On equilibrium tides in fully convective planets and stars

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ABSTRACT

We consider the tidal interaction of a binary consisting of a fully convective primary star and a relatively compact mass. Using a normal-mode decomposition we calculate the evolution of the primary angular velocity and orbit for arbitrary eccentricity. The dissipation acting on the tidal perturbation is assumed to result from the action of convective turbulence, the effects of which are assumed to act through an effective viscosity. A novel feature of the work presented here is that, in order to take account of the fact that there is a relaxation time $t_c$ (the turnover time of convective eddies) associated with the process, this is allowed to act non-locally in time, producing a dependence of the dissipation on tidal forcing frequency. Results are expressed in terms of the Fourier coefficients of the tidal potential, assumed periodic in time. We find useful analytical approximations for these valid for sufficiently large values of eccentricity $e > 0.2$.

We show that in the framework of the equilibrium tide approximation, when the dissipative response is frequency-independent, our results are equivalent to those obtained under the often used assumption of a constant time-lag between tidal response and forcing.

We go on to consider the case when the frequency dependence of the dissipative response is $\propto \frac{1}{1 + (\omega_m k t_c)^p}$, where $\omega_m$ is the apparent frequency (pattern speed $\times m$) associated with a particular harmonic of the tidal forcing as viewed in the frame corotating with the primary. We concentrate on the case $\omega_m k t_c \gg 1$, which is thought to be appropriate to many astrophysical applications. We study numerically and analytically the orbital evolution of the dynamical system corresponding to different values of the parameter $p$. We present results from which the time to circularize from large eccentricity can be found.

We find that when $p < 1$ the evolution is similar to that found under the constant time-lag assumption. However, the orbital evolution of the system with $p > 1$ differs drastically. In that case the system evolves through a sequence of spin–orbit corotation resonances with $\Omega_r/\Omega_{\text{orb}} = n/2$, where $\Omega_r$ and $\Omega_{\text{orb}}$ are the rotation and orbital frequencies and $n$ is an integer. When $p = 2$ we find an analytic expression for the evolution of semimajor axis with time for arbitrary eccentricity assuming that the moment of inertia of the primary is small.

We confirm the recent finding of Ivanov & Papaloizou, on the basis of an impulsive treatment of orbits of high eccentricity, that equilibrium tides associated with the fundamental mode of pulsation and dissipative processes estimated using the usual mixing-length theory of convection seem to be too weak to account for the orbital circularization from large eccentricities of extrasolar planets.

Generalizations and limitations of our formalism are discussed.

Key words: hydrodynamics – binaries: general – planetary systems: formation – stars: rotation.

1 INTRODUCTION

Tidal interaction between orbiting companions produces dissipation and orbital evolution towards a circular orbit that is synchronized with the internal rotations. Such a state is attained when the system is close enough for the tidal interaction to be strong enough [for a discussion in the case of close binary stars see Giuricin, Mardirossian & Mezzetti (1984a,b)]. The recently discovered extrasolar planets with periods less than about 5 d are in near-circular orbits, which is also possibly a result of tidal interaction (Marcy et al. 2001).

Zahn (1977) isolated two different types of tidal interaction. The first, applicable to fully convective stars or planets, described as
the equilibrium tidal interaction, applies when all relevant normal modes of oscillation of the tidally disturbed object have much higher oscillation frequencies than the orbital frequency. In this case dissipation is presumed to occur through a turbulent viscosity, which would act much like an anomalous Navier–Stokes viscosity. The second interaction, described as being through dynamic tides, occurs when normal modes have oscillation frequencies comparable to the orbital frequency and their excitation cannot be neglected.

The theory of the equilibrium tide with anomalous Navier–Stokes viscosity can be related to the classical theory of tides based on the idea that the effect of dissipation is to cause the tidal response to lag behind the forcing with a constant time-lag (Darwin 1879; Alexander 1973; Hut 1981).

The tidal theory based on a constant time-lag is often applied to problems where tidal interactions are important either in the case of stellar binaries (e.g. Lai 1999; Hurley, Tout & Pols 2002, and references therein) or of giant planets orbiting close to their central stars (e.g. Mardling & Lin 2002). It may be argued that this approach may be applicable when the dissipative process is one that acts as a standard Navier–Stokes viscosity (see e.g. Eggleton, Kiseleva & Hut 1998; Ivanov & Papaloizou 2004, hereafter IP; and below).

However, it has been noted on theoretical grounds that, if the assumed viscosity results from turbulence, there is a natural relaxation time associated with it, namely the eddy turnover time $t_c$. This may be longer than the orbital period, in which case the efficiency of the dissipation process should be reduced and it should be dependent on the tidal forcing frequency (see e.g. Goldreich & Nicholson 1977; Goldreich & Keeley 1977; Zahn 1989). As a result, tidal evolution may differ from that found under the constant time-lag assumption.

The importance of this effect for understanding the observed properties of a particular binary system has been stressed by Goldman & Mazeh (1994).

In this paper we investigate this problem in the simplest possible context. We study tidal evolution under the assumption that only the fundamental quadrupole modes are important for determining the tidal response and that their eigenfrequency is much larger than the orbital frequency so that an equilibrium tide approximation can be made. However, we allow the dissipation to have a characteristic time-scale associated with it, and through this produce a frequency-dependent response in the manner that has been suggested (e.g. Goldreich & Keeley 1977) might be appropriate for convective turbulence. We find then that only when the dissipative process acts instantaneously are results equivalent to the constant time-lag assumption obtained. In other situations, we find that there is the possibility of maintained spin–orbit resonances and accordingly that the state of pseudo-synchronization during the circularization process differs.

The plan of the paper is as follows. In Section 2 we formulate the basic equations for calculating the response to a tidal forcing potential in terms of a normal-mode expansion. We also formulate the treatment of dissipation using viscosity that incorporates a relaxation time $t_c$, identified with the turnover time of convective eddies. This allows the dissipative process to act non-locally in time and become weaker when the tidal forcing frequency as seen in a frame corotating with the tidally perturbed object becomes significantly larger than $t_c^{-1}$. We derive expressions for the rates of change of energy and angular momentum applicable to an orbit with arbitrary eccentricity. We introduce the equilibrium tide approximation applicable when the only important pulsation mode for the tidal response is the fundamental quadrupole mode and the orbital frequency is very much less than the frequency of this mode. We show that the results are equivalent to those under the approximation of a constant time-lag between tidal forcing and response of the perturbed object only in the special case $t_c = 0$, which corresponds to dissipation produced through the action of a standard Navier–Stokes viscosity.

In Section 3 we develop analytic approximations for the Fourier coefficients of the forcing potential. We consider the orbital evolution calculating the circularization from large eccentricities in Section 4. We consider the case when the angular momentum associated with rotation of the tidally perturbed object is much less than that of the orbit, so that pseudo-synchronization is achieved rapidly. We suppose that the frequency dependence of the dissipative response is $\propto 1/[1 + (\omega_{m,k} t_c)^2]$, where $\omega_{m,k}$ is the apparent frequency (pattern speed $\times m$) associated with a particular harmonic of the tidal forcing viewed in a corotating frame, $p$ is a parameter and $\omega_{m,k} t_c > 1$. We discuss when is is possible to attain spin–orbit resonance and the form pseudo-synchronization takes as $p$ is varied. Both numerical and analytic approaches are used. Finally in Section 5 we summarize our results and discuss them in the context of extrasolar planets.

2 BASIC EQUATIONS

For the case of a fully convective planet or star considered in this paper, the theory of tidal perturbation is relatively simple. When the body is not rotating, the pressure ($p$) modes do not contribute significantly to the interaction because of the large values of their oscillation frequencies, and the gravity ($g$) modes are absent. Therefore, only the fundamental ($f$) modes contribute significantly. For a rotating object there are also inertial or $r$ modes, which may give rise to a complicated spectrum and associated eigenfunctions. There could be significant excitation depending on the overlap of these with the forcing potential. For simplicity, this spectrum is neglected in this paper but in principle could be significant when considering the theory of the equilibrium tide (see e.g. Papaloizou & Pringle 1978, 1981; Ogilvie & Lin 2004, for a discussion).

We also assume that the rotation rate of the star $\Omega_i$ is uniform with axis of rotation directed perpendicular to the orbital plane, and is sufficiently small in magnitude when compared to a characteristic value of the eigenfrequency $\omega_0$ associated with the lowest-order $f$ mode with $l = 2$. It can be shown (see e.g. IP) that for our purposes it suffices to take into account only the leading and next-order terms in the expansion of all variables in the small parameter $\Omega_i/\omega_0$. Also, only the contribution of the leading quadrupole term in the expansion of the tidal potential in spherical harmonics and hence modes with $l = 2$ are considered. From now on we call the planet or star on which the tides are raised the primary and the perturbing companion the secondary.

