Exploring the expanding Universe and dark energy using the statefinder diagnostic

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ABSTRACT

The coming few years are likely to witness a dramatic increase in high-quality supernova data as current surveys add more high-redshift supernovae to their inventory and as newer and deeper supernova experiments become operational. Given the current variety in dark energy models and the expected improvement in observational data, an accurate and versatile diagnostic of dark energy is the need of the hour. This paper examines the statefinder diagnostic in the light of the proposed SuperNova Acceleration Probe (SNAP) satellite, which is expected to observe about 2000 supernovae per year. We show that the statefinder is versatile enough to differentiate between dark energy models as varied as the cosmological constant on one hand, and quintessence, the Chaplygin gas and braneworld models, on the other. Using SNAP data, the statefinder can distinguish a cosmological constant ($w = -1$) from quintessence models with $w \geq -0.9$ and Chaplygin gas models with $\kappa \leq 15$ at the 3$\sigma$ level if the value of $\Omega_m$ is known exactly. The statefinder gives reasonable results even when the value of $\Omega_m$ is known only to $\sim 20$ per cent accuracy. In this case, marginalizing over $\Omega_m$ and assuming a fiducial $\Lambda$-cold dark matter (LCDM) model allows us to rule out quintessence with $w \geq -0.85$ and the Chaplygin gas with $\kappa \leq 7$ (both at $3\sigma$). These constraints can be made even tighter if we use the statefinders in conjunction with the deceleration parameter. The statefinder is very sensitive to the total pressure exerted by all forms of matter and radiation in the Universe. It can therefore differentiate between dark energy models at moderately high redshifts of $z \lesssim 10$.

Key words: cosmological parameters – cosmology: theory.

1 INTRODUCTION

Supernova observations (Riess et al. 1998; Perlmutter et al. 1999), when combined with those of the cosmic microwave background (Benoit et al. 2003) and gravitational clustering (Percival et al. 2002), suggest that our Universe is (approximately) spatially flat and that an exotic form of negative-pressure matter called ‘dark energy’ (DE) causes it to accelerate by contributing as much as two-thirds to the closure density of the Universe – the remaining third consisting of non-relativistic dark matter and baryons. The simplest example of dark energy is the cosmological constant ($\Lambda$), with associated mass density

$$\rho_\Lambda = 6.44 \times 10^{-30} \left(\frac{\Omega_\Lambda}{0.7}\right)^2 \left(\frac{h}{0.7}\right)^2 \text{g cm}^{-3},$$

where $h$ is the Hubble constant $H_0$ in terms of 100 km s$^{-1}$ Mpc$^{-1}$ and $\Omega_\Lambda = 0.7 \pm 0.1, h = 0.7 \pm 0.1$. Although the cold dark matter model with a cosmological constant (hereafter LCDM) provides an excellent explanation for the acceleration phenomenon and other existing observational data, it remains entirely plausible that the dark energy density is weakly time dependent (see the reviews by Sahni & Starobinsky 2000; Peebles & Ratra 2003). Moreover, it is natural to suggest (in complete analogy with what has been done in the case of another type of ‘dark energy’ responsible for driving the expansion of the Universe during an inflationary stage in the early Universe) that the dark energy that we observe today might really be dynamical in nature and origin. This means that a completely new form of matter is responsible for giving rise to the second inflationary regime, which we are entering now.

Many models of dark energy have been proposed; in fact, any inflationary model [even a ‘bad’ one, i.e. without a ‘graceful exit’ to the subsequent radiation-dominated Friedmann–Robertson–Walker (FRW) stage] may be used for this purpose if one assumes different values for its microscopic parameters. The simplest of these models rely on a scalar field minimally interacting with Einstein gravity – quintessence (Peebles & Ratra 1988; Ratra & Peebles 1988; Frieman et al. 1995; Caldwell, Dave & Steinhardt 1998),
and bear an obvious similarity with the simplest variants of the inflationary scenario. Inclusion of a non-minimal coupling to gravity in these models together with further generalization leads to models of dark energy in a scalar–tensor theory of gravity (see Boisseau et al. 2000 and references therein). Other models invoke matter with unusual properties such as the Chaplygin gas (Kamenshchik, Moschella & Pasquier 2001) or $k$–essence (Armendariz-Picon, Mukhanov & Steinhardt 2000). Still others generate cosmic acceleration through topological defects (Bucher & Spergel 1999) or quantum vacuum polarization and particle production (Sahni & Habib 1998; Parker & Raval 1999). Lately it has been noticed that higher-dimensional ‘braneworld’ models could account for a late-time accelerating phase even in the absence of matter violating the strong energy condition (Dvali, Gabadadze & Porrati 2000; Alam & Sahni 2002; Deffayet, Dvali & Gabadadze 2002a; Deffayet et al. 2002b; Sahni & Shtanov 2002; see Sahni 2002 for a recent review of dark energy models). It is especially interesting that in the latter class of models ‘dark energy’ need not be an energy of some form of matter at all, but can have an entirely geometrical origin. Moreover, in these models the basic gravitational field equations do not have the Einstein form

\[ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi G (T_{\alpha\beta}^{\text{matter}} + T_{\alpha\beta}^{\text{radiation}} + T_{\alpha\beta}^{\text{DE}}) \]  

(2)

($c = h = 1$ is assumed here and below), and therefore the notions of ‘energy density’ and ‘pressure’ of DE lose their exact fundamental sense and become ambiguous and convention-dependent. A major ambiguity arises in models of scalar–tensor gravity as well as in braneworld models, both of which contain interaction terms between dark energy and non-relativistic matter. Interpreting such models within the Einstein framework (2) leads to the following dilemma: should these interaction terms be ascribed to dark matter (hence to $T_{\alpha\beta}^{\text{matter}}$ in equation 2) or to dark energy (to $T_{\alpha\beta}^{\text{DE}}$)?

Our answer to this question has the potential to alter the properties of dark energy including its density and pressure and hence also its equation of state. In marked contrast to such ambiguities which could arise if we are not careful with our usage of the term ‘equation of state’, the expansion factor of the universe in the physical frame $a(t)$, when expressed through the Hubble parameter $H \equiv \dot{a}/a$, is an unambiguous, fundamental and readily measurable quantity.

Given the rapidly improving quality of observational data and also the abundance of different theoretical models of dark energy, the need of the hour clearly is a robust and sensitive statistic that can succeed in differentiating cosmological models with various kinds of dark energy both from each other and, even more importantly, from an exact cosmological constant. In view of the non-fundamental nature of the notions of DE density and pressure pointed out above, we prefer to work with purely geometric quantities. Then such a sensitive diagnostic of the present acceleration epoch and of dark energy could be the statefinder pair $\{r, s\}$, recently introduced in Sahni et al. (2003). The statefinder probes the expansion dynamics of the universe through higher derivatives of the expansion factor $\dot{a}$ and $a$. Its important property is that $\{r, s\} = \{1, 0\}$ is a fixed point for the flat LCDM FRW cosmological model. Departure of a given DE model from this fixed point is a good way of establishing the ‘distance’ of this model from flat LCDM. As we will show in this paper, the statefinder successfully differentiates between rival DE models and, when combined with Supernova Acceleration Probe (SNAP) supernova data, can serve as a versatile and powerful diagnostic of dark energy.

The paper is organized as follows. In the next section we briefly review some theoretical models of dark energy. The behaviour of the statefinder pair for these models is discussed in Section 3, while the nature of data expected to become available from the SNAP experiment is the subject of Section 4. Section 4 also discusses model-independent parametric reconstructions of dark energy. Our conclusions are presented in Section 5.

2 DARK ENERGY MODELS AND THE ACCELERATION OF THE UNIVERSE

The rate of expansion of an FRW universe and its acceleration are described by the pair of equations

\[ \dot{a}/a = -\frac{4\pi G}{3} \sum_{i} (\rho_i + 3p_i), \]  

(3)

where the summation is over all matter fields contributing to the dynamics of the universe. Clearly, a necessary (but not sufficient) condition for acceleration ($\dot{a} > 0$) is that at least one of the matter fields in (3) should violate the strong energy condition $\rho + 3p \geq 0$. If for simplicity we assume that the dark energy pressure and density are related by the simple linear relation $p = w\rho$, then $w < -1/3$ is a necessary condition for the universe to accelerate. The acceleration of the universe can be quantified through a dimensionless cosmological function known as the ‘deceleration parameter’ $q = -\ddot{a}/aH^2$, equivalently

\[ q(x) = \frac{H''(x)}{H'(x)} - 1, \quad x = 1 + \frac{t}{\Omega_{0}}, \]  

(4)

where $q < 0$ describes an accelerating universe, whereas $q \geq 0$ for a universe that is either decelerating or expanding at the ‘coasting’ rate $a \propto t$. As will soon be shown, the deceleration parameter on its own does not characterize the current accelerating phase uniquely. The presence of a fairly large degeneracy in $q(z)$ is reflected in the fact that rival dark energy models can give rise to one and the same value of $q_0$ at the present time. This degeneracy is easily broken if, as demonstrated in Section 3, one combines $q(z)$ with one of the statefinders $r(z), s(z)$. The diagnostic pairs $\{r, q\}$ and $\{s, q\}$ provide a very comprehensive description of the dynamics of the universe and consequently of the nature of dark energy.

