scale and the deviation of the ratio \( \langle \beta_1 \rangle_{\text{vol}} / \langle \beta_1 \rangle_{\text{vol}} \) from unity. I also assume \( g_u = g_u = 1 \) to simplify the discussion, and a seed of \( h_3 \simeq 0.001 \). The associated solutions of (18) with \( R_M \) set to unity and assuming a constant \( \langle (\beta_1) \rangle_{\text{vol}} = \langle \beta_1 \rangle_{\text{vol}} \) are consistent with numerically simulations of Maron & Blackman (2002). Below we focus on the case \( f_u = 1 \) to emphasize the main ideas. We also assume \( g_u = 1 \) to simplify the discussion, and a seed of \( h_3 \simeq 0.001 \). The associated solutions of (18)–(20) will be discussed along with the comparison to solutions of (21) and (22).

3 Discussion of solutions

The critical helicity for growth of \( h_1 \) to beat diffusion is \( f_u \gtrsim k_1/k_2 \). For \( f_u \) below this value, \( h_1, h_2, h_3 \) all decay. This is consistent with numerical simulations of Maron & Blackman (2002). Below we focus on the case \( f_u = 1 \) to emphasize the main ideas. We also assume \( g_u = 1 \) to simplify the discussion, and a seed of \( h_3 \simeq 0.001 \). The associated solutions of (18)–(20) will be discussed along with the comparison to solutions of (21) and (22).

3.1 Large-scale field growth in the kinematic regime

By the kinematic regime we mean the growth regime in which \( h_1 \) grows independently of \( R_M \). This is essentially the time for \( h_2 \) to approach \( -f_u \) (see equation 18). At early times, before resistivity is important, \( h_3 \sim -h_1 \), so we can estimate the duration of the kinematic regime by determining the time at which \( h_1 \sim -f_u \). We have, from (18),

\[
h_{\text{kin}} \sim 3(k_2/k_3) \ln(f_0/h_1(0)),
\]

which is \( \sim 100 \) for \( f_0 = 1, k_2/k_1 = 5 \) and \( h_1(0) = 0.001 \), as used herein.

A careful look at (18)–(20) shows that for a small seed \( h_1 \) and \( h_1(0) \lesssim f_u (k_2/k_3)^{4/3} \), the kinetic helicity drives the growth of \( h_1 \) and the growth of the oppositely signed \( h_2 \). If \( h_2(0) \) exceeds the above critical value then \( h_1 \) and \( h_3 \) initially decay until \( h_3 \) falls below that value, after which \( h_1 \) starts to grow.] The growth of \( h_2 \) also supplies, from diffusion, a growth of \( h_3 \) from the fifth term in (20). For \( k_2 \ll k_1 \ll k_3 \), the value of \( h_1 \) at the end of the kinematic phase in the four-scale approach saturates at a slightly smaller value of \( h_1 \) than for the two-scale case. The slight difference between the two curves can be explained by ignoring diffusion and dissipation terms (those containing \( q \) and \( R_M \) in (18)–(20) and assuming a quasi-steady state. Setting time derivatives to zero in (18)–(20) then gives a system of three equations:
in the bottom left-hand plot of Fig. 1, which shows the growth of $h_1$.

For $k$, the value $h_1$ attains in the kinematic regime for the two-scale approach. For $k_1$, the current helicity at early times. This early-time dominance of the $h_1$ quantities is enhanced when there is an initial seed of $h_1$, as seen by comparing Figs 3 and 4. For a Kolmogorov velocity spectrum, however, the small-scale current helicity and associated magnetic energy end up being dominated by the forcing scale before $t_{\text{sw}}$, as discussed below (26), the maximum $h_1$ at the end of the kinematic regime is $h_1 \sim (k_2/k_3)^{\alpha_3}$ so that the ratio of current helicity at $k_1$ to that at $k_2$ is then $(k_1/k_2)^{\alpha_3} (h_1/h_2) \ll 1$, since $h_2 \sim 1$.

It is interesting to assess whether the times for crossover of small-scale dominance from $k_3$ to $k_2$ quantities depends on the magnetic Reynolds number. The answer is that for large $R_M$ it does not, but for low $R_M$ it does. One can be misled when applying the low-$R_M$ results to large $R_M$. To see this more explicitly, first note that the location of the crossovers for the case of Fig. 3 does depend on $R_M$: the lower $R_M$ is, the earlier the crossovers. However, for large $R_M$, the location of the crossover asymptotes to an $R_M$ independent value. It has been checked that the location of the crossovers for $R_M = 9000$ in Fig. 3 is indistinguishable for the location of the crossover for any higher $R_M$, keeping all other parameters fixed. This latter result is more directly seen in the comparisons of the top and bottom rows of Figs 3 and 4. There, the top rows in both cases correspond to a comparison between $R_M = 100$ and $2 \times 10^4$ for $k_3 = 160$. The bottom rows correspond to a comparison between $R_M = 100$ and $2 \times 10^4$ for $k_3 = 160$. The locations of the crossover for the $R_M = 2 \times 10^4$ and $2 \times 10^5$ cases are indistinguishable.

