On the Momentum Distribution of the Recoil Nucleus in the $^{12}$C ($p$, $2p$) $^{11}$B Reaction

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The momentum distribution of the recoil nucleus in the $^{12}$C ($p$, $2p$) $^{11}$B reaction is calculated in the general non-coplanar and non-symmetric case. We report our results and attempt to explain the experimental data obtained by Yuasa et al. with a propane bubble chamber.

§ 1. Introduction

In the beginning ($p$, $2p$) reactions were used for testing Serber's idea on direct interactions by high energy protons. Berkeley experiments, with 340 MeV incident protons on light nuclei, showed that the energy spectra of high energy outgoing protons and the angular distributions of the differential cross-sections are those expected from a quasi-elastic scattering of the incident protons by protons of the target nucleus. The same results were also obtained at Dubna, with 600 MeV incident protons. The experiments were soon extended to measurements with coincidence between two outgoing protons by many groups, at different energies: Chicago at 440 MeV, Harvard, Harwell, Orsay and Uppsala, at 150 MeV - 185 MeV. Tyrén, Hillman and Maris showed that quasi-free proton-proton scattering, with coincidence between two outgoing protons, can be used as a tool for investigating nuclear structure, and especially shell structure. These authors chose a definite and symmetric geometry for the outgoing protons and looked at summed energy spectra: They found clear peaks corresponding to the binding energies of the protons in the outer and inner shells against a continuous background due to multi-stage processes, when the protons are knocked out of an inner shell, leaving a hole, the final nuclei are left in high excitation energy states. These findings were also made by other experimental groups, which gave an unexpectedly clear confirmation of the shell structure. In parallel, much theoretical work has been carried out. By using the C. F. P. method, attempts were made to explain the energy spectra of the outgoing protons in quasi-free proton-proton scattering. However, our main interest is actually not the energy distributions of the emitted protons but the momentum distributions of the nuclear protons. In the ($p$, $2p$) reactions with coincidence between the two outgoing protons, the angular correlation distributions of differential cross-sections provide us with information on the
momentum distributions of protons in nuclei. Calculations, by using either plane waves\textsuperscript{10} or distorted waves\textsuperscript{11} or by partial wave analysis,\textsuperscript{12} have been performed, in the case of coincidence between the two outgoing protons, to obtain the momentum distributions of nuclear protons. In the experiments performed in the coplanar and symmetric case with $E_1 = E_2$ and $\theta_1 = -\theta_2 = \theta$, the recoil momentum $q = k_0 - k_1 - k_2$ is collinear with $k_0$ and has the absolute value

$$q = k_0 - 2k \cos \theta,$$

where the indices 0, 1 and 2 denote the incident and two outgoing protons, respectively; $E$ stands for energy; $\theta_i$ for the angle between $k_i$ and $k_0$. Zero recoil momentum, $q = 0$, is obtained for $\theta = \arccos (k_0 / 2k)$ (44° for $E_B = 0$, 34° for $E_B = 34$ MeV both at $E_0 = 150$ MeV, for example). Therefore, in the neighbourhood of $\theta$, one can expect that the recoil energy of the residual nucleus is negligible. In non-coplanar and non-symmetric cases, the condition for $q$ to vanish becomes complicated compared to that in the coplanar and symmetric case mentioned above. One cannot simply neglect the recoil energy for a certain angle; nevertheless, one had to do so entirely in view of the poor energy resolution. It was indeed true that, in many preliminary analyses of the experimental data\textsuperscript{6,7} on quasi-free proton-proton scattering, the recoil energy of the residual nucleus was neglected. Recently, Yuasa et al.\textsuperscript{13,14,15} have used a propane bubble chamber, with better energy and angular resolution ($\delta \theta \leq 2^\circ$ and $\delta E \leq 1$ MeV), to observe the $^{12}$C$(p, 2p)^{11}$ reactions in both coplanar and non-coplanar cases. Their interesting results have proved to be very useful in elucidating the momentum distribution of nuclear protons, for in the framework of the impulse approximation, the momentum $k$ of the struck proton before the reaction is simply equal to the recoil momentum of the residual nucleus with the opposite sign: $k = -q$. In the present note, we calculate the recoil momentum distributions of the residual nucleus and attempt to explain the experimental data obtained with a propane bubble chamber by Yuasa et al.\textsuperscript{13,14,15}