Now let us come to the issue of defining the energy density and pressure of DE. In view of the ambiguities discussed in the introduction, we shall define $p_{DE}$ and $\rho_{DE}$ by making use of the Einstein interpretation of gravitational field equations (not to be confused with the notion of the Einstein frame, which is used in scalar–tensor and string theories of gravity!). Namely, we assume that the gravitational field equations in a single-metric theory of $3+1$ gravity can be formally written in the form (2) where the Einstein tensor on the left-hand side is defined with respect to the physical space–time metric. All other terms are transferred to the right-hand side. Next, we subtract the energy–momentum tensor of dust (CDM + baryons) from the total energy–momentum tensor of matter and call the remaining part ‘the effective energy–momentum tensor of dark energy’ (in the Einstein interpretation). Combining this prescription with equation (3) and in the absence of spatial curvature, the energy density and pressure of dark energy can be defined as

\[ \rho_{DE} = \rho_{\text{critical}} - \rho_{\text{0}} = \frac{3H^2}{8\pi G}(1 - \Omega_m), \]  

\[ p_{DE} = \frac{H^2}{4\pi G} \left(q - \frac{1}{2}\right), \]  

(5)

where $\rho_{\text{critical}} = 3H^2/8\pi G$ is the critical density associated with an
Provided by quintessence—kinessence

For defining non-Einsteinian theories by physical quantity it does not provide us with an exhaustive description of how the universe can also be produced by a scalar field evolution governed by the equation of motion

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0,
\]

where

\[
H^2 = \frac{8\pi G}{3} \left( \rho_m (1 + z)^3 + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right).
\]

It is clear from (9) that \( w < -1/3 \) provided \( \dot{\phi}^2 < V(\phi) \). Models with this property can lead to an accelerating universe at late times. An important subclass of quintessence models displays the so-called ‘tracker’ behaviour during which the ratio of the scalar field energy density to that of the matter/radiation background changes very slowly over a substantial period of time. Models belonging to this class satisfy \( V^\phi/V(\phi)^2 \geq 1 \) and approach a common evolutionary ‘tracker path’ from a wide range of initial conditions. As a result, the present value of dark energy in tracker models is to a large extent (though not entirely) independent of initial conditions and is determined by parameters residing only in its potential – as in the case of the cosmological constant (for a brief review of tracker models see Sahni 2002). In this paper we will focus our attention on the tracker potential \( V(\phi) \propto \phi^{-\alpha}, \alpha \geq 1 \), which was originally proposed in Ratra & Peebles (1988). For this potential, the region of initial conditions for \( \phi \) for which the tracker regime has been reached before the end of the matter-dominated stage is \( \phi_m \ll M_F \equiv 1/\sqrt{\mathcal{G}} \), and the present value of quintessence is \( \phi(t_0) \sim M_F \).

For all quintessence models \( w > -1 \), and this inequality is saturated only if \( \phi = dV/d\dot{\phi} = 0 \). In order to obtain \( w < -1 \) matter must violate the strong energy condition \( \rho + 3p > 0 \), for some duration of time. It should be noted that DE with \( w < -1 \) is not excluded by observations (see Melchiorri et al. 2002 for a recent investigation). However, in order to have \( w < -1 \) one must look beyond quintessence models. Models based on scalar–tensor gravity (Boisseau et al. 2000) can have \( w < -1 \), so too can braneworld models (see Sahni & Stianov 2002 for a discussion of this issue and Alam & Sahni 2002 for a comparison of braneworld models with observational data).

(iv) Chaplygin gas. An interesting alternate form of dark energy is provided by the Chaplygin gas (Kamenshchik et al. 2001; Bilic, Tupper & Viollier 2002; Fabris, Goncalves & de Souza 2002; Alcaniz, Jain & Dev 2003; Avelino et al. 2003; Gorini, Kamenshchik & Moschella 2003), which obeys the equation of state

\[
p_c = -\Lambda /\rho_c.
\]

The energy density of the Chaplygin gas evolves according to

\[
\rho_c = \sqrt{A + B(1 + z)^\gamma},
\]

from where we see that \( \rho_c \to \sqrt{A} \) as \( z \to -\infty \) and \( \rho_c \to \sqrt{B(1 + z)^\gamma} \) as \( z \to \infty \). Thus, the Chaplygin gas behaves like pressureless dust at early times and like a cosmological constant during very late times. Note, however, that Chaplygin gas at \( z \gg 1 \) is not simply a new kind of CDM if we examine its inhomogeneities (i.e. if we apply this hydrodynamical equation of state to the inhomogeneous case, too! In contrast to CDM and baryons, the sound velocity in the Chaplygin gas \( v_s = \sqrt{\left(\frac{d\rho_c}{d\rho_m}\right)} = \sqrt{A/\rho_c} \) quickly grows \( \propto z^2 \) during the matter-dominated stage and becomes of the order of the velocity of light at present (it approaches light velocity asymptotically in the distant future). Thus, from the point of view of inhomogeneities, the properties of the Chaplygin gas during the matter-dominated epoch are very unusual and resemble those of hot dark matter, which has a large Jeans length, despite the fact that the Chaplygin gas formally carries negative pressure.
The Hubble parameter for a universe containing cold dark matter and the Chaplygin gas is given by
\[ H(z) = H_0 \left[ \Omega_m (1 + z)^3 + \frac{\Omega_m a}{\kappa} \sqrt{\frac{A}{B} + (1 + z)^6} \right]^{1/2}, \]
where \( \kappa = \rho_{dm}/\sqrt{\rho_c} \). It is easy to see from (14) that
\[ \kappa = \frac{\rho_{dm}}{\rho_c} (z \to \infty). \]
Thus, \( \kappa \) defines the ratio between CDM and the Chaplygin gas energy densities at the commencement of the matter-dominated stage. It is easy to show that
\[ A = B \left[ \kappa^2 \left( \frac{1 - \Omega_m}{\Omega_m} \right)^2 - 1 \right]. \]
In the limiting case when \( A = 0 \), the Chaplygin gas becomes indistinguishable from dust-like matter (if we examine its behaviour in an unperturbed FRW background). This limiting case corresponds to
\[ \kappa = \frac{\Omega_m}{1 - \Omega_m}, \]
and is shown as the outer envelope (dashed) to the Chaplygin gas models in Figs 1(a) and (b). In the other limiting case \( B = 0 \), the Chaplygin gas reduces to the cosmological constant.

The fact that the sound velocity in the Chaplygin gas is not small during the matter-dominated stage and becomes very large towards its end suggests that the parameter \( \kappa \) should be large in order to avoid damping of adiabatic perturbations. This requires \( A \gg B \). Recent investigations, which look at Chaplygin gas models in the light of galaxy clustering data and CMB anisotropies, show that this observation is correct if the equation of state \( p_c \propto -1/\rho_c \) is assumed to be universally valid (Carturan & Finelli 2002; Sandvik et al. 2002; Bean & Dore 2003). In our paper we consider the Chaplygin gas equation of state to be a phenomenological description of dark energy in an FRW background and do not assume that it remains true for perturbations. However, the fact that \( \kappa \) should be large for viable models will appear in our results, too. Finally, let us point out that the Chaplygin gas may be considered to be a specific case of \( k \)-essence with a constant potential and the Born–Infeld kinetic term. To illustrate this consider the Born–Infeld Lagrangian density
\[ \mathcal{L} = -V_0 \sqrt{1 - \phi_{,\mu} \phi^{,\mu}}, \]
where \( \phi_{,\mu} \equiv \partial \phi/\partial x^\mu \). For time-like \( \phi_{,\mu} \) one can define a four-velocity
\[ u^\mu = \sqrt{\phi^{,\mu} \phi_{,\mu}}, \]
this leads to the standard form for the hydrodynamical energy-momentum tensor
\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}, \]
where (Frolov, Kofman & Starobinsky 2002)
\[
\rho = \frac{V_0}{\sqrt{1 - \phi \dot{\phi}^2}}, \quad p = -V_0 \sqrt{1 - \phi \dot{\phi}^2},
\]
(21)
and we find that we have recovered (12) with \(A = V_0^2\).

(v) Braneworld models. Braneworld models provide an
interesting alternative to dark energy model building. According to
this higher-dimensional world view, we live on a (3 + 1)-dimensional
brane (‘brane’ being a multidimensional generalization of ‘membrane’),
which is either embedded in or bounds a higher-dimensional
space–time. The simplest example of a braneworld that can lead
to late-time acceleration is the model suggested by Deffayet et al.
(2002a) (henceforth we shall refer to this model as the DDG model):
\[
H = \sqrt{\frac{8\pi G D_m}{3} + \frac{1}{l_c^2} + \frac{1}{l_b^2}},
\]
(22)
where \(l_c = m^2/M^3\) is a new length-scale and \(m\) and \(M\) refer,
respectively, to the four- and five-dimensional Planck mass \((l_c = 2\epsilon_c\)
in the terminology of Deffayet et al. 2002a).
The acceleration of the Universe in this model is not caused by the presence of ‘dark
energy’ but is due to the fact that general relativity is formulated in five dimensions instead of the usual four. One consequence of this
is that gravity becomes five dimensional on length-scales \(R > l_c = 2H_b^{-1}(1 - \Omega_m)^{-1}\).  A more general class of
braneworld models is described by (Sahni & Shtanov 2002)
\[
\frac{H^2(z)}{H^2_0} = \Omega_m(1+z)^3 + \Omega_r + 2\Omega_b \pm 2\sqrt{\Omega_m \Omega_r(1+z)^3 + \Omega_b},
\]
(23)
where \(\Lambda_b\) is the bulk cosmological constant, \(\sigma\) is the brane tension and
\[
\Omega_m = \frac{\rho_m}{3M^2H^2_0}, \quad \Omega_r = \frac{\sigma}{3M^2H^2_0},
\]
\[
\Omega_b = \frac{1}{l_c^2H^2_0}, \quad \Omega_{\Lambda_b} = -\frac{\Lambda_b}{6H^2_0}.
\]
(24)
It is easy to see that \(l_c\) can be of the same order as the Hubble radius
\(l_H \sim H_0^{-1}\) if \(M \sim 100\) MeV. On short length-scales \(r \ll l_c\) and at early
times, one recovers general relativity, whereas on large length-
scales \(r \gg l_c\) and at late times brane-related effects begin to play
an important role. It is interesting that brane-inspired effects can lead
to the late-time acceleration of the Universe even in the complete
absence of a matter source that violates the strong energy condition
\(\rho + 3p \geq 0\) (Deffayet et al. 2002b; Sahni & Shtanov 2002).
The dimensionless value of the brane tension \(\Omega_r\) is determined
by the constraint relation
\[
\Omega_m + \Omega_r = 2\sqrt{\Omega_m \Omega_r + \Omega_{\Lambda_b}} = 1.
\]
(25)

The underlined terms in (23) and (25) make braneworld models
different from standard FRW cosmology. Indeed, by setting \(\Omega_i = 0\),
equation (23) reduces to the LCDM model
\[
\frac{H^2(z)}{H^2_0} = \Omega_m(1+z)^3 + \Omega_r
\]
(26)
which describes a universe containing matter and a cosmological
constant (7). The two signs in (23) correspond to the two separate
ways in which the brane can be embedded in the higher-dimensional
bulk. As shown in Sahni & Shtanov (2002), taking the upper sign in
(23) and (25) leads to the model called BRANE1, while the lower
sign in (23) and (25) results in BRANE2.