We adopt spherical polar coordinates $(r, \theta, \phi)$ with origin at the centre of mass of the primary. Under the above assumptions, the Lagrangian displacement of a fluid element of a fully convective primary can be written in the form:

$$\xi = \sum_{m=0}^{\infty} b_m(t) \xi_m^\theta.$$  (1)

Note that there could also be some non-standard modes like, for example, the mode appearing in a model of Jupiter with an assumed first-order phase transition between molecular and metallic hydrogen (e.g. Vorontsov 1984), etc.
Here $\xi_0^m$ are the solutions of the standard eigenmode equation for a non-rotating star (e.g. Tassoul 1978; Christensen-Dalsgaard 1998)
\begin{equation}
-\omega_0^2 \xi_0^m + C (\xi_0^m) = 0,
\end{equation}
where $\omega_0$ is the eigenfrequency of the fundamental mode in the non-rotating limit and $C$ is the standard self-adjoint operator accounting for the action of pressure and self-gravity forces on perturbations (e.g. Chandrasekhar 1964; Lynden-Bell & Ostriker 1967).

It is important to note that, since the displacement vector $\xi$ is a real quantity and the eigenfunctions $\xi_0^m$ have a trivial dependence $\propto e^{i\omega_0^m}$ on the azimuthal mode number $m$ and $\phi$, it follows from equation (1) that the mode amplitudes $b_m$ satisfy
\begin{equation}
b_m = (b_{-m}^*)^*,
\end{equation}
where from now on the superscript $*$ denotes the complex conjugate.

The eigenfunctions are orthogonal in the sense of the inner product
\begin{equation}
\langle \xi_0^m | \xi_0^m \rangle = \int d^3x \rho |(\xi_0^m)^* : \xi_0^m\rangle,
\end{equation}
where $\rho$ is the density of the primary star, and are normalized by the standard condition
\begin{equation}
\langle \xi_0^m | \xi_0^m \rangle = 1.
\end{equation}
They can be presented in the form
\begin{equation}
\xi_0^m = \xi(r) Y_{2m}(\theta, \phi)e_i + \xi(r)Y_{2m}(\theta, \phi),
\end{equation}
where $Y_{2m}$ are spherical harmonics and the components $\xi(r)_R$ and $\xi(r)_S$ are related to each other as
\begin{equation}
d/dr \xi_S = \frac{\xi_R - \xi_S}{r}.
\end{equation}
This condition follows from the fact that hydrodynamical motion induced in a non-rotating isentropic star by a forcing potential must be circulation-free.

The equation governing the Lagrangian displacement produced under tidal forcing is (see e.g. IP)
\begin{equation}
\ddot{\xi} + 2u_0 \cdot \nabla \dot{\xi} + C(\xi) = -\nabla \psi' + f',
\end{equation}
where $u_0$ is the rotational velocity of the primary and $f'$ is the viscous force per unit mass.

The evolution equation for the mode amplitudes $b_m$ obtained by projecting on to the eigenvectors $\xi_0^m$ is found to be (see IP)
\begin{equation}
\dot{b}_m + \omega_0 b_m + 2i m \gamma \Omega b_m = f_m^T + f_m'.
\end{equation}
Here the dimensionless coefficient $\beta$ determines the correction to $\omega_0$ due to rotation, and it has the form (e.g. Christensen-Dalsgaard 1998, and references therein):
\begin{equation}
\beta = 1 - \int_0^{R^o} r^2 dr \rho (2\xi_R \xi_S + \xi_S^2).
\end{equation}

The forcing amplitude $f_m^T$ is related to the forcing tidal potential $\psi'$ [assumed to be $\propto \exp(i m \phi(t))$] through
\begin{equation}
f_m^T = -\int d^3x \rho (\xi_0^m)^* \cdot \nabla \psi'.
\end{equation}

For our problem, the quantity $f_m^T$ can be written in the form
\begin{equation}
f_m^T = \frac{GMQ}{D(t)^3} W_m e^{-i m \phi(t)},
\end{equation}
where the overlap integral $Q$ is expressed as (Press & Teukolsky 1977)
\begin{equation}
Q = 2 \int_0^{R^o} dr \rho r^3 (\xi_R + 3\xi_S).
\end{equation}

Here the mass of the secondary star is $M$, $D(t)$ is the position vector of the secondary assumed to orbit in the plane $\theta = \pi/2$ with $D(t) = |D(t)|$ and $\Phi(t)$ being the associated azimuthal angle at some arbitrary moment of time $t$. For $|m| = 2$, $W_m = \sqrt{3\pi}/10$ and $W_0 = -\sqrt{3\pi}/5$. We find it useful below to represent $f_m^T$ as the product of a constant dimension factor and a time-dependent dimensionless function:
\begin{equation}
f_m^T = W_m C \phi_m(t), \quad C = \frac{GMQ}{a^3}, \quad \phi_m(t) = \frac{e^{-i m \phi(t)}}{D(t)},
\end{equation}
where $a$ is the semimajor axis and $D(t) = D(t)/a$.

### 2.1 The viscous force amplitudes

The amplitude associated with the viscous force is given by $f_m^v$ in equation (9). For the standard Navier–Stokes equations, $f_m^v$ can be directly calculated (see IP and below). However, in this paper we consider a more general type of viscous interaction presumed to be induced by convective turbulence and for which the expression for $f_m^v$ must be modified. In particular, as has been pointed out by a number of authors (e.g. Goldreich & Nicholson 1977; Goldreich & Keeley 1977), when the characteristic tidal forcing time is much smaller than the characteristic turnover time of convective eddies, $t_c$, the transport of energy and momentum by turbulent motions is ineffective and the effective viscosity coefficient should be relatively suppressed. In order to describe the mode energy dissipation arising from the turbulence, a time interval of extent $t_c$ centred on the local time $t$ should be considered. We would like to take this into account in a simple manner and accordingly introduce a non-local form of $f_m^v$ written in the form
\begin{equation}
f_m^v = -\int_{-\infty}^{\infty} dt' \gamma_m(t - t') [b_m(t') + im \Omega \xi_m(t')],
\end{equation}
where the fact that $b_m + im \Omega \xi_m$ is used means that the dissipation operates on the amplitude of the velocity field associated with a particular mode as viewed in the frame corotating with the primary. We specify a particular form of viscosity kernel $\gamma_m(t)$ below (see equation 69), and discuss here only its most general properties.

Since the tidal forcing time and the turnover time must be estimated in the corotating frame, and modes with different $m$ have different apparent frequencies in this frame, $\gamma_m(t)$ should depend on both $m$ and $\Omega$. For general linear perturbations, $\gamma_m(t)$ is a complex quantity, but the condition $\gamma_m(t) = (\gamma_m(t))^*$ must be valid for self-consistency of equation (9).

In the limit $t_c \to 0$ the viscous interaction is local in time and $\gamma_m(t) = \gamma(\phi(t))$, where $\gamma$ is the viscous mode damping rate. It has dimensions of inverse time for a normalized eigenfunction. In that case, which corresponds to standard Navier–Stokes viscosity, it may be explicitly expressed in terms of the components $\xi_R$ and $\xi_S$ of the density $\rho$ and kinematic viscosity $\nu$ of the star (IP):
\begin{equation}
\gamma_0 = 4 \int_0^{R^o} r^2 dr \nu \left\{ \frac{1}{3} \left( \frac{\xi_R - \xi_S}{r} + 3 \frac{\xi_S}{r} \right)^2 \\
+ \frac{1}{r^2} \left[ (\xi_R - 3 \xi_S)^2 + 2 \xi_R^2 + 3 (\xi_R - \xi_S)^2 \right] \right\},
\end{equation}
where the prime stands for differentiation with respect to $r$.

### 2.2 Energy and angular momentum exchange between primary and orbit

Recalling that for the problem on hand only modes with $|m| = 0, 2$ are excited, it follows from equation (9) that when $f_m^T$ and
\[ f_m^\alpha (|m| = 0, 2) \text{ are equal to zero, the following quantities are } \]
\[ E = b_2 b_2^* + \frac{1}{2} b_0^2 + \omega_0^2 (b_2 b_2^* + \frac{1}{2} b_0^2), \]
where we recall that \( b_0 \) is real and
\[ L = 2 \left \{ i [b_2^* b_2 - b_2(b_2^*)^*] - 4\Omega b_2 b_2^* \right \}. \]
These quantities represent the sum of the canonical energies and angular momenta associated with the different modes (e.g. Friedman & Schutz 1978).

When the effects of dissipation are included, the evolution equations for \( E \) and \( L \) follow from equation (9) as
\[ E = E_r + E_v, \quad L = L_r + L_v, \]
where
\[ E_u = b_2 (f_2^*)^2 + b_2^* f_2^* + b_0 f_0^u, \quad L_u = 2i b_2 (f_2^* - b_2 (f_2^*)^*), \]
where the index \( u \) is either \( T \) or \( v \).