§ 2. Theoretical consideration

In quasi-free proton-proton scattering in complex nuclei, the energy- and momentum-conservation equations are

$$E_0 = E_1 + E_2 + E_q + E_B$$

and

$$k_0 = k_1 + k_2 + q,$$

$k_0, k_1, k_2$ and $q$ stand for the momenta of the incident particle, two outgoing protons and the recoil nucleus, respectively; $E_0, E_1, E_2$ and $E_q$ for their corresponding kinetic energies; and $E_B = E_x + Q$, $E_x$ and $Q$ being the excitation energy of the residual nucleus and the $Q$ value of the reaction, respectively.
The relation (1) can also be written in terms of $k$ as
\[ k_0^2 - k_1^2 - k_2^2 - \frac{1}{A-1} q^2 - \frac{2ME_B}{\hbar^2} = 0, \tag{3} \]
where $A$ is the mass number of the target nucleus and $M$ the proton mass.

At sufficiently high energies ($\geq 100$ MeV or so), light nuclei become transparent with respect to the incident proton, the impulse approximation can thus be used; in the framework of this approximation, we have the cross section for quasi-free proton-proton scattering in the form
\[
\frac{d\sigma}{dq} = \frac{2\pi M}{\hbar k_0} \frac{2M}{(2\pi)^3} \left( \frac{2\pi}{\hbar k_1} \right) \delta\left( k_0^2 - k_1^2 - k_2^2 - \frac{1}{A-1} q^2 - \frac{2ME_B}{\hbar^2} \right) \times \delta(k_0 - k_1 - k_2 - q) |a|^2 N(q), \tag{4}\]
where $a = a(k_0, -q; k_1, k_2)$ is the scattering amplitude; $N(q)$ stands for the square of the overlap integral
\[
N(q) = \sum \left| \int \exp(-i\mathbf{q} \cdot \mathbf{r}) \varphi_{im}(\mathbf{r}) d\mathbf{r} \right|^2, \tag{5}\]
\[\sum\] indicates averaging over initial states and summing over final states. Assuming a single particle model, we have $N(q)$ in a simple form
\[
N(q) = \frac{n}{2l+1} \int \exp(-i\mathbf{q} \cdot \mathbf{r}) \varphi_{im}(\mathbf{r}) d\mathbf{r}^2, \tag{6}\]
where $n$ is the number of protons in the shell under consideration. If the scattering amplitude $a$ is assumed to be fairly constant, by integrating over $k_2$ and the angle between $k_1$ and $k_0 - q$, the cross section (4) becomes
\[
\frac{d^2\sigma}{d\Omega dq} = 4\pi M^2 \frac{k_1^{\text{max}}}{\hbar^2 k_0} \int \left( \frac{2\pi}{k_1^{\text{max}}} \right)^3 \frac{d(k_1^2)}{2|k_0 - q| \cdot k_1} |a|^2 \frac{N(q) \cdot q^2}{2|k_0 - q| \cdot k_1}. \tag{7}\]
\[2\mathcal{O}^2 + \mathcal{S}_2\] is the solid angle of the recoil nucleus, and $k_{1\text{max}}$ and $k_{1\text{min}}$ are determined as the roots of Eq. (3)
\[
k_1^2 = \frac{1}{2} \left\{ -\left( \frac{1}{A-1} q^2 + \frac{2ME_B}{\hbar^2} - k_0^2 \right) \right\}^{\pm} \sqrt{\left( \frac{1}{A-1} q^2 + \frac{2ME_B}{\hbar^2} - k_0^2 \right) - \left( \frac{A}{A-1} q^2 - 2k_0 \cos \theta_q + \frac{2ME_B}{\hbar^2} \right)^2}, \tag{8}\]
where $\theta_q$ is the angle between $k_0$ and $q$. Introducing the following quantities
\[
\Delta \equiv \frac{A}{A-1} q^2 - 2k_0 \cos \theta_q + \frac{2ME_B}{\hbar^2}, \quad m \equiv \frac{1}{A-1} q^2 + \frac{2ME_B}{\hbar^2} - k_0^2, \tag{9}\]
\[
\Delta = \frac{A}{A-1} q^2 - 2k_0 \cos \theta_q + \frac{2ME_B}{\hbar^2} \tag{10} \]