Three important classes of braneworld models deserve special
mention.

(i) BRANE1 models have an effective equation of state that is
more negative than that of the cosmological constant \(w < -1\).

(ii) BRANE2 models have \(w \geq -1\). For parameter values \(\Omega_m = \Omega_{\Lambda_b} = 0\), BRANE2 coincides with the dark energy model discussed in equation (22).

(iii) A class of braneworld models, called ‘disappearing dark energy’ (DDE) (Sahni & Shtanov 2002; Alam & Sahni 2002), have the important property that the current acceleration of the Universe is a transient phase, that is sandwiched between two matter-dominated epochs. These models do not have horizons and therefore help to reconcile an accelerating Universe with the demands of the string/M-theory (Sahni 2002) (as well as any theory that requires dark energy to decay in the future and transform into matter with \(w \geq -1/3\)).

Finally, we note that, for a spatially flat universe, the luminosity
distance for all models discussed above is given by the simple
expression
\[
\frac{D_L(z)}{1+z} = \int_0^z \frac{dz'}{H(z')},
\]
(27)
where \(H(z)\) is given by (7) for LCDM, by (8) for quiessence, by
(11) for quintessence, by (14) for the Chaplygin gas and by (23) for the
braneworld models.

3 THE STATEFINDER DIAGNOSTIC

As we have seen above, dark energy has properties that can be very
model dependent. In order to be able to differentiate between the
very distinct and competing cosmological scenarios involving dark
energy, a sensitive and robust diagnostic (of dark energy) is a must.
Although the rate of acceleration/deceleration of the Universe can be
described by the single parameter \(q = -\ddot{a}/aH^2\), a more sensitive
discriminator of the expansion rate and hence dark energy can be
constructed by considering the general form for the expansion factor of
the Universe
\[
a(t) = a(t_0) + \dot{a} (t - t_0) + \frac{\ddot{a}}{2} (t - t_0)^2 + \frac{\dddot{a}}{6} (t - t_0)^3 + \cdots.
\]
(28)

In general, dark energy models such as quiessence, quintessence,
k-essence, braneworld models, Chaplygin gas, etc. give rise to fami-
lies of curves \(a(t)\) having vastly different properties. Since we know that the acceleration of the Universe is a fairly recent phenomenon
(Sahni & Starobinsky 2000; Riess et al. 2001; Benitez et al. 2002)
we can, in principle, confine our attention to small values of \(t - t_0\)
in (28). We have shown in Sahni et al. (2003) that a new diagnostic
of dark energy called statefinder can be constructed using both the
second and third derivatives of the expansion factor. The second
derivative is encoded in the deceleration parameter, which has the
following form in a spatially flat universe:
\[
q = -\frac{\ddot{a}}{aH^2} = \frac{1}{2} (1 + 3w\Omega_X), \quad \Omega_X = 1 - \Omega_m.
\]
(29)

The statefinder pair \(\{r, s\}\) defines two new cosmological parameters (in addition to \(H\) and \(q\))
\[
\frac{r}{aH^3} = 1 + \frac{9w}{2\Omega_X (1 + w) - \frac{3}{2} \Omega_X \frac{w}{H}},
\]
(30)
\[
s = \frac{r - 1}{3(q - \frac{1}{2})} = 1 + w - \frac{1}{3} \frac{\dot{w}}{wH}.
\]
(31)
Clearly, an important requirement of any diagnostic is that it permits us to differentiate between a given dark energy model and the simplest of all models – the cosmological constant \( \Lambda \). The statefinder does exactly this. For the LCDM model, the value of the first statefinder stays pegged at \( r = 1 \) even as the matter density evolves from a large initial value (\( \Omega_m \approx 1 \), \( t < t_f \)) to a small late-time value (\( \Omega_m \to 0 \), \( t \gg t_f \)). It is easy to show that \( \{r, s\} = \{1, 0\} \) is a fixed point for LCDM.

The second statefinder \( s \) has properties that complement those of the first. Since \( s \) does not explicitly depend upon either \( \Omega_x \) or \( \Omega_m \), many of the degeneracies that are present in \( r \) are broken in the combined statefinder pair \( \{r, s\} \). For models with a constant equation of state (quiescence) \( s = 1 + w = \) constant, while the statefinder \( r \) is time-varying. For models with a time-dependent equation of state (kinessence), both \( r \) and \( s \) vary with time. As we will show in this paper, the statefinder pair \( \{r, s\} \) can easily distinguish between LCDM, quiescence and kinessence models. It can also distinguish between more elaborate models of dark energy such as braneworld models and the Chaplygin gas (see also Gorini et al. 2003; Sahni et al. 2003).

Interestingly, as demonstrated in Section 5, the statefinder pair \( \{r, s\} \) can easily distinguish between LCDM, quiescence and kinessence models. This is also very pronounced when we analyse evolutionary trajectories and see that the vertical line at \( q = 0.5 \) fixes the current position of LCDM. The quintessence, LCDM and the Chaplygin gas all end their evolution at the same common point in the future \( (q = -1, r = 1) \), which corresponds to a steady-state cosmology (SS) – de Sitter expansion. In Fig. 2 the LCDM model separates BRANE1 models (which have \( w_{\text{eff}} \approx -1 \)) from BRANE2 models as well as DDE models. BRANE2 models have \( w_{\text{eff}} \geq -1 \) generically, whereas DDE models consist of a transient accelerating regime which is sandwiched between two matter-dominated epochs. Thus, DDE both begins and ends its evolution at the SCDM point \( \{r, q\} = \{1, 0.5\} \) and its \( r-q \) space trajectory is a loop! BRANE1 and BRANE2 models, on the other hand, commence evolving at the SCDM point and tend to SS in the future. Figs 1(b) and 2 clearly demonstrate that the deceleration parameter cannot on its own differentiate between rival models of dark energy. The degeneracy that afflicts \( q(z) \) clearly also afflicts the equation of state \( w(z) \), since both \( q \) and \( w \) are related through (6). We therefore feel we have convincingly demonstrated that the statefinders can successfully differentiate between competing dark energy models as diverse as LCDM, quintessence, braneworld models and the Chaplygin gas. Statefinders can also be applied to other interesting candidates for dark energy including bigravity models (Damour, Kogan & Papazoglou 2002), generalised Chaplygin gas (Kamenshchik et al. 2001; Bento, Bertolami & Sen 2002), \( k \)-essence (Armendariz-Picon et al. 2000) scalar–tensor theories, etc.

Finally, we draw the readers attention to the following elegant relationship that exists between the statefinders, on the one hand,
and the total density $\rho = \sum \rho_i$ and total pressure $p = \sum p_i$ in the Universe:

$$q = \frac{3 \rho}{2 p}, \quad r - 1 = \frac{9 \rho + p}{2 \rho} \frac{\dot{p}}{p}, \quad \dot{s} = \frac{p + \rho}{p} \frac{\dot{p}}{p}.$$

(33)

From equation (33) we see that the statefinder $s$ is exceedingly sensitive to the total pressure $p$. This has some interesting consequences. At early times the presence of radiation ensures that the total pressure in the Universe is positive. Much later, the Universe begins to accelerate driven by the negative pressure of dark energy. In between these two asymptotic regimes, deep in the matter-dominated epoch, a stage is reached when the (negative) pressure of dark energy is exactly balanced by the positive pressure of radiation. At this precise moment of time $p \approx 0$ and $s \rightarrow \infty$! For LCDM this pressure balance is achieved at $z_e \sim 10$, consequently $|s| \gg 1$ when $z \sim z_e$. It can be shown that the redshift $z_e$ (at which $p = 0$) is quite sensitive to the form of dark energy. We therefore find that the statefinder $s$ diagnoses the presence of dark energy even at high redshifts when the contribution of DE to the total energy budget of the Universe is insignificant!

### 4 Model-Independent Reconstruction of Cosmological Parameters from SNAP Data

#### 4.1 The cosmological reconstruction of dark energy properties

Cosmological reconstruction is an effective statistical technique, which can be used in situations where a large number of theoretical models are to be compared with observations. Instead of estimating relevant parameters for each model separately, we can choose a model-independent fitting function and perform a maximum-likelihood parameter estimation for it. The resultant confidence levels can be used to rule out or accept the different models available. This technique is effective here because, as discussed in Section 2, a wide range of theoretical models have been suggested to explain dark energy.