The sum of the orbital energy \( E_{orb} \), the energy of the oscillation modes \( E \) and the energy of the primary star \( E_p \) is conserved during the evolution of the system, as is the sum of the corresponding angular momenta. If the slow change of orbital parameters due to tidal evolution is neglected, the pulsations of the primary star are strictly periodic, and the energy of the modes \( E \) and the associated angular momentum \( L \) time-averaged over a period does not grow with time. In this case we have
\[ \langle E_{orb} \rangle = -\langle E_r \rangle, \quad \langle L_{orb} \rangle = -\langle L_v \rangle, \]
where the angular brackets denote a time average such that, for any quantity \( Q \),
\[ \langle Q \rangle = \frac{1}{P_{orb}} \int_0^{P_{orb}} dr \, Q(t). \]

Note that the mean rate of energy dissipation \( \langle E_{diss} \rangle \) differs from \( \langle E_r \rangle \) for a rotating primary, being given by
\[ \langle E_{diss} \rangle = \langle E_v \rangle - \Omega_i \langle L_v \rangle, \]
and must be negative. On the other hand, the sign of \( E_v \) can be arbitrary.

2.3 Solution for the mode amplitudes

2.3.1 Fourier expansion of the mode and forcing amplitudes

In order to solve equation (9) for the mode amplitudes, we assume that the orbital parameters are fixed so that all quantities of interest are periodic functions of time with period \( P_{orb} \). Note that care should be taken with this assumption (see IP, and references therein, and also the discussion), especially for orbits with high eccentricity \( e \).\(^2\)

\(^3\) For simplicity, we assume hereafter that the tides are raised only in the primary star. The generalization to the case of two tidally interacting stars is straightforward. Also, note that the energy dissipated by tidal friction and finally radiated away from the primary is formally considered as a part of \( E_{orb} \).

\(^4\) For a highly eccentric orbit this assumption can break down when the excitation of so-called dynamic tides is significant (e.g. Press & Teukolsky 1977). Also the stochastic evolution of orbital parameters (e.g. Mardling 1995, IP, and references therein) may take place.

We use below a Fourier expansion for the mode amplitudes \( b_m \) and the dimensionless tidal forcing amplitudes \( \phi_m \) in the form:
\[ b_m = \sum_{k=-\infty}^{\infty} b_{m,k} e^{i\omega_k t}, \]
\[ \phi_m = \sum_{k=-\infty}^{\infty} \phi_{m,k} e^{i\omega_k t}. \]

Here we introduce the dimensionless time variable \( \tau = \Omega t \), and \( \Omega \) is the orbital frequency:
\[ \Omega = \sqrt{\frac{G(M + M_p)}{a^3}} = \sqrt{\frac{GM(1 + q)}{a^3}}, \]
where \( M_p \) is the mass of the primary, and \( q = M_p/M \) is the mass ratio.

From the condition \( b_m = (b_{-m})^* \) it follows that \( b_{2,k} = (b_{-2,-k})^* \). The amplitude \( b_0 \) is real and therefore \( b_{0,k} = (b_{0,-k})^* \).

The Fourier coefficients of the dimensionless tidal forcing amplitudes \( \phi_{m,k} \) are given by
\[ \phi_{m,k} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin[m\Phi(\tau)] \cos(k\tau) - \cos[m\Phi(\tau)] \sin(k\tau)}{D(\tau)^3} \, d\tau. \]

We note that \( D \) and \( \Phi \) are related to \( \tau \) through \( \dot{D}(\tau) = (1 - \cos \xi) \) and \( \tan \Phi = \sqrt{1 - e^2} \sin \xi / (\cos \xi - e) \) with \( \tau = \xi - \sin \xi \).

On account of the fact that \( D \) and \( \Phi \), respectively, are even and odd functions of \( \tau \), we note that \( \phi_{m,k} \) is a function only of the eccentricity and is real. We may therefore write
\[ \phi_{m,k} = \frac{1}{2} (\alpha_{m,k} - \beta_{m,k}), \]
where
\[ \alpha_{m,k} = \frac{1}{\pi} \int_0^{2\pi} d\tau \, \cos[m\Phi(\tau)] \cos(k\tau), \]
\[ \beta_{m,k} = \frac{1}{\pi} \int_0^{2\pi} d\tau \, \sin[m\Phi(\tau)] \sin(k\tau). \]

We note some obvious properties of the coefficients such as \( \alpha_{m,-k} = \alpha_{m,k} b_{m,-k} = -\beta_{m,k} \) and \( \alpha_{m,k} b_{m,k} = \beta_{m,k} \beta_{m,-k} = -\beta_{m,k} \). Furthermore, it is important to note that for the case of interest with \( m = 2 \) and positive values of \( k \) we have \( \alpha_{2,k} \approx \beta_{2,k} \), and for \( m = 0 \) we have \( \beta_{0,k} = 0 \).

We find approximate expressions for \( \alpha_{2,k} \) and \( \alpha_{0,k} \) for positive values of \( k \), these being the only ones required, in the Appendix. We find it convenient below to represent the dimensionless tidal forcing amplitudes \( \phi_{2,k} \) in terms of the real quantities \( (\phi_{+,} \phi_{-}) \) defined through
\[ \phi_{2,k} = \phi_k \mp i \phi_{-k}. \]

One can easily check that \( \alpha_{2,k}(\beta_{2,k}) \) are the Fourier cosine (sine) expansion coefficients of \( \phi_+ \) (\( \phi_- \)), and \( \alpha_{0,k} \) are the Fourier cosine expansion coefficients of \( \phi_0 \).

We introduce the Fourier transform of the kernel function
\[ \gamma_{m,k} = \int_{-\infty}^{\infty} d\tau \, \gamma_m(\tau) e^{-i\omega_k \tau}. \]
It is convenient to work with a dimensionless form \( \tilde{\gamma}_{m,k} \) defined through
\[ \tilde{\gamma}_{m,k} = \tilde{\gamma} \tilde{\gamma}_{m,k}. \]
where \( \tilde{\gamma} \) is some characteristic value of the damping rate. For the case of the standard Navier–Stokes viscosity, \( \tau_c \to 0 \), we have \( \tilde{\gamma} = \gamma_0 \) and \( \tilde{\gamma}_{m,k} = 1 \).

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In order to find solutions of equation (9), we first express \( f^m_\nu \) in terms of \( \gamma_{m,k} \) and \( b_{m,k} \) with the help of equations (15) and (23), so obtaining

\[
f^m_\nu = -i\Omega \sum_{-\infty}^{\infty} (k + m\sigma) \gamma_{m,k} b_{m,k} e^{ik\tau},
\]

(32)

where

\[
\sigma = \Omega/\Omega_1.
\]

(33)

After substitution of equations (23), (24) and (32) into equation (9), we obtain

\[
b_{m,k} = W_m C \phi_{m,k}/R,
\]

(34)

with

\[
R = \omega_0^2 - [k^2\Omega + m\beta\Omega_1 - i(k + 2m\sigma)\gamma_{m,k}] \Omega.
\]

(35)

We obtain expressions for \( \langle E_v \rangle \) and \( \langle L_v \rangle \) by substituting (23) and (32) in equations (19) and (20) and averaging the result over a period \( 2\pi/\tau \), so obtaining

\[
\langle E_v \rangle = -2\Omega^2 \sum_{1}^{\infty} \left[ k(k - 2\sigma)\gamma_{2-k,b_{2-k},b^*_{2-k}} + (k + 2\sigma)\gamma_{2-k,b_{2-k},b^*_{2-k}} + k^2\gamma_{0,k,b_{0,k},b^*_{0,k}} \right],
\]

(36)

\[
\langle L_v \rangle = -4\Omega^2 \sum_{1}^{\infty} \left[ (k - 2\sigma)\gamma_{2-k,b_{2-k},b^*_{2-k}} - (k + 2\sigma)\gamma_{2-k,b_{2-k},b^*_{2-k}} \right],
\]

(37)

where we have used the symmetry properties of \( b_{m,k} \) and \( \gamma_{m,k} \) as well as the fact that the Fourier coefficients of the potential vanish when \( k = 0 \) and \( m = 2 \), to obtain a sum over positive integers \( k \) only. It is easy to see that

\[
\langle E_{\text{diss}} \rangle = -2\Omega^2 \sum_{1}^{\infty} \left[ (k + 2\sigma)^2\gamma_{2-k,b_{2-k},b^*_{2-k}} + (k - 2\sigma)^2\gamma_{2-k,b_{2-k},b^*_{2-k}} + k^4\gamma_{0,k,b_{0,k},b^*_{0,k}} \right]
\]

(38)

is always negative when \( \gamma_{m,k} > 0 \).

Equations (36)–(38) and (34) together with expressions (26) for \( \phi_{m,k} \) and the standard Keplerian expressions for the orbital energy and angular momentum give a complete set of equations for our model.