In Eq. (6), $\mathcal{L}$ is the solid angle of the recoil nucleus, and $k_{1\text{max}}$ and $k_{1\text{min}}$ are determined as the roots of Eq. (3)
where $\theta_q$ is the angle formed by $\mathbf{q}$ with $k_0$. Substituting Eq. (7) into Eq. (6), we have the cross section in the form

$$
\frac{d^2\sigma}{d\Omega dE_q} = \frac{2(A-1)M^4}{4(2\pi)^3h^2k_0} |a|^2 \frac{1}{(2\pi)^3} N(q) \cdot q \times \sqrt{k_0^2 + 2k_0q \cos \theta_q - \frac{A+1}{A-1} q^2 - \frac{4ME_B}{h^2}}.
$$

(8)

By integrating Eq. (8) over $\Omega$, the cross section for quasi-free proton-proton scattering finally takes the following forms, for $0 \leq \theta_q \leq \pi/2$:

$$
\frac{d\sigma}{dE_q} = \frac{2(A-1)M^4}{4 \cdot 3h^6k_0^2} |a|^2 \frac{1}{(2\pi)^3} N(q)
\times \left[ \left\{ k_0^2 + 2k_0q - \frac{A+1}{A-1} q^2 - \frac{4ME_B}{h^2} \right\}^{3/2} - \left\{ k_0^2 - \frac{A+1}{A-1} q^2 - \frac{4ME_B}{h^2} \right\}^{3/2} \right],
$$

(9)

and for $\pi/2 \leq \theta_q \leq \pi$:

$$
\frac{d\sigma}{dE_q} = \frac{2(A-1)M^4}{4 \cdot 3h^6k_0^2} |a|^2 \frac{1}{(2\pi)^3} N(q)
\times \left[ \left\{ k_0^2 - \frac{A+1}{A-1} q^2 - \frac{4ME_B}{h^2} \right\}^{3/2} - \left\{ k_0^2 - 2k_0q - \frac{A+1}{A-1} q^2 - \frac{4ME_B}{h^2} \right\}^{3/2} \right].
$$

(10)

By setting $\theta_q$ equal to a certain angle, Eq. (8) gives us the momentum distribution of the recoil nucleus for that angle or, in the framework of the approximation used, the momentum distribution of the nuclear proton with the opposite sign; while Eqs. (9) and (10) give the momentum distribution summed over all directions of $\mathbf{q}$. Equation (6) contains the kinematical limits for the $(p, 2p)$ reaction as the upper and lower limits of the range of integration over $k_1$, therefore Eqs. (8), (9) and (10) satisfy the condition for energy and momentum conservation. This kinematical limitation was also considered by Yuasa and Hourany in the analysis of experimental data on the recoil momentum in the $^{12}_C(p, 2p)B^{11}$ reaction.