The basis of cosmological reconstruction rests on the observation that the expression for the luminosity distance (27) can be easily inverted (Starobinsky 1998; Huterer & Turner 1999; Nakamura & Chiba 1999):

$$H(z) = \left( \frac{d}{dz} \frac{D_L(z)}{1 + z} \right)^{-1}.$$

(34)

Thus, from a mathematical point of view, any given $D_L(z)$ defines $H(z)$. Equations (5) and (6) can then be used to obtain the dark energy density and the associated equation of state. Similarly, the statefinder pair $(r, s)$ can be determined by employing equation (32) together with equation (29). However, in practice the derivative with respect to $z$ may not be simply performed since $D_L(z)$ is noisy due to observational errors (mainly, due to variance in supernovae luminosity). Therefore, the smoothing of data over some interval $\Delta z$ is required ($\Delta z$ may depend on $z$). The value of $\Delta z$ is determined by estimated errors and by the required accuracy with which we want to determine $H(z)$. Of course, the resulting $H(z)$ will be smoothed, too, as compared with the genuine one. Note that our presentation here is very similar to that in Tegmark (2002).

Instead of actually dividing a measured range of $z$ into intervals, one may parametrize $H(z)$ by some fitting curve that depends on a number of free parameters. This leads to model-independent parametric reconstruction of $H(z)$, $\rho_{DE}(z)$, $w_{ef}(z)$ and other quantities. It is clear that the number of free parameters $N$ in such a fit just defines the equivalent smoothing interval $\Delta z$ (in particular, $\Delta z = z_{\text{max}}/N$ if $\Delta z$ is chosen to be independent of $z$ and we are considering the function $H(z)/H_0$, so that its value at $z = 0$ is known exactly).

Thus, the parametrization is equivalent to some kind of smoothing, with the actual method of smoothing (weighting) depending on the functional form of the parametric fit used. This refers even to such sophisticated methods as the ‘principal-component’ approach used in Huterer & Starkman (2003). Since decreasing $\Delta z$ (increasing $N$) results in a rapid growth of errors $[\Delta H(z) \propto (\Delta z)^{-3/2}]$ directly follows from equation (34), cf. Tegmark (2002), for a given $z_{\text{max}}$ there is no sense in taking $N$ to be large – this will merely result in a loss of accuracy of our reconstruction. Thus, we will consider only three-parameter fits for $H(z)$ [these will correspond to two-parameter fits for $w(z)$].

After the discovery that the Universe is accelerating, many different fitting function approaches were suggested and some are summarized below.

(i) Polynomial fit to dark energy. In this paper, we reconstruct dark energy using a very effective Ansatz introduced in Sahni et al. (2003) in which the dark energy density is expressed as a truncated Taylor series polynomial in $x = 1 + z$, $\rho_{DE} = A_1 + A_2 x + A_3 x^2$. This leads to the following Ansatz for the Hubble parameter:

$$H(x) = H_0 \left( \Omega_{m} x^3 + A_1 + A_2 x + A_3 x^2 \right)^{1/2},$$

(35)

which, when substituted in the expression for the luminosity distance (27), yields

$$D_L = \frac{c}{H_0} \int_{1}^{1+z} \frac{dx}{\sqrt{\Omega_{m} x^3 + A_1 + A_2 x + A_3 x^2}}.$$

(36)

The values of the parameters $A_1$, $A_2$, $A_3$ are obtained by fitting (36) to supernova observations by means of a maximum-likelihood analysis discussed in the next section. There are obvious advantages in choosing the Ansatz (35) namely, it is exact for the cosmological constant $w = -1$ ($A_2 = A_3 = 0$) as well as for quintessence with $w = -\frac{1}{3}$ ($A_1 = A_3 = 0$) and $w = -\frac{2}{3}$ ($A_1 = A_2 = 0$). Furthermore, the presence of the term $\Omega_{m} x^3$ in (35) ensures that the Ansatz correctly reproduces the matter-dominated epoch at early times ($z > 1$). The presence of this term also allows us to incorporate information pertaining to the value of the matter density and, as we shall soon demonstrate, permits elaborate statistical analysis with the introduction of priors on $\Omega_{m}$.

The statefinder pair for the polynomial fit (35) can be written in terms of $x = 1 + z$ as follows:

$$r(x) = \frac{\Omega_{m} x^3 + A_1}{\Omega_{m} x^3 + A_1 + A_2 x + A_3 x^2},$$

(37)

$$s(x) = \frac{2 (A_1 + 2 A_2 x + A_3 x^2)}{3 (3 A_1 + 2 A_2 x + A_3 x^2)}.$$

(38)

It is also straightforward to obtain expressions for the cosmological parameters $q$ and $w$ by substituting (35) in (4) and (6), respectively.

In Fig. 3 we show the maximum deviation between the exact value of the luminosity distance and the fit-estimated approximate value for a class of dark energy models. For LCDM ($w = -1$) and two quintessence models ($w = -2/3$, $w = -1/3$), the Ansatz (36) returns exact values. (The Ansatz is also exact for SCDM.) For the two tracker and Chaplygin gas models that we consider, the luminosity distance is determined to better than 1 per cent accuracy for a conservative range in $\Omega_{m}$ ($0.2 < \Omega_{m} < 0.5$). We therefore conclude that the polynomial fit (36) is very accurate and can safely be applied to reconstruct the properties of dark energy models.
In this paper we will use the polynomial fit (36) to perform a model-independent reconstruction of dark energy using the synthetic SNAP supernova data discussed earlier. Some details of our approach, which involves the maximum-likelihood method, will be discussed in Section 4.2. Our results for the cosmological reconstruction of dark energy using the statefinder will be presented in Section 5.

Although we will mainly work with the polynomial Ansatz (35) to reconstruct the properties of the statefinders, it is worthwhile to summarize some of the alternate approaches to the cosmological reconstruction problem.

(ii) Fitting functions to the luminosity distance $D_L$. An interesting complementary approach to the reconstruction exercise is to find a suitable fitting function for the luminosity distance. Such an approach was advocated in Huterer & Turner (1999) and Saini et al. (2000). In Huterer & Turner (1999) a polynomial fit for the luminosity distance was suggested, which had the form

$$D_L(z) = \sum_{i=1}^{N} a_i z^i.$$  

The Ansatz (39) was examined in Weller & Albrecht (2002), who demonstrated that this approximation does not give an accurate reconstruction of the equation of state of dark energy. Similar conclusions will also be reached by us later in this paper in connection with the reconstruction of the statefinder pair using (39).

A considerably more versatile and accurate fitting function to the luminosity distance is (Saini et al. 2000)

$$\frac{D_L(z)}{1+z} = \sum_{i=1}^{N} a_i z^i.$$  

The maximum deviation $|\Delta D_L/D_L|$ between the actual value of the luminosity distance in the redshift range $z = 0$–10 in a DE model and that calculated using the polynomial fit equation (36). The solid line at $\Delta D_L/D_L = 0$ represents models with $w = -1, -2/3$ and $-1/3$, for which the polynomial fit returns exact values. The dashed lines from top to bottom represent the tracker potential $V(\phi) = V_0/\phi^\alpha$ for $\alpha = 1$ and 2, respectively. The dotted lines represent Chaplygin gas models with $\kappa = 0.5$ and 2 (top to bottom).


\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{The maximum deviation $|\Delta D_L/D_L|$ between the actual value of the luminosity distance in the redshift range $z = 0$–10 in a DE model and that calculated using the polynomial fit equation (36). The solid line at $\Delta D_L/D_L = 0$ represents models with $w = -1, -2/3$ and $-1/3$, for which the polynomial fit returns exact values. The dashed lines from top to bottom represent the tracker potential $V(\phi) = V_0/\phi^\alpha$ for $\alpha = 1$ and 2, respectively. The dotted lines represent Chaplygin gas models with $\kappa = 0.5$ and 2 (top to bottom).}
\end{figure}

where $A_1$, $A_2$ and $A_3$ are parameters for which the values must be determined by fitting (40) to observations. Important properties of this function are that it is valid for a wide range of models and that it exactly reproduces the results both for SCDM ($\Omega_m = 1$) and the steady-state model ($\Omega_\Lambda = 1$). As demonstrated in Saini et al. (2000), an accurate analytical form for $D_L$ allows us to reconstruct the Hubble parameter by means of the relation (34). Cosmological parameters including $q(z)$, $w(z)$, $r(z)$, $s(z)$ can now be easily reconstructed using (4), (6) and (32).

(iii) Fitting functions to the equation of state. A somewhat different approach fits the equation of state of dark energy by the first few terms of a Taylor series expansion (Weller & Albrecht 2002):

$$w_{DE}(z) = \sum_{i=0}^{N} w_i z^i.$$  

For $N = 1$ the luminosity distance can be expressed as

$$D_L(z) = \frac{c}{H_0} \int_1^{1+z} \frac{dx}{\sqrt{\Omega_m x^3 + \Omega_x}},$$

$$\Omega_x = (1 - \Omega_m) x^{3(1+w_0)} \exp[3w_0(x - 1)].$$  

A modification of the above prescription was suggested in Gerke & Efstathiou (2002), which used a logarithmic expansion of the equation of state of dark energy:

$$w(z) = w_0 - \alpha \ln(1 + z),$$  

where $\alpha = dw/d(ln a)$. Yet another approach (Maor et al. 2002) advocated a quadratic fit to the total equation of state:

$$w_{T}(z) = w_0 + w_1 z + w_2 z^2,$$

where the total equation of state, $w_{T}(z)$, is defined in terms of the equation of state of dark energy, $w(z)$, as

$$w_{T}(z) = \frac{w(z)}{1 + (\Omega_m/1 - \Omega_m) \exp [-3 \int_1^{1+z} w(x) dx / x]}.$$  

Other approaches to the reconstruction problem can be found in Chiba & Nakamura (2000), Corasaniti & Copeland (2003) and Linder (2003).

4.2 Maximum-likelihood estimation of cosmological parameters

In order to determine how effective the statefinders are in discriminating between dark energy models, we adopt the method of maximum-likelihood estimation to our reconstruction exercise. Supernova data are expected to improve greatly over the next few years. This improvement will be spurred by ongoing efforts by the Supernova Cosmology Project\(^1\) and the High-z Supernova Search Team,\(^2\) as well as planned surveys such as the Nearby SN Factory\(^3\) (300 supernovae at $z \lesssim 0.1$) and SNAP\(^4\) (∼2000 supernovae at $z \lesssim 1.7$). We shall use data simulated according to the specifications of SNAP—a space-based mission, which is expected to greatly increase both the number of Type Ia supernovae observed and the accuracy of supernovae observations.

