Pion-Pion Correlation Effect in Two-Pion-Exchange Contribution for Nucleon-Nucleon Scattering

Susumu FURUICHI and Keiji WATANABE*

Department of Physics, Rikkyo University, Tokyo

*Department of Physics, Nagoya University, Nagoya

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The pion-pion correlation effect is investigated, starting from the pion-pion scattering phase shifts which show the presence of a $\rho$ meson in the $I=J=1$ state and a strong attraction without resonance in the $I=J=0$ state. By comparing with the OBEC amplitude, the roles of the pion-pion correlation and the normalization $f^+e^{i\phi}(0)=C_0$ in nucleon-nucleon scattering are clarified. Comparison with the empirical phase shifts in the iso-triplet state shows that $C_0\sim 0$ and a rather small value of $g_n^2$ are probable and not inconsistent with the analysis of pion-nucleon scattering.

§ 1. Introduction

Recently nucleon-nucleon scattering has been analysed by using one-boson-exchange models, and it has been found that the characteristic features in the so-called region II$^a$ are well explained by the one-boson-exchange contributions (hereafter referred to as the OBEC model). In order to make clear the role of the two-pion-exchange contribution in the region, the contribution from the uncorrelated two-pion state has been investigated in detail by using the partial-wave dispersion relation.$^{3,4}$ It has been shown that (i) the contribution from the $(3\, 3)$ resonance is large and mainly contributes to $L=0$ two-pion state$^3$ where $L$ is the angular momentum carried by the two pions exchanged, (ii) if we take the "proper" $2\pi$ contribution$^3$ defined by the left-hand-cut part of the nucleon-nucleon partial-wave dispersion relation, the two-pion-exchange contribution without pion-pion correlation shows very similar behaviour to the corresponding amplitudes in the OBEC model.$^3,4$

In this paper, we examine the pion-pion correlation effect in the two-pion-exchange contribution to nucleon-nucleon scattering. General formulations of the nucleon-nucleon partial-wave dispersion relation have been summarized in reference 3, where effects of pion-pion correlation are included through the evaluation of the helicity amplitude $f^\pm (\pm \Lambda)(t)$ for the $(\pi+\pi\rightarrow N+N)$ process.

In § 2 the helicity amplitudes $f^+ (\pm \Lambda)(t)$ and $f^- (\pm \Lambda)(t)$ are evaluated, being based on the investigation of pion-pion scattering. In this paper, we are mainly

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$^1$ Address after October, 1965: Dublin Institute for Advanced Studies, Dublin.

$^a$ For division of the regions, see Prog. Theor. Phys. Suppl. No. 3 (1956).
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concerned with how does the correlation effect change the features of the uncorrelated two-pion-exchange contribution summarized above. Therefore a simplified example of pion-pion scattering phase shifts will be used, for this we refer to our previous work. Contributions from \( f_+^{(+)}(t) \) and \( f_-^{(-)}(t) \) to nucleon-nucleon scattering are compared with those from the OBEC model.

In § 3 we compare these two-pion-exchange contributions with the empirical phase shifts for \( P \) and \( D \) partial waves in the iso-triplet state.

§ 2. \( f_+^{(+)} \) and \( f_-^{(-)} \) and their contribution to nucleon-nucleon scattering

At first, we consider the helicity amplitudes. As to the \( I=L=1 \) part \( f_-^{(-)}(t) \), an approximate formula has been obtained in a previous paper as follows:

\[
f_-^{(-)}(t) = -\frac{\gamma_\pm}{m_\rho^2 - t - i\gamma_\rho q^2} + \left[ f_-^{(-)}(t) \right]_{\text{CGLN}},
\]

where \( q = \sqrt{(t/4)-1} \), \( m_\rho \) is the observed \( \rho \)-meson mass and the parameters \( \gamma_\pm \) and \( \gamma_\rho \) are determined as

\[
\gamma_+ = 10.05, \quad \gamma_- = 6.38 \quad \text{and} \quad \gamma_\rho = 0.24.
\]

\( \left[ f_-^{(-)}(t) \right]_{\text{CGLN}} \) denotes the contribution from the pion-nucleon rescattering term and the formula (1) is convenient if we want to evaluate the correlated and the uncorrelated two-pion parts separately.