§ 3. Results

We calculate, in the present note, the momentum distribution of the recoil nucleus in $^{12}_C(p, 2p)B^{11}$ reactions induced by incident protons of 80, 100 and 150 MeV, with the residual nucleus $B^{11}$ left in its ground state; we have then $A=12$ and $E_B=Q=16$ MeV. By using harmonic oscillator wave functions for $\psi_{lm}(r)$ in Eq. (5), and in the case with $p$ protons ($l=1$), $N(q)$ becomes

$$
N(q) = \frac{4}{3} \frac{16\pi^{3/2}}{\alpha^{3/2}} q^3 \exp(-q^2/\alpha),
$$

(11)

where $\alpha$ stands for the spring constant of the harmonic oscillator wave function.
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Fig. 1. Momentum distributions of the recoil nucleus calculated with $\alpha = 0.372$ fm$^{-2}$. The solid curves represent the distributions summed over all directions of $q$, the dotted curves show the distributions at $\theta_q = 0$ and $\pi$.

Fig. 2. Variation of the momentum distributions with respect to angles $\theta_q$ calculated for $E_0 = 100$ MeV and with $\alpha = 0.372$ fm$^{-2}$.

Fig. 3. Variation of the momentum distributions with respect to angles $\theta_q$ calculated for $E_0 = 150$ MeV and with $\alpha = 0.372$ fm$^{-2}$.
We represent in Fig. 1 the momentum distributions of the recoil nucleus calculated with $\alpha = 0.372$ fm$^{-2}$ for $E_0 = 80, 100$ and $150$ MeV, respectively. The solid curves correspond to the distributions summed over all directions of $q$ ($0 \leq \theta_q \leq \pi/2$ and $\pi/2 \leq \theta_q \leq \pi$), and the dotted curves to those at $\theta_q = 0$ and $\pi$.

Figures 2 and 3 show the variation of the distributions with respect to the angles $\theta_q$, calculated with $\alpha = 0.372$ fm$^{-2}$ for $E_0 = 100$ and $150$ MeV, respectively.

The curves in Fig. 4 are calculated for $E_0 = 100$ MeV with $\alpha = 0.2076, 0.3$ and $0.372$ fm$^{-2}$, respectively.

In order to have an idea about of the experimental data, we reproduce in Fig. 5 the momentum distributions of the recoil nucleus obtained by Yuasa and Hourany$^{19}$, with a propane bubble chamber. In their data, the authors plotted the number of events of the $(p, 2p)$ reaction against the recoil momentum, while in our theoretical calculation, we consider the cross section. Unfortunately, the number of events used by Yuasa and Hourany has to be multiplied by $\sin \theta_q$, the solid angle correction, for comparison with the cross section. We cannot, therefore, directly compare our results with the experimental data. With the solid angle correction, the positions of the peaks of the momentum distributions for all events will shift toward smaller $q$, their pattern will accordingly change and become closer to our results, while the pattern of the momentum distribution at a certain angle, except for its amplitude, will remain the same.
4. Discussion and conclusion

As seen from the experimental data, the recoil momentum distributions, summed over all events or for coplanar and symmetric event at a certain angle, are nearly of the same patterns for \(70 < E_a < 90\) MeV and \(90 < E_a < 110\) MeV, and so are our results for \(E_0 = 80, 100\) and \(150\) MeV; this independence of the momentum distribution upon the incident energy is very consistent with the approximation and the model used in our calculation. However, at relatively low energies, \(\leq 70\) MeV or so, the effects of nuclear distortion become more sensible and the impulse approximation cannot be applied; these two facts would strongly affect the patterns of the momentum distributions. The experimental data for \(50 < E_a < 70\) MeV agree quite well with this statement. On the contrary, at sufficiently high energies, the influence of the nuclear distortion on the momentum distributions of the outgoing protons is known to be relatively small; moreover, since we are dealing with the momentum distribution of the recoil nucleus \(B^{11}\) which is much heavier than proton, the distortion effects become thereby very much smaller and can be practically neglected.