\(^1\)http://www-supernova.lbl.gov
\(^2\)http://cfa-www.harvard.edu/cfa/ori/Research/supernova/HighZ.html
\(^3\)http://snfactory.lbl.gov
\(^4\)http://snfactory.lbl.gov
Table 1. Expectations from SNAP for a single-year period of observation.

<table>
<thead>
<tr>
<th>Redshift interval</th>
<th>Number of supernovae</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z = 0.0 - 0.2$</td>
<td>50</td>
</tr>
<tr>
<td>$z = 0.2 - 1.2$</td>
<td>1800</td>
</tr>
<tr>
<td>$z = 1.2 - 1.4$</td>
<td>50</td>
</tr>
<tr>
<td>$z = 1.4 - 1.7$</td>
<td>15</td>
</tr>
</tbody>
</table>

SNAP

The SNAP mission is expected to observe about 2000 Type Ia supernovae each year, over a period of 3 yr, according to the specifications given in Table 1. We assume a Gaussian distribution of uncertainties and an equidistant sampling of redshift in four redshift ranges. The errors in the redshift are of the order of $\delta z = 0.002$. The statistical uncertainty in the magnitude of supernovae is assumed to be constant over the redshift range $0 \leq z \leq 1.7$ and is given by $\sigma_{\text{mag}} = 0.15$. The systematic uncertainty limit is $\sigma_{\text{sys}} = 0.02$ mag at redshift $z = 1.5$. For simplicity we assume a linear drift from $\sigma_{\text{sys}} = 0$ at $z = 0$ to $\sigma_{\text{sys}} = 0.02$ at $z = 1.5$, so that the systematic uncertainty on the model data is given by $\sigma_{\text{sys}}(z) = (0.02/1.5)z$.

Optimizing the model with 2000 data points is somewhat time consuming, therefore we produced a smaller number of binned supernovae luminosity distances by binning the data in a redshift interval $\Delta z = 0.02$. This interval is comparable to the statistical uncertainty in the redshift measurement of high-$z$ supernovae due to the peculiar velocities of the galaxies in which they reside, which is typically of the order of $v_{\text{peculiar}} \approx 1000 \, \text{km} \, \text{s}^{-1}$. In our experiment we smoothed the data in the first three redshift bins of Table 1 by binning, the last interval had relatively fewer supernovae and was left unbinned. The statistical error in magnitude, and hence in the luminosity distance is weighed down by the factor of $1/\sqrt{N_{\text{bin}}}$, where $N_{\text{bin}}$ is the number of supernovae in each bin.

We use SNAP specifications to construct mock supernovae catalogues. We may then use the method of maximum-likelihood parameter estimation on this mock data to estimate the different cosmological parameters of interest.

Maximum-likelihood estimation

The observable quantity for a given supernova is its bolometric or ‘apparent’ magnitude $m$, which is a measure of the light flux received by us from the supernova. To convert from $m$ to cosmological distance, we use the well-known relationship between the luminosity distance $D_L$ and the bolometric magnitude

$$m = M_0 + 25 + 5 \log D_L,$$

where $M_0$ is the absolute magnitude of the supernovae and the luminosity distance $D_L$ is measured in the units of Mpc. (For Type Ia supernovae, the typical apparent magnitude at $z = 1$ is about 25, which shows that we are dealing with very faint objects at that redshift.) Type Ia supernovae are excellent standard candles, and the dispersion in their apparent magnitude is $\sigma_{\text{mag}} = 0.15$, which is nearly independent of the supernova redshift. To relate this to the dispersion in the measured luminosity distance, we use equation (46) to obtain

$$\frac{\sigma_{\text{dist}}}{D_L} = \frac{\ln 10}{5} \sigma_{\text{mag}} = 0.069.$$

While constructing mock supernovae catalogues we shall assume that the errors in the luminosity distance are Gaussian with zero mean and dispersion given by the above expression ($\sim 7$ per cent), the normalized likelihood function is therefore given by

$$L(y_i, p_k) = \prod_{i=1}^{N_{\text{dat}}} \left( \frac{1}{\sqrt{2\pi\sigma_{\text{dist}}(z_i)}} \right) \times \exp \left\{ -\frac{1}{2} \left[ \frac{y_i - D_L^0(z_i; p_k)}{\sigma_{\text{dist}}(z_i)} \right]^2 \right\},$$

where the index $i$ ranges from 1 to $N_{\text{dat}}$, which is the number of supernovae in our sample, and we have denoted the fiducial supernovae luminosity distance at a redshift $z = z_i$ as $y_i = D_L(z_i)$, where $D_L(z)$ is the luminosity distance simulated with SNAP specifications for a chosen background model using (27). $p_k$ are the parameters of the fitting function. (We shall mostly exploit the fitting function (36) for which $p_k = A_1$). We maximize the Likelihood function $L$ to obtain the maximum-likelihood values of the parameters of the fitting function. In practice we minimize the negative of the log-likelihood, which is given by

$$\mathcal{L} = -\log(L) = \frac{1}{2} \sum_i \left[ \frac{y_i - D_L^0(z_i; p_k)}{\sigma_{\text{dist}}(z_i)} \right]^2,$$

where a constant term arising from the multiplicative factor is ignored. We are interested in estimating the statefinder pair $r(z)$ and $s(z)$ and the deceleration parameter $q(z)$ from synthetic SNAP data.

The priors that we have used for our reconstruction exercise are the following.

The values of $H_0$ and $M_0$ (the absolute magnitude of supernovae) are assumed to be known. We consider a flat universe, so that the present-day value of $\Omega_k$ is given by $\Omega_k = 1 - \Omega_m = A_1 + A_2 + A_3$. Also, when optimizing the model, we may assume priors on $\Omega_m$ using information from other observations. This leaves only three free parameters (including $\Omega_m$ on which bounds can be specified). (Optimizing without priors we found the variances of $A_1$ to be much larger if no bounds were specified on $\Omega_m$).

Reconstruction of cosmological parameters

Using the procedure described in detail above we now propose to reconstruct different cosmologically important quantities using SNAP data. We shall focus our attention to the statefinder pair $\{r(z), s(z)\}$, the deceleration parameter $q(z)$ and the cosmic equation of state $w(z)$. Using SNAP specifications, we generated 1000 data sets $\{z_i^*, D_L^i, \sigma_{\text{dist}}^i, \sigma_{\text{sys}}^i, \sigma_{\text{mag}}^i\}$, where the index $i$ runs from 1 to 1000 and the index $j$ from 1 to 2000, with the LCDM as our fiducial model. For each of these experiments, the best-fitting parameters, $A_{1j}$ and other cosmological quantities is computed as

$$r(z) = \frac{1}{1000} \sum_{i=1}^{1000} r_i(z),$$

$$s(z) = \frac{1}{1000} \sum_{i=1}^{1000} s_i(z),$$

and so on for other quantities. Here the angular brackets denote an ensemble average. We may also calculate the covariance matrix of these quantities at different redshifts which is given by

$$[C_{ij}] = \begin{pmatrix} C_{rr} & C_{r\ell} & C_{r\ell} \\ C_{\ell r} & C_{\ell\ell} & C_{\ell\ell} \\ C_{\ell r} & C_{\ell\ell} & C_{\ell\ell} \end{pmatrix},$$


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where
\[ C_{rr} = \langle r(z)^2 \rangle - \langle r(z) \rangle^2, \]  
\[ C_{ss} = \langle s(z)^2 \rangle - \langle s(z) \rangle^2, \]  
\[ C_{rs} = \langle r(z) s(z) \rangle - \langle r(z) \rangle \langle s(z) \rangle, \]  
and the angular averages are evaluated as in (50).

5 RESULTS AND DISCUSSION

From the results we can estimate the accuracy with which the Ansatz recovers model-independent values of different cosmological parameters, especially the statefinder pair introduced in Sahni et al. (2003). We can also determine whether this pair is useful in discriminating the cosmological constant model from other models of dark energy.

5.1 Cosmological reconstruction for an LCDM fiducial model

Synthetic supernova data are generated for a fiducial LCDM model with $\Omega_\Lambda = 0.7$, $\Omega_m = 0.3$ and assuming SNAP specifications summarized in the previous section. Next, we determine the statefinder pair and other cosmological parameters as functions of the redshift using the polynomial fit to dark energy (35). Our results can be represented in two complementary ways. First, we show the confidence levels in the $r_0 - s_0$ space, where the subscript '0' denotes the present-day value of the statefinders. We also find it useful to consider the integrated, averaged quantities:
\[ \bar{r} = \frac{1}{z_{\text{max}}} \int_{0}^{z_{\text{max}}} r(z) \, dz, \]  
\[ \bar{s} = \frac{1}{z_{\text{max}}} \int_{0}^{z_{\text{max}}} s(z) \, dz. \]  

For the LCDM model, $r$ and $s$ do not evolve with time, therefore we find that $\bar{r} = 1$ and $\bar{s} = 0$. However, for most other models of dark energy the statefinder pair evolves and the averaged quantities differ from their present-day values. As a result of averaging over redshift, the averaged parameters $\bar{r}$, $\bar{s}$ are in many cases less noisy than $r_0$, $s_0$. The maximum redshift used for our reconstruction is $z_{\text{max}} = 1.7$. One of the results of our analysis is that the deceleration parameter $q$ is very well determined (see Fig. 9 later), therefore we also construct a second statefinder pair, $\{s, q\}$, which will be shown to be an excellent diagnostic of dark energy.