### 2.4 The equilibrium tide limit

#### 2.4.1 General energy and angular momentum exchange rates

In the approximation scheme corresponding to calculation of the equilibrium (or quasi-static) tide, the orbital and rotational frequencies and the mode damping rate are very much smaller in magnitude than the oscillation frequency \( \omega_0 \). In this case these can be neglected in the denominator \( R \) of equation (34). The Fourier mode amplitudes \( b_{m,k} \) are then simply proportional to \( \phi_{m,k} \) such that

\[
b_{m,k} = W_m C \phi_{m,k}/\omega_0^2.
\]

(39)

Substituting (39) in (36) and (37) we obtain

\[
\langle E_v \rangle = -S_1 E^0_v,
\]

(40)

\[
\langle L_v \rangle = -S_1 L^0_v,
\]

(41)

where

\[
E^0_v = \frac{3\pi \Omega^2 C^2 \bar{\gamma}}{10 \omega_0^2}, \quad L^0_v = \frac{3\pi \Omega C^2 \bar{\gamma}}{5 \omega_0^2},
\]

and

\[
S_1 = \sum_{1}^{\infty} \left[ k(k - 2\sigma)\phi_{2-k,\phi^2_{2-k}} + (k + 2\sigma)\phi_{2-k,\phi^2_{2-k}} + \frac{2}{3}k^3\gamma_{0,k,\phi_{0,k}} \right],
\]

(42)

\[
S_2 = \sum_{1}^{\infty} \left[ (k - 2\sigma)\phi_{2-k,\phi^2_{2-k}} - (k + 2\sigma)\phi_{2-k,\phi^2_{2-k}} \right].
\]

(43)

Here we use the definitions of \( \bar{\gamma} \) and \( \bar{\gamma}_{m,k} \), the explicit form of \( W_m \) and the fact that \( \phi_{m,k} \) are real. The quantities \( E^0_v \) and \( L^0_v \) provide characteristic values of the rates of change of the energy and angular momentum and set a characteristic time-scale of tidal evolution. Taking into account the definition of \( C \) (equation 14) and the normalization condition for the eigenmodes (5), one can easily see that these quantities have the correct dimensions of energy and angular momentum per unit of time, respectively.

As discussed above, \( |\phi_{2-k}| \ll |\phi_{2-k}| \) and in a good approximation we can neglect these quantities have the correct dimensions of energy and angular momentum per unit of time, respectively.

Equations (36)–(38) and (34) together with expressions (26) for \( \phi_{m,k} \) and the standard Keplerian expressions for the orbital energy and angular momentum give a complete set of equations for our model.

#### 2.4.2 An alternative form

There is a form of equations (40)–(44) that is useful for relating the work here to that of Hut (1981) in which he studied tidal interactions of general orbits using a constant response time-lag approximation together with the assumption of equilibrium tides. We write them in the form:

\[
\langle E_v \rangle = -\bar{E}_1^0 (\Psi_1 - 2\sigma \Psi_2),
\]

(47)

\[
\langle L_v \rangle = -\bar{L}_1^0 (\Psi_2 - 2\sigma \Psi_3),
\]

(48)

where

\[
\Psi_1 = \frac{1}{2} \sum_{k}^\infty \left[ \bar{\gamma}_{2-k,\phi_{2-k}}(\alpha_{2-k} + \beta_{2-k})^2 + \bar{\gamma}_{2-k,\phi_{2-k}}(\alpha_{2-k} - \beta_{2-k})^2 \right]
\]

(49)

\[
\Psi_2 = \frac{1}{2} \sum_{k}^\infty k \left[ \bar{\gamma}_{2-k,\phi_{2-k}}(\alpha_{2-k} + \beta_{2-k})^2 - \bar{\gamma}_{2-k,\phi_{2-k}}(\alpha_{2-k} - \beta_{2-k})^2 \right],
\]

(50)

\[
\Psi_3 = \frac{1}{2} \sum_{k}^\infty \left[ \bar{\gamma}_{2-k,\phi_{2-k}}(\alpha_{2-k} + \beta_{2-k})^2 + \bar{\gamma}_{2-k,\phi_{2-k}}(\alpha_{2-k} - \beta_{2-k})^2 \right].
\]

(51)

Here we have expressed \( \phi_{2-k} \) in terms of \( \alpha_{2-k} \) and \( \beta_{2-k} \) using equation (27). Obviously, we have \( S_1 = \Psi_1 - 2\sigma \Psi_2 \) and \( S_2 = \Psi_2 - 2\sigma \Psi_3 \).
2.4.3 The limit $t_\circ \to 0$

The limit $t_\circ \to 0$ corresponds to local behaviour and a Navier–Stokes viscosity. Assuming in this limit that $\gamma_m(t) = \gamma_0\delta(t)$, we may adopt $\gamma_m = 1$ in equations (49)–(51). In this case the infinite series (49)–(51) can be exactly summed by use of arguments exploiting Parseval’s theorem.

Then we directly obtain from equations (49)–(51) that

$$\Psi^0 = \sum_{\alpha=1}^{\infty} k^2 (\alpha^2_{2,k} + \beta^2_{2,k} + \frac{1}{3} \alpha^0_{2,k})$$

(52)

(\Psi^2 = 2 \sum_{\alpha=1}^{\infty} \kappa \alpha_2 \beta_{2,k},

(53)

$$\Psi^3 = \sum_{\alpha=1}^{\infty} (\alpha^2_{3,k} + \beta^2_{3,k})$$

(54)

Using the definitions of $\alpha_{m,k}$ and $\beta_{m,k}$ (equation 28) and also of $\phi_0$, $\phi_-$ and $\phi_0$ (equation 29), one can easily show that the series (52)–(54) can be expressed in terms of integrals:

$$\Psi^0 = \frac{1}{\pi} \int_0^{2\pi} d\tau \left[ \left( \frac{d\phi_-}{d\tau} \right)^2 + \left( \frac{d\phi_0}{d\tau} \right)^2 + \frac{1}{3} \left( \frac{d\phi_0}{d\tau} \right)^2 \right].$$

(55)

$$\Psi^2 = \frac{1}{\pi} \int_0^{2\pi} d\tau \left( \phi_+ - \phi_0 \right) \left( \phi_+ - \phi_0 \right).$$

(56)

$$\Psi^3 = \frac{1}{\pi} \int_0^{2\pi} d\tau \left( \phi^2_+ + \phi^2_0 \right).$$

(57)

The evaluation of the integrals (55) and (56) is straightforward, with the result:

$$\Psi^0 = \frac{8}{\epsilon^5} \left( 1 + \frac{31}{2} \epsilon^2 + \frac{255}{8} \epsilon^4 + \frac{185}{16} \epsilon^6 + \frac{25}{64} \epsilon^8 \right).$$

(58)

$$\Psi^2 = \frac{4}{\epsilon^{11}} \left( 1 + \frac{15}{2} \epsilon^2 + \frac{45}{8} \epsilon^4 + \frac{5}{16} \epsilon^6 \right).$$

(59)

$$\Psi^3 = \frac{2}{\epsilon^9} \left( 1 + 3 \epsilon^2 + \frac{3}{8} \epsilon^4 \right).$$

(60)

where $\epsilon = \sqrt{1 - e^2}$.

When expressions (50)–(60) are inserted into equations (47) and (48), the resulting exchange rates of energy and angular momentum are identical to the corresponding expressions obtained by Hut (1981) provided we make the identification

$$k_H = \frac{4\pi \gamma M_\ast (GQ)^2}{5 \omega_0^2 R_p^8},$$

(61)

where $R_p$ is the primary radius, $k_H$ is the apsidal motion constant defined as in Hut (1981), and $T_H$ is Hut’s typical time of tidal evolution.

Thus, in the limit of constant (frequency-independent) viscosity, our normal-mode approach to the problem is equivalent to the standard constant time-lag approach (Alexander 1973; Hut 1981). However, it has the advantage of enabling us to express the unspecified evolution time $T_H$ in terms of micro-physical quantities such as the kinematic viscosity $\nu$ and the primary density $\rho$ and quantities related to the normal mode (see also Eggleton et al. 1998). Note that equation (61) has been obtained by IP in the limit of highly eccentric orbits4 (see also equations 23–25 of Kumar, Ao & Quataert 1995).

3 THE FOURIER EXPANSION OF THE TIDAL FORCE AMPLITUDE

Here we briefly compare the approximate values for the Fourier coefficients $\alpha_{2,k}$ and $\alpha_{0,k}$ obtained with our analytic approximations derived in the Appendix with the exact coefficients obtained numerically. The various coefficients are plotted in Figs 1, 2 and 3.

In Fig. 1 we compare values of the Fourier coefficients obtained analytically with those obtained numerically for small $e = 0.2$ and 0.3. One sees that $\alpha_{0,k}$ decreases monotonically with $k$ and that the analytic and numerically obtained values are in good agreement for practically all values of $k$.

The difference is largest and of order of 10 per cent at $k = 1$ and rapidly decreases thereafter with $k$. The coefficient $\alpha_{2,k}$ has a maximum at $k_{max} = 2$ for $e = 0.2$ and at $k_{max} = 3$ for $e = 0.3$. For values of $k > k_{max}$ the analytic and numerically obtained values differ by at most 10 per cent and approach each other rapidly with increasing $k$. For values of $k < k_{max}$ the agreement between the analytic and numerically values is less good.