As to the \( I=L=0 \) part, on the other hand, we calculate \( f_+^{(+)}(t) \) directly using an appropriate two-pion scattering phase shift \( \delta^{\rho}_{\pi\pi}(t) \), because the experimental information on the pion-pion interaction in the \( S \) state is not so sufficient as that in the \( P \) state. Now, let us assume the following dispersion relation for \( f_+^{(+)}(t) \):

\[
f_+^{(+)}(t) = \left[ C_0 + \frac{t}{\pi} \int dt' e^{-u(t')} \frac{\text{Im} f_+^{(+)}(t')}{t'(t-t')} \right] e^{u(t)},
\]

where \( a = 4(1 - 1/4m^2) \), \( m \) being the nucleon mass and \( C_0 \) is a parameter. Here \( u(t) \) is expressed as

\[
u(t) = \frac{t}{\pi} \int_0^\infty dt' \frac{\delta^{\rho}_{\pi\pi}(t')}{t'(t-t')} ,
\]

and \( \text{Im} f_+^{(+)}(t) \) is approximated by the CGLN term.

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\* The neutral pion mass is taken as the unit of mass.
\* The notation corresponds with that of reference 3).
\* Strictly speaking, two subtractions are required when we take the \( N \) pole and the \((3\ 3)\) resonance in \( \text{Im} f_+^{(+)}(t) \). For simplicity, we adjust \( f_+^{(+)}(0)' \) to that of the CGLN term, which is very close to the value estimated from pion-nucleon scattering by Menotti.
The parameter \( C_0 = f_+^{(+)0}(0) \) has been examined in relation to the forward pion-nucleon scattering. If \( f_+^{(+)0}(t) \) tends to \( [f_+^{(+)0}(t)]_{C0=0} \) as \( \delta_{\pi\pi}^0 \to 0 \), \( C_0 \) must be 42. On the other hand, the low energy limit of the forward pion-nucleon scattering gives \( C_0 = -1.9 \pm 3 \). In the meantime we use \( C_0 \) as a running parameter rather than assigning it a fixed value, and examine the consistency between nucleon-nucleon and pion-nucleon scatterings.

For simplicity, we use the value of \( \delta_{\pi\pi}^0 \) given by Jacob, Mahoux and Omnès. It roughly corresponds to that with \( \lambda \approx -0.13 \) in reference 5), as shown in Fig. 1.

In order to estimate the contributions from \( f_+^{(+)0} \) and \( f_-^{(-)1} \) to nucleon-nucleon scattering, we firstly calculate the following quantities:

\[
\begin{align*}
& b_1^{a}(t) = \sqrt{\frac{t - 4}{t}} \frac{3\pi}{2} q_\pi^4 |f_+^{(+)0}(t)|^2, \\
& b_1^{(0)}(t) = \sqrt{\frac{t - 4}{t}} \frac{3\pi}{4} q_\pi^4 V_2 \left| f_-^{(-)1}(t) - f_+^{(-)1}(t) \right|^2, \\
& b_2^{(0)}(t) = -\sqrt{\frac{t - 4}{t}} \frac{3\pi}{2} \frac{q_\nu^2}{q_\pi^2} \text{Re} \left\{ \frac{m}{V_2} \left[ f_-^{(-)1}(t) - f_+^{(-)1}(t) \right] \right\}, \\
& b_1^{(0)}(t) = -\sqrt{\frac{t - 4}{t}} \frac{3\pi}{2} q_\pi^4 |f_-^{(-)1}(t)/V_2|^2, \\
& \text{where} \quad q_\pi^2 = m_\pi^2 - t/4. \quad \text{As } is \text{ discussed in reference 3)}, \text{ the contributions from } f_+^{(+)0}(t) \text{ and } f_-^{(-)1}(t) \text{ can be interpreted as the contributions from the } I=0 = 0 \text{ boson and } I=1 = 1 \text{ boson, respectively, where the distributions of the respective masses are given by } b_1^{a}(t) \text{ and } b_1^{(0)}(t). \text{ Numerical values of } b_1^{(0)} \text{ and } b_1^{(0)} \text{ obtained from Eqs. (1)-(4) are plotted in Figs. 2 and 3, respectively.}
\end{align*}
\]