Fig. 4 shows the 99.73 per cent confidence level in $\{\bar{s}, \bar{r}\}$ (left-hand panel) and $\{\bar{s}, q\}$ (right-hand panel) for the fiducial LCDM model with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$. For comparison we also show values of $r$, $s$, $q$ for quiessence, kinessence and Chaplygin gas models. From this figure we see that the statefinders can easily distinguish LCDM from: (i) quiessence with $w \gtrsim -0.9$; (ii) the Chaplygin gas with $\kappa \lesssim 15$; (iii) the quiessence potential $V(\phi) \propto \phi^{-\alpha}$, $\alpha \gtrsim 1$; and (iv) the DDG braneworld models discussed in Deffayet et al. (2002a).

The above analysis assumed that the value of $\Omega_m$ is known exactly. However, in practice it will be some time before $\Omega_m$ is known to 100 per cent accuracy and it is only natural to expect some amount of uncertainty in the observational value of this important physical parameter. We incorporate this uncertainty by marginalizing over

\[ \bar{r} = \frac{1}{z_{\text{max}}} \int_{0}^{z_{\text{max}}} r(z) \, dz, \]  
\[ \bar{s} = \frac{1}{z_{\text{max}}} \int_{0}^{z_{\text{max}}} s(z) \, dz. \]  

Figure 4. This figure shows 3σ confidence levels in the averaged statefinders (a) $\{\bar{s}, \bar{r}\}$ and (b) $\{\bar{s}, q\}$. The polynomial fit to dark energy, equation (35) has been used to reconstruct the statefinders for an LCDM fiducial model with $\Omega_m = 0.3$. The dashed line above the LCDM fixed point represents the family of quiessence models having $w = \text{constant}$. The dashed line below the LCDM fixed point shows Chaplygin gas models. It should be noted that the best-fitting point in both (a) and (b) coincides with the LCDM fixed point (solid star). In the upper half of both panels, the solid rhombi correspond to tracker potentials $V = V_0/\phi^k$, while triangles show $w = \text{constant}$ quiessence models. In the lower half of both panels, solid hexagons show Chaplygin gas models with different values of $\kappa$. (The constant $\kappa$ gives the initial ratio between cold dark matter and the Chaplygin gas. Only models with $\kappa \gtrsim \Omega_m/(1 - \Omega_m)$ are permitted by theory, see equations 15 and 17). All models, with the exception of the braneworld model, have $\Omega_m = 0.3$ currently. The braneworld model is marked by a cross and corresponds to the DDG model (22) with $\Omega_m = 0.24$, which best fits current supernova data. Comparing the left- and right-hand panels we find that $\{\bar{s}, q\}$ is a slightly better diagnostic than $\{\bar{s}, \bar{r}\}$ for tracker and quiessence models and can be used to rule out a constant equation of state $w < -0.9$ at the 3σ level if the value of $\Omega_m$ is known exactly.
the value of $\Omega_m$. Two priors will be incorporated into our analysis, the weak Gaussian prior: $\Omega_m = 0.3 \pm 0.05$ and the stronger Gaussian prior: $\Omega_m = 0.3 \pm 0.015$.

Figs 5(a)–(d) show the confidence levels in the statefinder pairs $\{\bar{r}, \bar{s}\}$, $\{q, \bar{q}\}$, $\{r_0, s_0\}$ and $\{q_0, s_0\}$, respectively. For the purposes of discrimination we also show the values of $r, s, q$ for quiessence, kinessence, Chaplygin gas and braneworld models. Fig. 5(a) shows that the diagnostic $\{\bar{r}, \bar{s}\}$ permits the LCDM model to be distinguished from quiessence models with $w > -0.8$, quintessence models with $\alpha \geq 1$, Chaplygin gas models with $\kappa \lesssim 6$ and braneworld models at the 99.73 per cent confidence level and after applying the strong Gaussian prior of $\Omega_m = 0.3 \pm 0.015$. The discriminatory power of the statefinder clearly worsens for the weaker prior $\Omega_m = 0.3 \pm 0.05$.

The situation can be dramatically improved if, instead of working with $\{\bar{r}, \bar{s}\}$ we use the diagnostic $\{\bar{q}, \bar{q}\}$ (see Fig. 5b). We find in this case that the fiducial LCDM model can be distinguished from quiessence with $w > -0.85$ and the braneworld model at the...
99.73 per cent confidence level even for the weak prior $\Omega_m = 0.3 \pm 0.05$. In Figs 5(c) and (d) we plot the confidence levels for current values of the pair $\{s_0, r_0\}$ and $\{s_0, q_0\}$. A few important points need to be noted here.

(i) The semimajor axis of the confidence ellipse for $\{s, q\}$ is tilted away from the dashed curve, representing constant-$w$ models (quiescence). This enables the second statefinder pair $\{r, s\}$ to be somewhat better at discriminating between LCDM and quintessence models than $\{r, s\}$. For instance, the current value $\{s_0, q_0\}$ can discriminate the cosmological constant model from quintessence models having $w \gtrsim -0.9$, quintessence models with $\alpha \geq 1$, Chaplygin gas models with $\kappa \lesssim 2$ and the braneworld model even after applying the weak Gaussian prior $\Omega_m = 0.3 \pm 0.05$.

(ii) For Chaplygin gas models averaging over redshift considerably enhances the discriminatory prowess of both $\{s, r\}$ and $\{s, q\}$.

(iii) From Fig. 5(d) we find that marginalization over $\Omega_m$ has only a small effect on the diagnostic $\{s_0, q_0\}$, which contributes to making this statefinder pair a much better discriminator of dark energy than $\{r_0, s_0\}$ if the value of $\Omega_m$ is uncertain.

Our results shown in Figs 4 and 5 clearly demonstrate that both $\{r, s\}$ as well as $\{s, q\}$ are excellent diagnostics of dark energy with the latter being somewhat more sensitive than the former.

We now proceed to examine the information content in the cosmological parameters when examined individually. In Fig. 6, we plot the variation of the ensemble-averaged value $\langle r(z) \rangle$ with redshift. The $1\sigma$ error bounds are shown for two different priors on $\Omega_m$ and for the case when the value of $\Omega_m$ is known exactly. This figure shows that $\langle r(z) \rangle$ is a good diagnostic of dark energy and allows us to discriminate (at the 68.3 per cent confidence level, CL) between different dark energy models and LCDM. The discrimination improves at higher redshifts ($z \gtrsim 0.8$), especially if the uncertainty in the value of $\Omega_m$ is small. The sweet spot for this parameter, i.e. the point at which $\langle r(z) \rangle$ is most accurately determined, is at $z \sim 1.4$. (For earlier work on the sweet spot see Huterer & Turner 1999; Weller & Albrecht 2002; Huterer & Starkman 2003.)

Fig. 7 shows the variation of the ensemble-averaged value of the second statefinder $\langle s(z) \rangle$ with redshift. Again $1\sigma$ errors for the two priors $\Omega_m = 0.3 \pm 0.05$, $0.3 \pm 0.015$ and when the value of $\Omega_m$ is known exactly ($\Omega_m = 0.3$) are shown. We see that $\langle s(z) \rangle$ is determined even more accurately than $\langle r(z) \rangle$, and therefore can serve as a better diagnostic of dark energy. For the strong Gaussian prior $\Omega_m = 0.3 \pm 0.015$ (or when $\Omega_m$ is known exactly) the value of $\langle s(z) \rangle$ is very well determined even at higher redshifts, its sweet spot being at $z \sim 1.4$. For the weak prior $\Omega_m = 0.3 \pm 0.05$, $s$ is not so well determined at high redshifts, but it is still accurate enough to distinguish between rival models of dark energy. Two points are of interest here. First, $r$ and $s$ are both much more accurately determined at higher redshifts if the value of $\Omega_m$ is accurately known. This explains why the parameters $\{r, s\}$ perform better as discriminators than $\{r_0, s_0\}$. Secondly, the sweet spot for both of these parameters appears at $z \sim 1.4$, only if the value of $\Omega_m$ is accurately known. Upon
The dashed line above the LCDM point represents the quiescence models, and that below the LCDM point represents different Chaplygin gas models. The solid triangles represent model values for constant-\(\omega\) quiescence models, and the solid hexagons represent Chaplygin gas models with different values of \(\kappa\). Only those Chaplygin gas models with \(\kappa \gtrsim \Omega_m/(1 - \Omega_m)\) are allowed. For all the dark energy models, \(\Omega_m = 0.3\) is used. The ellipses represent the 3\(\sigma\) confidence levels in the \(\bar{r} - \bar{s}\) space for the exact prior \(\Omega_m = 0.3\). In (a), the dark grey contour in solid outline represents the confidence level for the cosmological constant fiducial model obtained when \(r, s\) are averaged over the redshift range \(z = 0.1-1.7\). The light grey contour with dotted outline is the confidence level for the \(\alpha = 1\) tracker model obtained when the averaging is over the entire redshift range. In (b), we show confidence levels for the cosmological constant model (dark grey contour) and for the \(\alpha = 1\) tracker model (light grey contour) with the averaging done from \(z = 1\) to 1.7. Remarkably, using the statefinders \(\bar{r}, \bar{s}\) one can rule out quintessence models with \(w \gtrsim -0.95\) and Chaplygin gas models with \(\kappa \lesssim 24\) at 3\(\sigma\) if only very high-redshift supernovae belonging to the redshift interval \(1 \lesssim z \lesssim 1.7\) are considered. The reason for this is that both \(r(z)\) and \(s(z)\) are determined with great accuracy at \(z \gtrsim 1\). Indeed, a ‘sweet spot’ at \(z_s \approx 1.4\) ensures that both \(r(z_s)\) and \(s(z_s)\) are known with great accuracy at that point – see Figs 6 and 7.