In Fig. 2 the same comparison is given for intermediate values of eccentricity $e = 0.4, 0.5$ and 0.6. In this case the agreement between numerical and analytic curves is much better. The disagreement clearly decreases with increasing $e$, and for $e = 0.6$ the curves for $\alpha_{0,k}$ differ by less than or of the order of $\sim 1$ per cent. In the case of $\alpha_{2,k}$ the agreement is less good. However, in contrast with the case of smaller $e$, the disagreement remains small even for $k < k_{max}$ provided that $k$ is sufficiently large. For example, the curves corresponding to $e = 0.6$ differ by less than $\sim 10$ per cent for $k > 4$.

In Fig. 3 we show the comparison for $e = 0.7, 0.8$ and 0.9. The disagreement between analytic and numerically obtained values is very small for all interesting values of $k$. The numerical and analytic curves corresponding to $\alpha_{0,k}$ practically coincide with each other.

4 Note a misprint in IP. Their coefficients $\beta_1$ and $\beta_2$, and accordingly the numerical factor entering their equation (46), analogous to equation (61), must be divided by 2.

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4 ORBITAL EVOLUTION

The evolution equations for the orbital parameters follow from equations (21) and equations (47) and (48) giving two equations for the rate of change of orbital semimajor axis and specific angular mo-

mentum in the form

\[ \dot{a} = -\frac{2E_0^0a^2}{GM}\left(\Psi_1 - 2\sigma\Psi_2\right) \]
\[ = -\frac{3\sigma}{5}\frac{(GM)^3}{M_p^0}\frac{Q^2}{a^4}\frac{\gamma(1+q)}{a^{15/2}}\left(\Psi_1 - 2\sigma\Psi_2\right), \]
\[ \dot{L}_{\text{orb}} = \frac{L_0}{M_p}(\Psi_2 - 2\sigma\Psi_3) \]
\[ = \frac{3\sigma}{5}\frac{(GM)^3}{M_p^0}\frac{Q^2}{a^4}\frac{\gamma(1+q)}{a^{15/2}}\left(\Psi_2 - 2\sigma\Psi_3\right), \]

where \( L_{\text{orb}} \) is the specific orbital angular momentum and we recall that \( a \) is the semimajor axis, \( M_p \) is the mass of the primary, and \( q = M_p/M \).

To determine the ratio of the primary rotation rate and the orbital mean motion, \( \sigma = \Omega_1/\omega \), and hence close the set of equations (62) and (63), we use the law of conservation of angular momentum in the form

\[ L_{\text{orb}} + I\Omega_t = L_{\text{ini}}, \]

where \( I \) is the moment of inertia of the primary per unit mass and \( L_{\text{ini}} \) is taken to be the ‘initial’ value of the specific orbital angular momentum at the beginning of the orbital evolution due to equilibrium tides. Note that in addition to \( \sigma \), the quantities \( \Psi_1 \), \( \Psi_2 \) and \( \Psi_3 \) depend on eccentricity \( e \), which itself may be expressed in terms of \( a \) and \( L_{\text{orb}} \) by use of the standard relation \( e = \sqrt{1 - (L_{\text{orb}}/GMa)} \).

It is convenient to rewrite equations (62) and (63) in dimensionless form by use of a length-scale, \( a_0 \), which is defined to be the semimajor axis corresponding to the tidal equilibrium state of the dynamical system found from the conditions (\( da/dt = dL_{\text{orb}}/dt = 0 \)).

When \( a = a_0 \) is determined in this way, we also have \( e = 0, \sigma = 1 \) in equilibrium. We define the dimensionless semimajor axis, \( x \), to be \( x = a/a_0 \) and the dimensionless angular momentum to be \( y = L_{\text{orb}}/L_0 \), where \( L_0 = \sqrt{GMa_0}/(1 + q) \).

In terms of these variables, equations (62) and (63) take the form

\[ \frac{dx}{dt} = -\frac{1}{x^3}\left(\Psi_1 - 2\sigma\Psi_2\right), \]
\[ \frac{dy}{dt} = \frac{1}{x^{15/2}}\left(\Psi_2 - 2\sigma\Psi_3\right), \]

where we introduce the dimensionless time \( \tau = t/t_0 \) with

\[ t_0 = \frac{5}{3\pi}\frac{M_0\alpha_0^2}{(GM)^2\gamma(1+q)}, \]

The time \( t_0 \) defines a characteristic time-scale for tidal evolution of the orbital elements, in this case due to equilibrium tides.

We note that, in terms of the dimensionless variables, the eccentricity \( e = \sqrt{(1 - y^2/x)} \) and the law of conservation of angular momentum (64) takes the form

\[ y + \frac{1}{x^{15/2}} = y_{\text{ini}}, \]

where \( y_{\text{ini}} = L_{\text{ini}}/\sqrt{GMa_0} \). We recall that \( \sigma = \Omega_1/\Omega \) and a dimensionless moment of inertia \( I = I(1 + q)/a_0^{15/2} \).

As for most cases of astrophysical interest, the angular momentum content of the orbit dominates that associated with the rotation of the primary; for simplicity, we consider below only the case when the primary effectively has low inertia assuming that \( I \ll 1 \).

As we see below, in this case, the orbital evolution of the binary is essentially independent of the value of \( I \). When the viscosity Fourier transform coefficients \( \gamma_{m,k} \) are specified, the set of equations (65), (66) and (68) becomes complete.

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we shall allow for general values of viscosity. When the potential pattern associated with the Fourier component \( \gamma_k \) is sufficiently large, the contribution of the Fourier term in the expansion of the tidal forcing amplitude,\footnote{We stress that in the case of forced oscillations, the corresponding global disturbance with particular \( m \) and \( k \) oscillates with the frequency of the Fourier term in the expansion of the tidal forcing amplitude.}\n
\[
\gamma_{m,k} = \frac{\gamma_0}{1 + (\omega_{m,k} \omega_{k})^p},
\]

where \( \omega_{m,k} = |k \Omega + m \Omega_i| \)

is the apparent frequency associated with the global disturbance as viewed in the frame corotating with the primary, it being assumed that that is well defined.\footnote{We stress that in the case of forced oscillations, the corresponding global disturbance with particular \( m \) and \( k \) oscillates with the frequency of the Fourier term in the expansion of the tidal forcing amplitude.}\n
### 4.1 Explicit form of the viscosity coefficients

Before solving the equations of tidal evolution we should specify the viscosity Fourier transform coefficients \( \gamma_{m,k} \). As we have mentioned above, it was suggested by Goldreich & Nicholson (1977) and Goldreich & Keeley (1977) that the action of an effective viscosity arising from convective turbulence should be suppressed when the typical time-scale associated with convective eddies is much larger than the time-scale associated with the global disturbance on which it is presumed to act. To take this effect into account in the simplest manner possible, we adopt expressions for \( \gamma_{m,k} \) of the form

\[
\gamma_{m,k} = \frac{\gamma_0}{1 + (\omega_{m,k} \omega_{k})^p},
\]

where \( \omega_{m,k} = |k \Omega + m \Omega_i| \)

is the apparent frequency associated with the global disturbance as viewed in the frame corotating with the primary, it being assumed that that is well defined.\footnote{We stress that in the case of forced oscillations, the corresponding global disturbance with particular \( m \) and \( k \) oscillates with the frequency of the Fourier term in the expansion of the tidal forcing amplitude.}\n
Goldreich & Nicholson (1977) and Goldreich & Keeley (1977) adopted \( p = 2 \). However, Zahn (1989) has argued that \( p \sim 1 \). Here we shall allow for general values of \( p \). Equation (69) can be brought into the standard form (31) with

\[
\hat{\gamma}_{m,k} = \frac{T_p^p}{(1 + T_p^p |k + m \sigma| x^{-3p/2})^p}, \quad \hat{\gamma} = \frac{\gamma_0}{T_p^p},
\]

where the dimensionless parameter \( T_p = t_c \sqrt{G M / a_0} \) determines the magnitude of the suppression of the viscosity. We assume below that \( T_p \gg 1 \). In this limit, many details of the orbital evolution are essentially independent of the particular value of this parameter.

### 4.2 Effective spin–orbit resonances

A feature arising in the limit \( T_p \gg 1 \) is that the action of viscosity is most effective for a particular forcing term when the rotational and orbital frequencies are such that \( (k + m \sigma) = 0 \). When this is satisfied, the potential pattern associated with the Fourier component \( (m, k) \) of the forcing potential corotates with the primary. The relative forcing frequency is thus zero and there is no suppression of the viscosity. When \( T_p \gg 1 \), viscosity acts strongly only very close to the corotation resonances with \( (k + m \sigma) = 0 \), occurring when \( (k = 1, 2, 3, \ldots, m = -2) \). Hence, the terminology associated with a resonance is used.

As we shall see below, the qualitative nature of the orbital evolution depends on whether the parameter \( p \) exceeds or is less than unity. This is related to the resonant behaviour considered above.