Assuming the impulse approximation and a single-particle model, and neglecting nuclear distortion effects, one generally has the momentum distribution of the recoil nucleus for the coplanar and symmetric case in the form

\[
\frac{d^2\sigma}{dE_d d\Omega} \propto q^3 \exp(-q^2/\alpha).
\] (12)

On the other hand, in our formulation for non-coplanar and non-symmetric cases with \(p\) protons knocked out and the residual nucleus left in low-lying states, it
is given by Eq. (8) and Eqs. (9) and (10) respectively for the distribution at a certain angle and that summed over all directions of \( q \), \( N(q) \) in Eqs. (8), (9) and (10) being given explicitly by Eq. (11). It can be easily seen that (8), (9) and (10) fall off at \( q \sim 0 \) faster than (12); this makes the momentum distributions in the general case of non-coplanarity and non-symmetry vanish at \( q \sim 0 \). Since the term \( N(q) \) of Eqs. (8), (9) and (10) takes the same value for \( q \) and \(-q\), as seen from its explicit form in Eq. (11), the maxima of the summed distributions for \( \theta_q < \theta_0 < \theta_1 \) and \( \theta_2 < \theta_q < \theta_3 \) are determined by the kinematical conditions 
\[
\sqrt{k_0^2 + 2k_0 q - ((A+1)/(A-1)) q^2 - 4ME_B/h^2} = \sqrt{k_0^2 - ((A+1)/(A-1)) q^2 - 4ME_B/h^2},
\]
respectively, while the maximum of the distribution at a certain angle is determined by the kinematical condition 
\[
\sqrt{k_0^2 + 2k_0 q \cos \theta_q - ((A+1)/(A-1)) q^2 - 4ME_B/h^2}.
\]
In the latter case, we note that even if we confine ourselves to \( \theta_q = 0 \) and \( \pi \), Eq. (8) does not give the coplanar and symmetric quasi-free proton-proton scattering, for we can make up a certain \( q \) of the recoil nucleus by a different set of momenta \( k_1' \) and \( k_2' \), with \( \theta_1' \neq \theta_2' \). Coplanar and symmetric scattering implies the conditions \( |k_1| = |k_2| \) and \( \theta_1 = -\theta_2 \) (Fig. 6).

We further remark that the ratios among the maxima determined by the above kinematical conditions, say, the peaks for \( 0 \leq \theta_q \leq \pi/2 \) and \( \pi/2 \leq \theta_q \leq \pi \) divided by those at \( \theta_q = 0 \) and \( \pi \), respectively, and the peaks for \( 0 \leq \theta_q \leq \pi/2 \) and at \( \theta_q = 0 \) divided by those for \( \pi/2 \leq \theta_q \leq \pi \) and at \( \theta_q = \pi \), respectively, agree well quantitatively with the experimental data of Yuasa and Hourany (Fig. 5). Gooding and Pugh,\(^9\) by their experiments, found that the recoil energies of the residual nucleus \( B'^4 \) in \( ^{12}C(p,2p) \) reactions are situated mainly in the region of the order of 1 MeV; this corresponds to recoil momenta of \( \sim 160 \text{ MeV}/c \) and is entirely consistent with our results, where the maxima of the momentum distributions for \( 0 \leq \theta_q \leq \pi/2 \) (Fig.1) and at \( \theta_q = 0, \pi/6 \) and \( \pi/3 \) (Figs. 2 and 3) are found at \( \sim 0.8 \text{ fm}^{-1} \). However, the positions of these maxima may be slightly changed with the values of \( \alpha \), as seen in Fig. 4; for large values of \( \alpha \), the maximum shifts toward large momenta and the distribution becomes broader, this point was also discussed by Johansson and Sakamoto.\(^1\) To conclude our note, we would like to emphasize that, as far as the asymmetric patterns of the momentum distributions, the positions of maxima and their various ratios are concerned, the agreement between the experimental data and our calculated
results is very good; but as for the determination of the number, nature and position of the peaks found experimentally, further experimental as well as theoretical work has to be done so that the recoil momentum of the residual nucleus in quasi-free proton-proton scattering could be used to elucidate the structure of light nuclei, in particular, the momentum distribution of nuclear protons.

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