Note that the case of $k_3 = 160$ and $R_M = 100$ for $k_2 = 5$ is the case of $k_3 \approx k_1 \approx k_2$ for Kolmogorov turbulence, in other words, $P_{\text{FM}} \approx 1$. This follows because choosing $R_M = 100$ for $k_3 = 160$ also corresponds to choosing $(k_1/k_2) = R^{\alpha_3/2} \approx 100$, where $R$ is the hydrodynamic Reynolds number and where $k_3$ is taken to be the viscous wavenumber. The power of $\frac{1}{2}$ follows from the Kolmogorov spectrum. This can be distinguished from any viscous wavenumber.

Thus, restating the results of this section in terms of $P_{\text{FM}}$, we can say that for the $P_{\text{FM}} = 1$ case the crossover time for the small-scale quantities to that at $k_2$ is then $(k_1/k_2)^{\alpha_3} (h_1/h_2) \ll 1$, since $h_2 \sim 1$.

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Thus, restating the results of this section in terms of $P_{\text{FM}}$, we can say that for the $P_{\text{FM}} = 1$ case the crossover time for the small-scale quantities to that at $k_2$ is then $(k_1/k_2)^{\alpha_3} (h_1/h_2) \ll 1$, since $h_2 \sim 1$.
current helicity at \(k_3\) to deplete below that at \(k_2\) is earlier than for the \(Pr_M \gg 1\) case in which \(k_3 \ll k_2\). This is expected, since the resistivity is not effective at dissipating \(k_3\) for \(Pr_M \gg 1\). At large \(R_M\) for fixed \(k_3 = k_v\) (or \(Pr_M \gg 1\)), the crossover is the result of the dynamical depletion of \(h_3\) from the first two terms on the right of (20), and becomes independent of \(R_M\) or \(Pr_M\).

### 3.3 Saturation: the doubly maximal inverse transfer state

The growth term for \(h_3\) at early times is the fifth term in (20), but eventually this is offset by the fourth term, after which the only remaining terms for \(h_1\) are resistive and inverse transfer loss terms. Thus, as we have seen, \(h_1\) eventually decays, whilst \(h_2\) continues to grow. The latter takes over the role of compensating negative helicity, and the saturation proceeds exactly in the two-scale approach: positive helicity at \(h_1\) and negative helicity \(h_2\) at the forcing scale. This is demonstrated in the right-hand column of Fig. 1, where the late-time evolution of the two-scale and four-scale approaches are seen to be indistinguishable for all \(R_M\). Accordingly, at late times, the current helicity at \(k_2\) asymptotes to equal that at \(k_1\), just as in the two-scale approach (Brandenburg 2001; Blackman & Field 2002; Field & Blackman 2002).

The features of the four-scale model discussed herein can be used as an aid to understanding the full saturated spectrum of the helical dynamo both above and below the forcing scale. Consider \(k_L \leq k_1 < k_t \leq k_2 < k_3 \leq k_v\), where \(k_L\) is the wavenumber of the largest scale available, \(k_t\) is the forcing scale and \(k_v\) is the viscous scale and we allow \(k_1, k_2, k_3\) to take intermediate values. The time for the spectrum to evolve depends on the ratios \(k_t/k_L\) and \(k_t/k_v\), but the qualitative evolution can be understood simply. At early times, a seed spectrum forced with negative kinetic helicity with \(f_b > k_t/k_L\), can grow positive magnetic helicity at \(k_1\). (For \(f_b = 1\) the maximum growth of the large-scale magnetic helicity initially occurs at \(k_1 \approx k_L/2\).) The magnetic helicity will inverse transfer from \(k_1 > k_2\) to successively smaller \(k\), eventually all the way to \(k_t \approx k_L/2\).
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3.4 Implications and comparison with fractionally helical dynamo spectra

Maron & Blackman (2002) showed that as \( f_h \) exceeds \( k_f/k_l \) and approaches 1 for \( Pr_M > 1 \), the saturated magnetic energy spectrum changes from peaking on the resistive scale to peaking on two scales, with one peak at the forcing scale and one at the system scale. (Their \( f_h = 1 \) results matched Brandenburg 2001.) For \( f_h > k_f/k_l \), \( f_h(k_f/k_l) \) times the equipartition energy ends up in magnetic energy at the \( k_1 \) scale after the kinematic regime. At saturation, \( f_h k_f/k_l \) of the equipartition energy ends up at \( k_f \).