Let us suppose that \( T_p \gg 1 \) and we are close to a resonance such that the value of the dimensionless rotational frequency \( \sigma \) is close to some integer, \( n \), divided by 2, thus \( \sigma \approx n/2 \). Then it follows from equations (43) and (44) that the quantities \( S_1 = (\Psi_j - 2 \sigma \Psi_2) \) and \( S_2 = (\Psi_2 - 2 \sigma \Psi_3) \) are mainly determined by the terms in the respective summations for these expressions with \( k = n \) and \( m = -2 \), which are resonant in the above sense. Later we call these terms the ‘resonant terms’.

We have \( S_1 = S_1' + S_1^{\text{res}} \) and \( S_2 = S_2' + S_2^{\text{res}} \), where

\[
S_1' = \frac{2 T_p^p n(n - 2 \sigma) p_{2,n}^2}{(1 + T_p^p |n - 2 \sigma| x^{-3p/2})},
\]

\[
S_2' = \frac{2 T_p^p n(n - 2 \sigma) p_{2,n}^2}{(1 + T_p^p |n - 2 \sigma| x^{-3p/2})},
\]

and \( S_1^{\text{res}} \) and \( S_2^{\text{res}} \) stand for all other (non-resonant) terms in the series.

When we have exact equality, \( \sigma = n/2 \), the resonant terms are equal to zero. However, their absolute values increase very sharply with increase of the difference \( |\sigma - n/2| \). For example, when \( T_p \gg 1 \) and \( |\sigma - n/2| \sim T_p^{-1} \ll 1 \), the resonant terms are proportional to \( T_p^{-p-1} \) in absolute magnitude. Therefore, for the case \( p > 1 \) they can, in principle, be much larger than the non-resonant terms. On the other hand, in the case of \( p < 1 \) the contribution of the resonant terms to \( S_1 \) and \( S_2 \) is insignificant. We shall see below that, in the case \( p > 1 \), the orbital evolution can proceed through a stage where the condition \( \sigma \approx n/2 \) is maintained for a long time. Thus, the system can evolve in a state that maintains a specific spin–orbit corotation resonance.

### 4.3 The case \( p < 1 \)

When the parameter \( p \) is smaller than 1, the orbital evolution of a low-inertia primary star is, in many details, similar to the case of assumed constant time-lag between response and forcing potential considered by Alexander (1973) and Hut (1981). This also corresponds to the standard Navier–Stokes viscosity \( (t_c = 0) \). If the initial value of \( \sigma \) is sufficiently small, because of the small inertia of the primary, the system quickly relaxes to a state where the orbital angular momentum is approximately conserved and we can assume that \( y \approx y_{in} \approx 1 \). In this case the eccentricity and the semimajor axis are related to each other as

\[
e \approx (1 - 1/x).
\]

Also because of the low inertia of the primary, its rotation rapidly approaches the value given by \( \Omega_i \approx \sigma \Omega \), where \( \sigma_{ps} \) corresponds to a state of so-called ‘pseudo-synchronization’ (Hut 1981) and is obtained from equation (66) after setting \( dy/dx = 0 \).

In a state of pseudo-synchronization, \( \sigma_{ps} \) is given by

\[
\sigma_{ps} = \frac{\Psi_2}{2 \Psi_3}.
\]

However, note that both \( \Psi_2 \) and \( \Psi_3 \) are, in general, functions of \( \sigma_{ps} \), and therefore equation (74) should be considered as a non-linear algebraic equation for \( \sigma_{ps} \). It is convenient to express its solution in the form \( \sigma_{ps} = \sigma_{ps} \sigma_H \), where \( \sigma_H \) represents the solution of (74) for the case of constant time-lag or the standard Navier–Stokes viscosity \( (t_c = 0) \). This is given by (Hut 1981)

\[
\sigma_H = \frac{\Psi_2}{2 \Psi_3} = \frac{1}{e^3} \left( 1 + \frac{12}{5} e^2 + \frac{28}{15} e^4 + \frac{2}{3} e^6 \right),
\]

\[
e = \sqrt{(1 - e^2)}
\]

and we have made use of equations (59) and (60).
We show the dependence of $\sigma_1$ on $e$ obtained numerically for different values of $p$ in Fig. 4. We see that the difference between $\sigma_{ps}$ and $\sigma_H$ is rather small for sufficiently large eccentricities. This comes about, in simple terms, because the inverse stellar rotation frequency $\Omega_{ps}^{-1} = 1/(\sigma_{ps} \Omega)$ at pseudo-synchronization is always of the order of a characteristic time of periastron passage.

To obtain an approximate equation for the evolution of the semimajor axis, we substitute equation (74) into equation (65) to obtain

$$\frac{dx}{dt} = \frac{1}{x^7} \left( \Psi_1 - \frac{\Psi_2}{\Psi_3} \right).$$

(76)

As we have mentioned above, resonant effects are not important when $p < 1$. That means that to a good approximation we may neglect unity in the denominator of expressions (71) in the limit $T_+ \to \infty$. Then the functions $\Psi_1$, $\Psi_2$ and $\Psi_3$ in equation (76) do not depend on the parameter $T_+$. Taking into account equation (73) we see that for a given $p$ the right-hand side of equation (76) is only a function of the semimajor axis $x$. In that case, the variables in equation (76) can be separated and equation (76) can easily be integrated by numerical means. We show the result of integration of equation (76) in Figs 5 and 6 together with the results of numerical integration of the full set of equations (65), (66) and (68). One can see that the approximate semi-analytic approach gives a very good approximation to the solutions of the full set of equations.

Note that for our numerical integration we set $e = 0.9$ initially and terminated the integration when $e = 0.2$. We use a very large value of $T_+ = 10^4$ and a small value of the dimensionless moment of inertia $\tilde{I} = 10^{-5}$. We have checked that other reasonable values of these parameters do not lead to a different evolution of our dynamical system. We have used our approximate expressions for the Fourier coefficients of the tidal forcing amplitude in numerical integration of equations (65), (66) and (68), and good agreement between numerical and semi-analytic calculations implies that use of the approximate analytic expressions for the Fourier coefficients is justified.

In Fig. 5 we show numerical solutions of our equations setting the initial value of the rotational parameter $\sigma_{in}$ equal to zero, and in Fig. 6 we consider the case of a high $\sigma_{in} = 100$.

4.4 The case $p > 1$

In this case the evolution is qualitatively different from when $p < 1$. This is because the effects of spin–orbit resonances become important. To see this, let us consider the case $p = 2$ where a simple analytic approach is possible in detail.

4.4.1 Pseudo-synchronization

Assume that the moment of inertia is very small and the rotation of the primary is close to a spin–orbit resonance of the order $n$ so that $\Delta \sigma = \sigma - n/2 \ll 1$. Then just as for the case $p < 1$ the assumption...
of low primary inertia means that we can consider the orbital angular momentum to be a conserved quantity and set $dy/d\theta = 0$.

It then follows from equation (66) that $S_2 = \Psi_2 - 2z \Psi_3 = 0$. As discussed above we divide $S_2$ into resonant and non-resonant parts; writing $S_2 = S_2^r + S_2^w$ we must then have $S_2^r = -S_2^w$. Using the explicit form of $S_2^r$ (see equation 72) we obtain

$$\Delta \sigma = \frac{S_2^w}{1 + 4T_s^2x^{-3/2}(|\Delta \sigma|)^2} = \frac{S_2^w}{4T_s^2\phi_2^{\perp-n}}.$$  \hspace{1cm} (77)

The solution of this equation gives the deviation of $\sigma$ from the value corresponding to exact resonance as

$$\Delta \sigma = \frac{x^3\phi_2^{\perp-n}}{2S_2^w} \left[ 1 - \sqrt{1 - \left( \frac{S_2^w}{x^3T_s^2\phi_2^{\perp-n}} \right)^2} \right],$$  \hspace{1cm} (78)

where we do not consider the second unphysical root. For a physically meaningful solution the expression contained within the square root in equation (78) must be positive. This implies that the dynamical system can stay in the resonance only if

$$\frac{|S_2^w|}{\phi_2^{\perp-n}} \leq x^{3/2}T_s,$$  \hspace{1cm} (79)

and therefore

$$|\Delta \sigma| \leq \Delta \sigma_{\max} = \frac{x^{3/2}}{T_s}.$$  \hspace{1cm} (80)

Condition (79) tells us that, for a given $T_s$ and $x$, evolution of the dynamical system is possible only if the ratio $|S_2^w|/\phi_2^{\perp-n}$ is sufficiently small. This can always be satisfied if $T_s$ is sufficiently large.

However, for a sufficiently large $n$, this ratio increases with decrease of eccentricity $e$. Therefore, during the process of tidal circularization, a resonance with some particular order $n = 2\sigma$ becomes at some point impossible. The dynamical system may then rapidly relax to some other resonance with $n' < n$ where the ratio is smaller and the condition (79) is satisfied. Thus, the analytic approach indicates that a low-inertia primary evolves through a sequence of spin–orbit resonances with decreasing $n$ during the process of tidal circularization. As we shall see below, this character of the evolution is confirmed by numerical simulation. Note that the qualitative character of the tidal evolution remains the same for any $p > 1$.

