On Excitations of Cyclotron Harmonic Waves Propagating Perpendicular to an External Magnetic Field

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As is well known, a magneto-active plasma with bi-maxwellian velocity distribution is stable against a longitudinal wave propagating perpendicular to an external magnetic field. In this note we shall show that under a more general condition some waves propagating exactly perpendicular to an external magnetic field exhibit instabilities at frequencies near the cyclotron harmonics.

Assuming the uniform external magnetic field \( B_0 \) to be in the \( z \) direction and all variables to be of the form \( \exp(i(k \cdot r - \omega t)) \), we have the dielectric tensor \( \varepsilon \) given by

\[
\varepsilon = 1 + \sum_{n=1}^{\infty} \frac{\omega_p^2}{\omega^2 + \omega_p^2} \int_0^\infty v_\perp dv_\parallel,
\]

\[
\times \int_{-\omega}^{\omega} \frac{dv_\parallel}{\omega} \left( \frac{mv_\parallel}{\eta} J_n(\eta) \right) - iv_\perp J_n(\eta),
\]

\[
\times \left[ \frac{n(\frac{\partial f_0}{\partial v_\perp} + k_\parallel A)}{\omega} J_n(\eta),
\right.
\]

\[
\left. \frac{i(\frac{\partial f_\perp}{\partial v_\perp} + k_\parallel A)}{\omega} J_n(\eta),
\right.
\]

\[
\frac{\partial f_\parallel}{\partial v_\parallel} - \frac{n k_\perp}{\omega A} J_n(\eta).
\]

(1)

where \( \omega_p \) is the electron plasma frequency, and \( A \) is the electron cyclotron frequency,

\[
A = v_\parallel \frac{\partial f_0}{\partial v_\perp} - v_\parallel \frac{\partial f_\parallel}{\partial v_\perp}, \quad \eta = k_{\perp} v_\parallel / \omega_p,
\]

and \( f_0 \) is the unperturbed electron distribution (normalized to unity). The indices // and \( \perp \) are referred to the direction of the external magnetic field. The ions are assumed to be a uniform background providing neutralization. A self-consistent wave is then described by the dispersion equation

\[
|\varepsilon + \frac{c^2}{\omega^2} (k_k - k)^2| = 0.
\]

In the case \( k_\parallel = 0 \), the integration with respect to \( v_\parallel \) in Eq. (1) can easily be carried out. Then the argument in the integral of each component of \( \varepsilon \) has no singularities in the whole complex \( v_\perp \) plane, and therefore we can expect neither Landau damping arising from the wave-particle interaction nor the process inverse to it. In the case where \( \omega \) is close to \( N\omega_H \), where \( N \) is an integer, we can see from the following examples that in a certain situation the dispersion equation has a root in \( \omega \) having a positive imaginary part and that the system is unstable. (The instability is interpreted as due to a wave-wave interaction.)

Example 1. Ordinary wave \( (E // B_0) \)

The dispersion equation is reduced to

\[
\omega^2 = c^2 k^2 + \omega_p^2
\]

\[
\times \left\{ 1 - 2 \left( 1 - \frac{T_\perp}{T_\parallel} \right) \sum_{\omega = -\infty}^{\infty} \frac{n \omega_p^2}{\omega - \omega_0^2} A_n(\lambda) \right\},
\]

(3)

where \( f_0 \) is assumed to be a function of \( v_\parallel/T_\parallel \) and \( v_\perp/T_\perp \) only. \( A_n(\lambda) \) denotes \( I_n(\lambda) \exp(-\lambda) \) where \( I_n(\lambda) \) is the modified Bessel function and \( \lambda = k_\parallel^2 T_\parallel / (\omega_0^2 \omega_p) \). Substituting \( \omega = \omega_r + i\omega_i \) into Eq. (3) and assuming \( |\omega_r - N\omega_H| < |\omega_i| < |\omega_H| \), we have

\[
N^2 \omega_B^2 = c^2 k^2 + \omega_p^2
\]

\[
\times \left\{ 1 - 2 \left( 1 - \frac{T_\perp}{T_\parallel} \right) \sum_{\omega = -\infty}^{\infty} \frac{n}{N - n} A_n(\lambda) \right\}
\]

(4)

and

\[
\omega_i^2 = \frac{T_\perp}{T_\parallel} \omega_p^2 A_n(\lambda)
\]

\[
\times \left\{ 1 - 2 \left( 1 - \frac{T_\perp}{T_\parallel} \right) \sum_{\omega = -\infty}^{\infty} \frac{n}{N(N - n)} A_n(\lambda) \right\}^{-1}
\]

(5)

Numerical calculation shows that the value \( \lambda \) obtained from Eq. (4) is much smaller than unity and that the result of the summa-
tion in Eq. (5) is also smaller than unity. Therefore we conclude that if $T_\tau > T_1$, the system is unstable, thus an ordinary wave with $\omega \sim N\omega_H$ is excited. It may be pointed out that the above conclusion holds even in the case of finite collision frequency $\nu$ provided that $|\omega_r - N\omega_H| \ll |\omega_1 + \nu|$. Example 2. \textit{Longitudinal wave ($E \perp B_0$, $E \not\parallel k$)}

When $\omega$ becomes close to $N\omega_H$, an extraordinary wave often tends to a longitudinal wave\textsuperscript{3)} ($|E \cdot k| \gg |E \times k|$). Then the dispersion equation is

$$1 + \frac{\omega_0^2}{\omega} \sum_{n = -\infty}^{\infty} \frac{n^2}{\omega - n\omega_H} K_n(k) = 0,$$  

(6)

where $K_n(k)$ is defined by

$$(\omega_H/k)^2 \int_0^\infty J_n^2(\eta) \frac{\partial f_0}{\partial v_L} dv_L.$$

The substitution of $\omega = \omega_r + i\omega_1$ into Eq. (6) gives

$$1 + \frac{\omega_0^2}{\omega_H} \sum_{n = -\infty}^{\infty} \frac{n^2}{N(N-n)} K_n(k) = 0,$$  

(7)

and

$$\omega_1 = N^2 K_H(k) \omega_0^2$$

$$\times \left\{1 - \frac{\omega_0^2}{\omega_H} \sum_{n = -\infty}^{\infty} \frac{n^2}{N(N-n)^2} K_n(k) \right\}^{-1},$$  

(8)

where we have assumed $|\omega_r - N\omega_H| \ll |\omega_1| \ll |\omega_H|$.

Once $f_0$ is given, the wave number $k$ is determined from Eq. (7) and the summation in Eq. (8) can also be performed; this usually yields a result smaller than unity. In this case we have

$$\int_0^\infty J_n^2(\eta) \frac{\partial f_0}{\partial v_L} dv_L > 0$$

as the condition for excitation of a longitudinal cyclotron harmonic wave.\textsuperscript{4)}

For the case $k \perp B_0$ and $E \perp B_0$, we may have a transverse wave ($|E \cdot k| \ll |E \times k|$) under certain conditions. The behaviour of this wave will be discussed elsewhere.

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