Fig. 8 shows how sweet spot information can be used to improve the statefinders as a diagnostic tool. For both \(r\) and \(s\) the sweet spot appears at high redshifts. Therefore, one expects that the discriminatory prowess of the statefinders will improve considerably if only data at \(z \gtrsim 1\) is considered. This is indeed the case. Fig. 8 shows 3\(\sigma\) confidence levels in \(\bar{r}, \bar{s}\) for two cases: (a) the statefinder pair is averaged over the full redshift range \(0 \leq z \leq 1.7\), (b) the statefinder pair is averaged over the high-redshift range \(1 \leq z \leq 1.7\); \(\Omega_m = 0.3\) for both cases. The dark grey ellipses represent the confidence level for the LCDM model, and the light grey ellipses represent the confidence level for the \(\alpha = 1\) quiescence model. We see that there is a dramatic improvement in the determination of the statefinder pair in Fig. 8(b), where the statefinders have been averaged only for \(z \gtrsim 1\). From Fig. 8(a) we see that \(\bar{r}, \bar{s}\) can discriminate between LCDM and quiescence models with \(w \gtrsim -0.90\) and Chaplygin gas models with \(\kappa \lesssim 15\), whereas Fig. 8(b) shows that \(\bar{r}, \bar{s}\) can discriminate between LCDM and quiescence models with \(w \gtrsim -0.95\) and Chaplygin gas models with \(\kappa \lesssim 24\). We therefore conclude that high-redshift supernovae can play a crucial role in constraining properties of dark energy and our results support the views expressed in Linder & Huterer (2003). We must, however, note that in order to use sweet spot information optimally the value of \(\Omega_m\) must be known to very high accuracy. Indeed, for the Gaussian prior of \(\Omega_m = 0.3 \pm 0.05\), a consideration of only high-redshift supernovae does not lead to any improvement in the results. This is because, as seen from Figs 6 and 7, after marginalization over \(\Omega_m\), the sweet spot for both \(r(z)\) and \(s(z)\) disappears. The second point to note is that the angle of inclination of the semimajor axis of the ellipse to the \(w = \text{constant curve (quiescence)}\) appears to depend upon the redshift range over which the statefinder pair is being averaged.

Fig. 9 shows the variation of the mean deceleration parameter \(q(z)\) with redshift. We see that \(q(z)\) is very well determined over the entire range \(0 \leq z \leq 1.7\). This justifies our choice of \(\{s, q\}\) as the second statefinder pair. Indeed, the behaviour of \(r(z)\) and \(s(z)\), on one hand, and \(q(z)\), on the other hand, is in some ways complementary. While both \(r(z)\) and \(s(z)\) are determined to increasing accuracy at higher redshifts, the deceleration parameter is very well determined at lower redshifts and the sweet spot for this parameter appears at the redshift \(z_s \approx 0.25\). It is interesting that, in sharp contrast to what was earlier observed for \(r\) and \(s\), the sweet spot in \(q(z)\) persists even after we marginalize over \(\Omega_m\)!

From Fig. 9 we can also determine the value of the acceleration epoch (the redshift at which the Universe began accelerating). We find that the acceleration epoch is determined quite accurately: \(z(q = 0) = 0.66 \pm 0.06\).

Fig. 10 shows maximum-likelihood contours for the pair \(\bar{r}, \bar{q}\), where \(\bar{q}\) has been averaged over the redshift interval \(0 < z < 0.4\) while \(\bar{r}\) has been averaged over \(1 \leq z \leq 1.7\). This figure clearly...
models with the tracker potential between dark energy models at low redshifts $\lesssim 1$.

The solid line shows the best-averaged $(q(z))$ with $\sigma/Omega_1$ for fitting case the best-model can be distinguished up to $\sigma/Omega_1$ since it can distinguish LCDM from quiessence models with $w_{10}$ show that the ability of the averaged state fits to dark energy, equation (35), for the prior $Omega_1 = 0.3$ exactly. Shaded regions represent the 1 level provided we restrict ourselves to lower value than the $q(z)$ for the cosmological constant model. We now briefly examine the accuracy of the statefinder pair and the Ansatz (35) in determining the statefinder pair for a fiducial model other than LCDM. We know that the Ansatz returns exact values for the cosmological constant $w = -1$ as well as for quiessence having the constant equation of state $w = -2/3$ and $-1/3$ (see Fig. 3). It is therefore important to study the accuracy of the statefinder in reconstructing the properties of dark energy in models in which both the dark energy density as well as the equation of state vary with time and for which the Ansatz (35) is approximate. For this purpose we shall work with a fiducial dark energy model that evolves under the influence of the tracker potential $V = V_0/\phi$ and use the Ansatz (35) in tandem with the statefinders (32) to reconstruct the properties of dark energy.

Fig. 12 shows our results in terms of $3\sigma$ confidence levels in $(\bar{r}, \bar{s})$. We find that, if the value of $Omega_1$ is known to reasonable accuracy ($Omega_1 = 0.3 \pm 0.015$) then the averaged statefinder pair $(\bar{r}, \bar{s})$ is able to distinguish the model $V = V_0/\phi$ from the model $V = V_0/\phi^2$ at the $3\sigma$ level. As expected, a large uncertainty in the current value of the matter density reduces the efficiency of this diagnostic and the two models $V = V_0/\phi$ and $V_0/\phi^2$ cannot be clearly distinguished if the uncertainty in $Omega_1$ is increased to $Omega_1 = 0.3 \pm 0.05$.

Figs 13 and 14 show the performance of the ensemble-averaged statefinders $(r(z))$ and $(s(z))$. As in the earlier case when our fiducial model was assumed to be LCDM, we find that the errors in $r(z)$ and $s(z)$ are small. However, a slight bias in the determination of the statefinders exists at low redshifts so that for $z \lesssim 0.4$ the value of the best fit $(r, s)$ is larger (smaller) than the exact fiducial value. Averaging over the entire redshift range can significantly reduce this bias and we conclude that the Ansatz (35) works well even for those dark energy models for which it does not return exact values. One should also note the reappearance of the sweet spot for the statefinders $r$ and $s$ in Figs 13 and 14. For the statefinder $r(z)$ the sweet spot appears at $z \simeq 1.6$, while for $s(z)$ the sweet spot is at $z \simeq 1.2$. As in the case of the LCDM model, one can try and constrain the properties of dark energy further by evaluating the averaged statefinders using supernovae data only from $z \geq 1$. Our results shown in Fig. 8 demonstrate that the confidence ellipse for $(\bar{r}, \bar{s})$ becomes much smaller when the averaging is done over the redshift range $1 \leq z \leq 1.7$ than when the averaging is over the entire redshift range. The performance of the deceleration parameter $(q(z))$ for this quiessence model is shown in the Fig. 15. We see that the deceleration parameter is very accurately determined. The sweet spot for the deceleration parameter occurs at lower redshifts, at $z \simeq 0.2$, and by combining higher-redshift data in determining $\bar{s}$ with lower-redshift data for determining $\bar{q}$ we can significantly improve the errors on the second statefinder pair $(s, q)$, as demonstrated in Fig. 10. As was the case for the LCDM model, the sweet spot gradually disappears if we incorporate the prevailing uncertainty in the value of the matter density by marginalizing over large values of $Omega_1$.
5.3 Cosmological reconstruction using other fitting functions

For comparison, we also carry out the reconstruction exercise using two of the fitting functions described in Section 4.1. In Fig. 16 we show the results for \( \{ \tilde{r}, \tilde{s} \} \) using the polynomial fit to the luminosity distance (39) with \( N = 5 \). In this case, because of the nature of the Ansatz, it is not possible to place any priors on \( \Omega_m \).

We find that this Ansatz does not perform well for the statefinder pair. First, it does not determine \( \tilde{r}, \tilde{s} \) with the accuracy seen in the case of the polynomial fit to dark energy. Secondly, the best-fitting value for \( \{ \tilde{r}, \tilde{s} \} \) is biased with respect to the fiducial LCDM value. Additionally, the errors on both \( r \) and \( s \) are unacceptably large due to which one cannot distinguish between the cosmological constant model and kinessence models with \( \alpha < 6 \), quiescence models with \( w \leq -0.4 \) and Chaplygin gas models with \( \kappa \geq 2 \) at the 3\( \sigma \) confidence level. Even at 1\( \sigma \) (68.3 per cent CL), one can only discriminate LCDM from kinessence models with \( \alpha \geq 3 \), quiescence models with \( w \gtrsim -0.6 \), and Chaplygin gas models with \( \kappa \leq 3 \). We therefore conclude that the polynomial fit to the luminosity distance (39) is not very useful for the reconstruction of the statefinders.

We also carry out a similar reconstruction exercise using the polynomial fit to the equation of state (41) with \( N = 1 \). This Ansatz can accommodate priors on \( \Omega_m \) and we expect it to perform better than the polynomial fit to the luminosity distance. Indeed, Fig. 17 clearly demonstrates that a two-parameter Taylor expansion in the equation of state is better than a five-parameter expansion in the luminosity distance (Our results in this context support the earlier observations of Weller & Albrecht 2002.) From Fig. 17 we find that the Ansatz (41) can distinguish the cosmological constant model from quiescence models with \( w \gtrsim -0.6 \), kinessence models with \( \alpha \geq 3 \) and Chaplygin gas models with \( \kappa \leq 3 \) at the 99.73 per cent confidence limit after we have marginalized over \( \Omega_m \) with a Gaussian prior of \( \Omega_m = 0.3 \pm 0.05 \). However, a comparison of Fig. 17 with Fig. 5 shows that the equation of state expansion (41) is not quite as accurate as the polynomial fit to dark energy (35) in reconstructing the statefinder pair \( \{ \tilde{r}, \tilde{s} \} \). We therefore conclude that the statefinders can be reconstructed using several complementary methods. The polynomial fit for dark energy (35), by providing a good reconstruction of the parameters \( \tilde{r}, \tilde{s}, \tilde{q} \), can successfully be used for the model-independent reconstruction of dark energy.