### 4.4.2 Evolution of the semimajor axis

Now let us discuss the evolution of the semimajor axis. We assume that condition (79) is satisfied and the primary evolves in a state of spin–orbit resonance with some particular order $n = 2\sigma$. In order to find an evolution law for the semimajor axis, we have to evaluate $S_1 = \Psi_1 - 2z \Psi_2$. In general, this quantity can be represented only in terms of complicated summations (see equation 43). However, for the case $p = 2$ these series can be performed in the limit $T_s \rightarrow \infty$.

The resulting expression is remarkably simple. Then we can go on to obtain an analytic expression for the dependence $x(t)$ (see below).

The quantity $S_1$ can also be divided into resonant and non-resonant parts (see equation 72): $S_1 = S_1^r + S_1^w$. It is easy to see that $S_1^r = nS_2^r$ and from the condition of constant orbital angular momentum it follows that $S_2^r = -S_2^w$, and we have

$$S_1 = S_1^w - nS_2^w.$$  \hspace{1cm} (81)

Now the explicit form of $S_1$ can be written down with the help of equations (43) and (44) in the form

$$S_1 = 2\sum_{k=n}^{\infty} \left[ (k-n)(k-2\sigma) \right] \eta_{2,-k} \phi_2^{\perp-k}$$

$$\quad + (k+n)(k+2\sigma) \eta_{2,+k} \phi_2^{\perp+k} + \frac{3}{2} k^2 \zeta_{0,k} \phi_0^{\perp,k}.$$  \hspace{1cm} (82)

In equation (82) we can set $2\sigma = n$. We can also neglect unity in the denominators of the expressions for $\eta_{m,k}$ and set $2\sigma = n$ there (see equations 71), so as to obtain

$$\eta_{2,\pm k} = x^{3/2} (k \mp n)!^2, \quad \zeta_{0,k} = x^{3/2}.$$  \hspace{1cm} (83)

Using equation (83) in equation (82) gives

$$S_1 = 2x^3 \sum_{k=1}^{\infty} \left( \alpha_{2,k}^2 + \beta_{2,k}^2 + \gamma_{2,k}^2 \right)$$

$$\quad = x^3 \sum_{k=1}^{\infty} \left( \alpha_{2,k}^2 + \beta_{2,k}^2 + \gamma_{2,k}^2 \right).$$  \hspace{1cm} (84)

where in the last equality we use expressions for $\phi_{m,k}$ in terms of $\alpha_{m,k}$ and $\beta_{m,k}$ (see equation 27). Note that the series (84) does not contain any divergence at $k = n$ and we include the term with $k = n$ in the series, assuming with justification when $n$ is large that it gives a negligible contribution.

According to Parseval’s theorem we have

$$\sum_{k=1}^{\infty} (\alpha_{2,k}^2 + \beta_{2,k}^2) = \frac{1}{\pi} \int_0^{2\pi} d\phi_2 (\phi_2^++\phi_2^-).$$

$$2\alpha_{0,0}^2 + \sum_{k=1}^{\infty} \alpha_{0,k}^2 = \frac{1}{\pi} \int_0^{2\pi} d\phi_2 \phi_2^2.$$  \hspace{1cm} (85)

Here we recall that the coefficients $\alpha_{2,0}$ and $\beta_{2,0}$ vanish and the inclusion of these terms in the summations is redundant. Furthermore the effect of $\alpha_{0,0}$ in the second summation is negligible for orbits of significant eccentricity. From the definitions of the quantities $\alpha_\pm$ and $\beta_\pm$ (see equation 29) we have $\phi_2^+ + \phi_2^- = \phi_2^0$. Accordingly, both integrals in equation (85) may be taken to be $\Psi_0^2 = 2(1 + 3e^2 + 5e^4)/e^9$ (see equation 60) with $e = \sqrt{(1 - e^2)}$, and

$$S_1 = \frac{4}{3} x^3 \Psi_0^2 = \frac{8}{3} x^3 e^2 \left( 1 + 3e^2 + \frac{3}{8} e^4 \right).$$  \hspace{1cm} (86)

Taking into account that in our case the orbital angular momentum is approximately conserved, we can express $e$ in terms of $x$ using equation (73), and substituting equation (86) in equation (65) we obtain a remarkably simple equation for the evolution of the semimajor axis, which takes the form

$$\frac{dx}{dt} = -\frac{35}{3} x^{-3/2} \left( x^2 - \frac{6}{7} x + \frac{3}{35} \right).$$  \hspace{1cm} (87)

Note that equation (87) can be easily generalized to take into account the term $\alpha_{0,0}$ and also the terms determined by a non-zero rotational angular momentum of the primary. The solution of equation (87) can be expressed in terms of elementary functions

$$\tilde{t} = \left( \frac{6}{35} \sqrt{x} + \frac{1}{10} \sqrt{\frac{15}{2} \ln(z)} + \tilde{C} \right),$$  \hspace{1cm} (88)

where $\tilde{C}$ is a constant of integration and

$$z = \frac{\sqrt{x} - \sqrt{x+1}}{\sqrt{x} + \sqrt{x+1}},$$  \hspace{1cm} (89)
where \( x_\pm = \frac{1}{3}(1 \pm 2\sqrt{\frac{e}{2}}) \) and \( c_\pm = x_\pm^{3/2}/2 \). It is very important to note that solution (88) does not depend on the order \( n \) of some particular resonance. Therefore it describes approximately not only the evolution of the dynamical system in a state of spin–orbit resonance with a particular value of \( n \), but also a system evolving from one resonance to another but spending most of its time in a state of spin–orbit resonances with different values of \( n \).

4.5 Numerical calculations

We solve equations (65) and (66) numerically for different initial values of the rotational parameter \( \sigma_{\text{ini}} \) and \( p = 2 \), and also for \( \sigma_{\text{ini}} = 0 \) and \( p = 1.5 \) and 2.5. The dimensionless parameter \( T_* \) is set to 500 and the dimensionless moment of inertia \( I \) is set to 1/300 for all numerical calculations. We have checked that other values of these parameters do not change the dynamical evolution significantly provided that \( T_* \) is sufficiently large and \( I \) is sufficiently small. All calculations are started with \( e = 0.9 \) and terminated when \( e = 0.2 \).

We start by discussing the case \( p = 2 \). In Fig. 7 we show the dependence of \( x = a/a_\text{eq} \) on time. It is seen from this figure that the analytic solution (88) is in very good agreement with the numerical solutions corresponding to \( \sigma_{\text{ini}} = 0 \) and 10. It also approximates rather closely the case of intermediate \( \sigma_{\text{ini}} = 50 \). Results of calculation with high initial values of \( \sigma = 75 \) and 100 are in a good agreement with our analytic expression only for sufficiently large values of time \( \tilde{t} \). In the case of a large \( \sigma_{\text{ini}} \), initially, the semimajor axis (and eccentricity) grows with time.\(^6\) Accordingly, the orbital energy also increases at the beginning of evolution. It turns out that the rotational energy of the primary is transferred to the orbital energy on a short time-scale and this drives an increase of the semimajor axis. In Fig. 8 we show the evolution of the orbital and rotational energies per unit of mass and their sum with time for the case \( \sigma_{\text{ini}} = 75 \). The sum is gradually decreasing with time. That means that it is dissipated by the frictional processes. However, the orbital energy sharply increases and the rotational energy sharply decreases at time \( \tilde{t} < \tilde{t}_l \approx 0.05 \). As seen from Fig. 7 the moment of time \( \tilde{t}_l \) corresponds to the maximum of the curve \( x(t) \) with \( \sigma_{\text{ini}} = 75 \). In Fig. 9 we show the dependence of \( \sigma \) on time for the cases with different \( \sigma_{\text{ini}} \). The solid curve corresponding to \( \sigma_{\text{ini}} = 0 \) shows a sharp growth of \( \sigma \) with time to the value \( \sigma = 8 \). Then the dynamical system is evolving in the \( \sigma = 8 \) resonance for a period of time \( \tilde{t} < \tilde{t}_l \approx 0.23 \). After the time \( \tilde{t}_l \), the system evolves through a sequence of resonances with decreasing values of \( \sigma \). The solution corresponding to \( \sigma_{\text{ini}} = 10 \) evolves in the \( \sigma = 10 \) resonance from the very beginning. The late-time evolution is similar to the previous case. The initial evolution of \( \sigma \) for the case \( \sigma_{\text{ini}} = 50 \) looks like a monotonic decrease of \( \sigma \). In fact, as we see in Fig. 10, the system is evolving through a large number of resonances with decreasing resonance order \( n = 2\sigma \), but does not stay in any of these resonances for a long time. The curves corresponding to \( \sigma_{\text{ini}} = 75 \) and 100 are similar to the previous case with the exception that these curves show that the initial value of \( \sigma \) is maintained for a short initial period of time. This period of time corresponds to the stage of initial increase of the semimajor axis. In Fig. 10 we show the evolutionary tracks of our dynamical system on the plane \((e, \sigma)\). As seen from this figure, the late-time evolution proceeds through a sequence of spin–orbit resonances with decreasing resonance order \( n \). The dotted curve shows the analytic dependence \( \sigma(t/e) \) corresponding to the case of the standard viscosity. It is important to point out that all curves corresponding to \( p = 2 \) have \( \sigma > \sigma_{\text{ini}} \) for, practically, all values of eccentricity. Thus, a delayed response of convective motions to the tidal forcing can lead to a high state of rotation of the primary star. It is also very interesting to note that the dynamical system can evolve in a state of spin–orbit resonance even when the eccentricity is relatively small \( \sim 0.2 \).