The connection between the \( b_j \)'s and the nucleon-nucleon partial-wave amplitudes \( h_J \) has been given by one of the present authors\(^{(a)}\) as described below. First, the discontinuity across the left-hand cut of \( h_J \) is derived as
Fig. 3a.

\[ \text{Im } h_{J}^{(a)S}(v) = \left( \frac{1}{8\pi} \right) \left( \frac{m^2}{2v} \right)^{2} \int_{t}^{\infty} dt \ b_{1}^{(a)}(t) \rho_{J}^{1}(v, 1 + \frac{t}{2v}) \theta \left( -v - \frac{t}{4} \right), \]  

(7)

\[ \text{Im } h_{J}^{(b)P}(v) = \left( \frac{1}{8\pi} \right) \left( \frac{m^2}{2v} \right)^{2} \int_{t}^{\infty} dt \left[ -b_{1}^{(b)}(t) \left( 4m^2 + 8v + t \right) \rho_{J}^{1}(v, 1 + \frac{t}{2v}) \right. \]

\[ + b_{2}^{(b)}(t) \rho_{J}^{2}(v, 1 + \frac{t}{2v}) + b_{3}^{(b)}(t) \rho_{J}^{3}(v, 1 + \frac{t}{2v}) \left] \theta \left( -v - \frac{t}{4} \right), \right) \]  

(8)

where \( v \) is the squared barycentric momentum, \( \rho_{J}(v, 1 + t/2v) \) represents the operation of partial-wave expansion and is given in reference 10), \( \theta(x) \) is the usual step function and the superscripts \( S \) and \( P \) denote \( L=0 \) and \( 1 \) respectively.

Then, the left-hand-cut contribution to \( h_{J} \) is defined as

\[ h_{J}^{(i)L}(v) = \frac{\gamma_{i}}{\pi} \int_{l}^{\infty} dv' \text{Im } h_{J}^{(i)L}(v') \frac{\rho_{J}(v')}{{v'}^{2} (\gamma'-\gamma)}, \]  

(9)

where \( i \) and \( L \) stand for \( a, b \) and \( S, P \) respectively. Here, \( h_{J} \) and \( \rho_{J} \) for a given value of the total angular momentum \( J \) are matrices with respect to the spin and orbital angular momentum \( l \).

Next, we compare the contribution from the \( b_{l} \) 's with the corresponding OBEC contribution. In the \( I=L=1 \) part, the contribution from the first term in Eq. (1), the \( \rho \)-meson part, can be related with a vector-boson-exchange contribution in a somewhat direct manner, and is characterized by the following parameters:
The contribution from the second term in Eq. (1) to nucleon-nucleon scattering has also very similar behavior to a vector-boson-exchange contribution with the parameters

\[ m_\sigma = 740 \text{ MeV}, \quad G_\sigma^2/4\pi = 26, \quad f_\sigma^2/4\pi = 3.6(1/m)^2, \]  

as shown in reference (11). These parameters (10) and (11) are very similar to each other, and it may be expected that the total contribution from Eq. (1) is very close to the simple sum of (10) and (11). In fact, if we apply the same procedures as those of reference 4), the following parameters are obtained for the total contribution:

\[ m_\sigma = 750 \text{ MeV}, \quad G_\sigma^2/4\pi = 40, \quad f_\sigma^2/4\pi = 5.5(1/m)^2. \]  

In the \( I=L=0 \) part, the two-pion-exchange contribution can be compared with a scalar-boson-exchange contribution. In this case, the method given for the uncorrelated part of the two-pion-exchange contribution may also be suitable, since there is no apparent peak in \( b_\rho(t) \). Following reference 4), this is done by comparing the \( \nu \)-dependences of the \( P \)-wave partial-wave amplitudes \( h_{l_1l_2}(\nu)|_{l_1l_2} \) derived from \( b_\rho(t) \) with those of the one-scalar-boson-exchange contribution. The effective mass \( m_\sigma \) of the corresponding one-scalar-boson-exchange contribution is fixed by the comparison of the following quantity for the two contributions:

\[ \left( \begin{array}{c} -2h_{00} - 3h_{11} + 5h_{12} \\ h_{10} + 3h_{11} + 5h_{12} \end{array} \right), \]  

and then the effective coupling constant \( g_\sigma^2/4\pi \) is determined by comparing each \( h_{l_1l_2} \) with the corresponding OBEC amplitude with this value of \( m_\sigma \). Results are summarized in Table I. The last row represents the uncorrelated part given in reference 4).