6 CONCLUSIONS AND DISCUSSION

This paper contains an in-depth study of properties of the statefinder diagnostic introduced in Sahni et al. (2003). The statefinder pairs \( \{ r, s \} \) and \( \{ s, q \} \) have the potential to successfully discriminate between a wide variety of dark energy models, including the cosmological constant, quintessence, the Chaplygin gas and braneworld models. The statefinders play a particularly important role in the case of modified gravity theories such as string/M-theory-inspired scalar–tensor models and braneworld models of dark energy, for which the equation of state is not a fundamental physical entity and therefore does not provide us with an adequate description of
Figure 11. Variation of \((w(z))\) with \(z\) for the cosmological constant model. The solid line shows best-fitting \((w(z))\) averaged over all realizations calculated with the polynomial fit to dark energy, equation (35), for the prior \(\Omega_m = 0.3\) exactly. The dot-dashed line represents the exact value of \((w) = -1\) for the cosmological constant model. Shaded regions represent the 1\(\sigma\) confidence levels for \((w)\). The dark grey outer contour is for the Gaussian prior \(\Omega_m = 0.3 \pm \sigma_{\Omega_m}\) with \(\sigma_{\Omega_m} = 0.05\), the grey contour in the middle uses the Gaussian prior \(\sigma_{\Omega_m} = 0.015\) and the light grey contour uses \(\Omega_m = 0.3\) exactly. The dotted and dashed lines represent the model values of \((w)\) for different constant-\(w\) quiescence models and for kinetics models with the tracker potential \(V(\phi) = V_0/\phi^\alpha\) for different values of \(\alpha\), respectively. We see that \(w\) can distinguish between the cosmological constant model and other dark energy models only at \(z \lesssim 1\).

the accelerating Universe. Our results, summarized in Figs 1 and 2, show that the statefinders \(r, s\) considerably extend and supplement traditional measures of cosmological dynamics such as the deceleration parameter \(q\). To give a concrete example of how this can happen consider two (or more) cosmological dark energy models that have identical (hence degenerate) values of \(q_0\). Although such models will have the same current value of \(\dot{a}/a\), the value of the third derivative \(\dddot{a}\) (hence \(r\) and \(s\)) will in general be different in both models. The statefinder pairs \(\{r, q\}\) and \(\{s, q\}\) therefore provide us with a ‘phase-space’ picture of dark energy that distinguishes dynamical dark energy models both from each other as well as from the cosmological constant and helps break cosmological degeneracies present in rival models of dark energy. The statefinder \(s\) is remarkably sensitive to the total pressure of all forms of matter and radiation in the Universe. As a result \(s\) remains sensitive to the presence of dark energy even at moderately high redshifts \(z \sim 10\), when the Universe is matter-dominated.

Forthcoming space-based missions such as SNAP are expected to greatly increase and improve the current Type Ia supernova inventory. Anticipating this development we have carried out a maximum-likelihood analysis that combines the statefinder diagnostic with realistic expectations from the SNAP experiment. Our results, summarized in Figs 4–10, show that both \(\{r, s\}\) as well as \(\{s, q\}\) are excellent diagnostics of dark energy. If the value of \(\Omega_m\) is known exactly, then the averaged-over-redshift statefinder pair \(\{\bar{s}, \bar{q}\}\) can distinguish between the cosmological constant model \((w = -1)\) and a dark energy model having \(w = -0.9\) at the 99.73 per cent CL. It can also distinguish (at the same level of confidence) the cosmological constant (LCDM) from Chaplygin gas models with \(k \lesssim 7\).

Bearing in mind the fact that the current observational data determine \(\Omega_m\) to a finite level of accuracy, we have probed how well the statefinder fares as a diagnostic after one incorporates the prevailing uncertainty in the value of the matter density by marginalizing over values of \(\Omega_m\) that are uncertain. Somewhat surprisingly, the statefinder fares rather well even for the relatively weak prior \(\Omega_m = 0.3 \pm 0.05\). In this case, by employing the diagnostic \(\{s_0, q_0\}\), the LCDM model can be differentiated from the \(w = -0.9\) model on the one hand, and from the tracker potential \(V(\phi) \propto \phi^{-1}\) and the DDG braneworld model (Deffayet et al. 2002b) on the other, at the 99.73 per cent CL.

Finally, we should mention that of the two statefinders \(s(z)\) appears to be better constrained by observations, especially if the uncertainty in \(\Omega_m\) is small. Interestingly both \(r(z)\) and \(s(z)\) show less scatter at higher redshifts \((z \gtrsim 1)\) and thereby complement the behaviour of the deceleration parameter \(q(z)\) and the cosmic equation of state \(w(z)\), which are better constrained at lower \(z (z \lesssim 0.4)\). One is therefore tempted to construct a new diagnostic \(\{\bar{s}, \bar{q}\}\), where \(\bar{s}\) is averaged over the redshift range \(1 \lesssim z \lesssim 1.7\), whereas \(\bar{q}\) is averaged over the redshift range \(0 < z \lesssim 0.4\). From Figs 8 and 10 we find that \(\{\bar{s}, \bar{q}\}\) provides an extremely potent diagnostic of dark energy since it can distinguish a fiducial LCDM model from a dark energy model with \(w \gtrsim -0.95\), on the one hand, and from a Chaplygin gas model?
the Gaussian prior \(\Omega_m\)

**Figure 13.** Variation of \((\bar{r}(z))\) with \(z\) for the kinessence model with the tracker potential \(V(\phi) = V_0/\phi^a\) for \(a = 1\). The solid line shows best-fitting \((\bar{r}(z))\) averaged over all realizations calculated with the polynomial fit to dark energy, equation (35), for the prior \(\Omega_m = 0.3\) exactly. Shaded regions represent the 1\(\sigma\) confidence levels for \((\bar{r})\). The dark grey outer contour is for the Gaussian prior \(\Omega_m = 0.3 \pm \sigma_{\Omega_m}\) with \(\sigma_{\Omega_m} = 0.05\), the grey contour in the middle uses the Gaussian prior \(\sigma_{\Omega_m} = 0.015\) and the light grey contour uses \(\Omega_m = 0.3\) exactly. The dotted, dashed and dot-dashed lines represent the model values of \((\bar{r})\) for different constant-\(w\) kinessence models, for tracker kinessence models for different values of \(a\), and for Chaplygin gas models with different \(\kappa\), respectively. The thick solid line represents LCDM.

**Figure 14.** Variation of \((\bar{s}(z))\) with \(z\) for the kinessence model with the tracker potential \(V(\phi) = V_0/\phi^a\) for \(a = 1\). The solid line shows best-fitting \((\bar{s}(z))\) averaged over all realizations calculated with the polynomial fit to dark energy, equation (35), for the prior \(\Omega_m = 0.3\) exactly. Shaded regions represent the 1\(\sigma\) confidence levels for \((\bar{s})\). The dark grey outer contour is for the Gaussian prior \(\Omega_m = 0.3 \pm \sigma_{\Omega_m}\) with \(\sigma_{\Omega_m} = 0.05\), the grey contour in the middle uses the Gaussian prior \(\sigma_{\Omega_m} = 0.015\) and the light grey contour uses \(\Omega_m = 0.3\) exactly. The dotted, dashed and dot-dashed lines represent the model values of \((\bar{s})\) for different constant-\(w\) kinessence models, for tracker kinessence models for different values of \(a\), and for Chaplygin gas models with different \(\kappa\), respectively. The thick solid line represents LCDM.

**Figure 15.** Variation of \((q(z))\) with \(z\) for the kinessence model with the tracker potential \(V(\phi) = V_0/\phi^a\) for \(a = 1\). The solid line shows best-fitting \((q(z))\) averaged over all realizations calculated with the polynomial fit to dark energy, equation (35), for the prior \(\Omega_m = 0.3\) exactly. Shaded regions represent the 1\(\sigma\) confidence levels for \((q)\). We find here that the use of exact \(\Omega_m = 0.3\) and the Gaussian prior \(\Omega_m = 0.3 \pm \sigma_{\Omega_m}\) with \(\sigma_{\Omega_m} = 0.015\) gives us almost the same bounds, represented by the light grey contour, the dark grey outer contour uses the Gaussian prior \(\sigma_{\Omega_m} = 0.05\). The dashed and dot-dashed lines represent the model values of \((q)\) for tracker kinessence models for different values of \(a\), and for Chaplygin gas models with different \(\kappa\), respectively. The thick solid line represents LCDM.

**Figure 16.** 3\(\sigma\) confidence levels in the parameter space \(\bar{r}, \bar{s}\) are shown for the cosmological constant model using the polynomial fit to luminosity distance, equation (39). The solid star represents the model value of the parameter pair for the cosmological constant model. The dashed line above the LCDM point represents the quasiessence models, and that below the LCDM point represents different Chaplygin gas models. The solid rhombi represent tracker kinessence models with different \(a\), the solid triangles represent model values for constant-\(w\) quasiessence models, and the solid hexagons represent Chaplygin gas models with different values of \(\kappa\). Only those Chaplygin gas models with \(\kappa \geq \Omega_m/(1 - \Omega_m)\) are allowed. For all the dark energy models, \(\Omega_m = 0.3\) is used. The best-fitting point for the reconstruction is represented by the solid square. The ellipses represent the 1\(\sigma\), 2\(\sigma\), 3\(\sigma\) confidence levels in the \(\bar{r}-\bar{s}\) space.
Figure 17. 3σ confidence levels in the parameter space $f$, $\delta$ are shown for the cosmological constant model using the polynomial fit to the equation of state, equation (41). The solid star represents the model value of the parameter pair for the cosmological constant model using the polynomial

$$
\Omega_m = 0.3
$$

with $\kappa \leq 25$, on the other hand, at the 99.73 per cent confidence level.

We therefore believe we have convincingly demonstrated that the statefinder pair $[\delta, f]$ and $[\delta, \bar{q}]$ provides an excellent diagnostic of dark energy, which will be used to successfully differentiate between the cosmological constant and dynamical models of dark energy.

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