The fraction of small-scale magnetic energy associated with the helicity dynamics is also determined by \( f_h \). Whatever the dynamics of the non-helical magnetic energy spectrum, the helical fraction of the magnetic spectrum seems to be explicable by the independent dynamics discussed above, suggesting that the helical and non-helical parts of the spectrum are rather decoupled. Recall that the non-helical magnetic energies on the \( k_1 \) and \( k_2 \) scales do not enter the theory, only that on the \( k_2 \) scale (see the discussion between equations 17 and 18). In the present theory, the expected helical fraction of the small-scale magnetic energy that follows the helicity dynamo dynamics is given simply by \( f_h \). This follows because the small-scale current helicity in saturation is \( f_h \nu^2 k_2 \sim \langle b \cdot \nabla \times b \rangle \) and since \( \langle \nu^2 \rangle \sim \langle \nu^3 \rangle \) in saturation, we have the associated helical magnetic energy fraction = \( f_h \). If we include the non-helical component of the magnetic energy, then \( f_h \) is a lower limit to the fraction of the total small-scale magnetic energy that winds up at the forcing scale.

4 CONCLUSIONS

A four-scale non-linear magnetic dynamo theory was presented using the approximation that correlations of mixed scales are assumed to vanish. The goal was to develop a simple theory that sheds light on the evolution of the full magnetic spectrum for the helical dynamo and to show how the presence of kinetic helicity influences the spectrum both above and below the dominant scale of the turbulent kinetic energy. The velocity spectrum was assumed to be Kolmogorov, with kinetic helicity input only at the forcing scale. The results of the simple theory are consistent with existing numerical simulations and make additional predictions that can be tested. The theory includes the dynamical backreaction of the growing magnetic field on the turbulence driving the field growth.

The growth of the large-scale field in the four-scale theory is consistent with that predicted in the two-scale theory (Blackman & Brandenburg 2002; Blackman & Field 2002; Field & Blackman 2002) at late times, but previous simulations do not have enough resolution to test the dynamics of the kinematic regime and so higher-resolution simulations will be needed. The four-scale theory herein predicts that as the large helical scale field grows, the small-scale helical field of the opposite sign and its associated current and magnetic helicity will first grow at the smallest scales where both \( \nu \) and \( b \) are finite (for \( Pr > 1 \), this is the viscous scale). This cannot be seen in a two-scale approach, since there the system scale and forcing scales are the only ones present. The source of helical field for the very smallest scales is diffusion from above. These very small
(viscous) scale helical fields drain by inverse transfer, and the helical magnetic energy below the forcing scale peak at successively larger scales, arriving at the forcing scale before \( \tau_{\text{kin}} \). If the viscous scale is much larger than the resistive scale, then resistivity plays little role in this process; the process is independent of \( \alpha \) for large \( \alpha \). At \( \tau_{\text{kin}} \) the growth of both the large- and small-scale helical fields proceed exactly as in the two-scale theory: large-scale current and magnetic helicities are primarily carried by \( k_1 \) and the oppositely signed small-scale quantities at scale \( k_2 \).

That high-resolution simulations are needed to test the kinematic regime also means that they are needed for testing the assumption made above equation (7). If future simulations for \( \Pr_{\text{M}} \gg 1 \) show that the current helicity piles up at the resistive scale rather than the viscous scale at very early times, this would suggest that there exist mixed correlations that can grow helicity on the resistive scale. In the present, four-scale theory, the helical magnetic energy on the resistive scale decays, and the non-helical component is decoupled from the helical component.

I emphasize that the model presented herein is a simplified theory, meant to guide physical understanding of helical MHD turbulence. It must be subject to further testing.

The larger the fractional kinetic helicity \( f_h \), the more the overall magnetic spectrum behaves like its helical component. If the non-helical component prefers to pile-up at small scales, the helical component will provide at least some fraction that wants to be doubly peaked at the forcing and system scales. It is thus important to emphasize that the driving turbulence with kinetic helicity on a scale \( k_2 \) affects the shape of the overall helical magnetic energy spectrum and thus that of the overall magnetic energy spectrum above and below the forcing scale.

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