### Table I.

<table>
<thead>
<tr>
<th>( C_0 )</th>
<th>( m_\sigma ) (MeV)</th>
<th>( g_\sigma^2/4\pi )</th>
<th>( g_{\sigma^2}/4\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>560</td>
<td>24</td>
<td>31</td>
</tr>
<tr>
<td>29</td>
<td>600</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>10</td>
<td>630</td>
<td>17</td>
<td>15</td>
</tr>
<tr>
<td>-1.9</td>
<td>700</td>
<td>16</td>
<td>9</td>
</tr>
<tr>
<td>CGLN</td>
<td>730</td>
<td>86</td>
<td>44</td>
</tr>
</tbody>
</table>

As is seen from Table I, \( \delta_{\sigma^2} \) reduces the effective mass \( m_\sigma \). It has some similarity to the cutoff of the \( t \) integration in Eq. (7). On the other hand, the variation of \( C_0 \) mainly causes a variation of \( g_\sigma^2/4\pi \). We cite the effective coupling constant \( g_{\sigma^2}/4\pi \) roughly estimated for \( m_\sigma = 600 \text{ MeV} \) in the last column, in order to compare the rates of suppression.

Comparing the parameters given in Table I with those obtained from an-
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alysis using the OBEC model, we find that $C_5 \approx 10$ roughly corresponds to the iso-scalar scalar contribution. On the other hand, $G_s^2/4\pi$ in Eq. (12) is considerably larger than that for the iso-vector vector contribution in the OBEC model, whereas $f^2_v/4\pi$ in Eq. (12) is slightly smaller than that in the OBEC model.

§ 3. Comparison with empirical phase shifts

Now, let us compare the contribution from two-pion exchange with the empirical phase shifts. For such a purpose, other possible contributions have to be considered. The one-pion-exchange and one-$\omega$-meson-exchange are taken as possible competing contributions in region II, and $\gamma$-meson effects and other possible many particle effects are tentatively neglected.

<table>
<thead>
<tr>
<th>$I \ L$</th>
<th>pion-pion correlation</th>
<th>deviation from the CGLN part</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0</td>
<td>correlation without resonance</td>
<td>cutoff or suppression due to $\delta_{\pi\pi}^g$ and $C_0$</td>
</tr>
<tr>
<td>1 1</td>
<td>correlation with resonance</td>
<td>simple enhancement</td>
</tr>
<tr>
<td>$L \geq 2$</td>
<td>without correlation</td>
<td></td>
</tr>
</tbody>
</table>

The two-pion-exchange contribution under consideration may be classified as in Table II. In the case $L \geq 2$, we assume the correlation effects to be negligible, since it is not likely to discuss the effects in comparison with the present experimental data. The energy dependences of each part are shown in Figs. 4a~4d for the $^1D_2$ and $^3P_J$ states. The comparison is carried out by adjusting $g^4_u$ for each value of $C_0$ and for each partial wave. The relations between $C_0$ and $g^4_u/4\pi$ when we use YLAM for the empirical phase shifts are plotted in Fig. 5. It seems probable from Fig. 5 that an overall fit to $^3P_J$ and $^1D_2$ states is realized for

$$C_0 \sim 0, \quad g^4_u/4\pi = 3 \sim 9.$$  \hspace{1cm} (14)

This is not inconsistent with the value of $C_0$ which is determined by the pion-nucleon scattering. However, the present investigation of pion-pion correlation is still in a preliminary stage, especially for the $I=L=0$ part, and further investigations of the pion-pion scattering will be required to confirm this consistency.

In the present paper, we consider only the iso-triplet nucleon-nucleon states, or equivalently proton-proton scattering. The iso-singlet amplitudes or neutron-proton scattering will be considered in a future investigation.
Fig. 4a. Each part of the $^1D_2$ amplitude with respect to $L$ is plotted.

Fig. 4b. $^3P_0$ amplitude.

Fig. 4c. $^3P_1$ amplitude.

Fig. 4d. $^3P_2$ amplitude.
Fig. 5.

References

4) S. Furuichi and W. Watari, to be published.