Finally, let us briefly discuss the case of \( p \neq 2 \). In Figs 11 and 12 we show the dependence of \( x \) on time \( \tilde{t} \) and the evolutionary tracks on the plane \((e, \sigma)\) for \( p = 1.5 \) and 2.5 in comparison with the previous case of \( p = 2 \). The initial value of the rotational parameter \( \sigma_{\text{ini}} \) is set to zero. From Fig. 11 it is seen that the case \( p = 1.5 \) \((p = 2.5)\) evolves faster (slower) than the case \( p = 2 \) with respect to the time \( \tilde{t} \) towards the low-eccentricity state. This effect is similar to what was found in the case of small \( p < 1 \) considered above (see Fig. 5). Fig. 12 shows that the effect of evolution through a sequence

\(^6\) Note that for all cases considered hereafter the orbital angular momentum is conserved in a good approximation and equation (73) is valid.

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of spin–orbit resonances remains valid for the different values of $p > 1$. The resonances are less prominent for the $p = 1.5$ case, and therefore the corresponding curve follows the curve representing the standard Navier–Stokes viscosity case on a large scale. On the other hand, the resonances corresponding to $p = 2.5$ case are very strong, and the corresponding dynamical system evolves in $\sigma = 4$ resonance until a small eccentricity $e \sim 0.3$ is reached.

5 DISCUSSION AND CONCLUSIONS

In this paper we have developed a new self-consistent formalism for the calculation of the rate of energy and angular momentum transfer from an eccentric binary orbit to a fully convective primary planet or star. This is based on Fourier expansion in time of the tidal forcing potential and a normal-mode expansion of the tidal response. To undertake calculations of orbital evolution, we have considered only the $(l = 2)$ fundamental mode, which dominates the response when the primary does not rotate, or when any potentially resonant inertial modes present when it rotates have poor overlap with the forcing potential.

We have assumed that the dissipation of the tidal response is due to an effective viscosity induced by convective turbulence. This is presumed to have an associated relaxation time or eddy turnover time that causes a delocalization of the dissipation process in time. It results in a weakening of the dissipation for tidal forcing at relative frequencies exceeding $1/t_c$. We considered the case when the frequency dependence of the dissipative response is $\propto 1/[1 + (\omega_{m,k} t_c)^p]$, where $\omega_{m,k}$ is the apparent frequency associated with the tidal forcing as viewed in the frame corotating with the primary in detail.

We use the fact that the orbital frequency is significantly smaller than the eigenfrequency of the $(l = 2)$ $f$ mode to introduce the equilibrium tide approximation. We note that, in general, this approximation is not equivalent to the classical assumption of a constant time-lag between tidal forcing and response (e.g. Alexander 1973; Hut 1981). However, we do show that they are equivalent only for the standard Navier–Stokes viscosity law with instantaneous action of the viscosity ($p = 0$).
We determine the orbital and rotational evolution of the primary numerically and analytically assuming that the angular momentum associated with the primary is much less than the orbital angular momentum for a range of values of $p$. We found that the evolution of the primary does not depend significantly on $T_\ast = \Omega t_\ast$ with $\Omega$ being the orbital frequency, provided that $T_\ast \gg 1$ and a scaled time $\tilde{t} \propto T_\ast^{-1/2} t$ is used.

However, the dependence on the parameter $p$ was found to be significant. In the case of small $p < 1$, the evolution is similar to the case of the standard Navier–Stokes viscosity ($p = 0$). When $p > 1$ and $T_\ast \gg 1$, the system evolves through a sequence of specific spin–orbit corotation resonances with $\Omega_\ast/\Omega = n/2$, where $\Omega_\ast$ is the rotation frequency and $n$ is an integer. For $p = 2$ we found an analytical expression for the evolution of semimajor axis given by equation (88).

We stress that for $p > 1$ the primary rotation frequency at ‘pseudo-synchronization’ is larger than obtained in the standard constant time-lag approximation ($p = 0$) and that the primary can evolve in a state of spin–orbit resonance even when the eccentricity is small ($e \sim 0.2$). This may have observational consequences. In principal, one could use observations of the rotation of a sufficiently bright star in a close eccentric binary to test different models for response of convective turbulence to the tidal forcing.

It should be possible to extend our formalism in several respects. For example, it can be generalized to take into account binaries with misaligned directions of orbital and rotational angular momenta [for the constant time-lag case, such a generalization has already been made by, for example, Alexander (1973) and Hut (1981)]

One can also extend the use of equations (34) and (35) in any situation where the contribution of other terms in the denominator is significant. In the case of small $\epsilon$ the standard Navier–Stokes viscosity ($p = 0$) is valid.

Now let us consider the resonance $k_\ast \approx k_\ast \approx k_\ast$, which is found from the condition that the denominator $R$ (equation 35) is close to zero, and the corresponding $b_{\nu_k}$ could be large. When $k_\ast > k_{\text{max}}$, the corresponding term in series (23) is not periodic, and the corresponding amplitude may be amplified with time. From the condition $k_\ast > k_{\text{max}}$ we find $\omega_0 \Delta P_{\text{orb}} > 1$,

$$\omega_0 \Delta P_{\text{orb}} > 1,$$

which, to within a numerical factor, coincides with a condition for stochastic growth of the mode amplitude due to dynamic tides found by IP. When this condition is fulfilled and the energy transfer from the orbit to the mode is larger than the energy transfer associated with the equilibrium tide, our approach is not valid.

Finally we comment that our results for the time to circularize an orbit starting from large eccentricity, $t_{\text{circ}}$, can be summarized for the various values of $p$ in the compact form

$$t_{\text{circ}} \approx \frac{5M_\ast \alpha_0^2 \tilde{a}_0^7}{3\pi GM \Omega^2} \gamma(1 + q).$$

Using equation (71) for $\gamma$, this becomes

$$t_{\text{circ}} \approx \frac{5M_\ast \alpha_0^2 \tilde{a}_0^7}{3\pi GM \Omega^2} \gamma(1 + q) (t_\text{circ} \sqrt{GM(1 + q)/\alpha_0^3})^p.$$ (93)

The dissipation rate $\gamma$ is given in IP as $\gamma = 0.1/t_\text{circ}$; inserting this into (93) and making some other reductions we obtain

$$t_{\text{circ}} \approx \left[ \frac{5}{3\pi} \left( \frac{\alpha_0^2 R_0^3}{GMp} \right)^{1/2} \left( \frac{M_\ast R_0^3}{Q^2} \right)^{1/2} \tilde{a}_0^8 \left( \frac{10}{1 + q} \tilde{t} \right) \right] \times (t_\text{circ} \sqrt{GM(1 + q)/\alpha_0^3})^p.$$ (94)

The product of the two quantities in square brackets in (94) can be estimated for a Jupiter-mass protoplanet in the late stages of its evolution from data given in IP to be $\sim 8$. Similarly IP gives $t_\text{circ} > 1$ yr. Using these values and the representative value $\tilde{t} = 0.2$, we find for $q = 10^{-3}$ that

$$t_{\text{circ}} > 10^{11} \left( \frac{2 \times 10^6 \text{ cm}}{R_0} \right)^8 \left( \frac{\alpha_0}{0.05 \text{ au}} \right)^8 (t_\text{circ} \sqrt{GM/\alpha_0^3})^p \text{ yr.}$$ (95)

This rather long time-scale confirms the finding of IP on the basis of an impulsive treatment of orbits with very high eccentricity that equilibrium tides associated with the fundamental mode may not account for the circularization of the orbits of the recently discovered extrasolar planets with orbital periods of a few days ($\alpha_0 = 0.05$ au) even when $p = 0$.

Use of more plausible values, say, $p > 1$, would give circularization times well beyond a lifetime of the planetary systems of a few gigayears. This may indicate that dynamic tides in the protoplanet (and possibly also the star for higher-mass protoplanets) and tidal inflation arising from dissipation in the protoplanet is important (see e.g. IP, and references therein). Note, however, that the physics of convective turbulence is very poorly understood and the validity of the mixing-length theory used is an issue. Also the contribution of the spectrum of inertial modes to the tide response may significantly shorten the circulation time-scale (see e.g. Ogilvie & Lin 2004).

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**REFERENCES**


APPENDIX A: APPROXIMATE ANALYTIC FORM FOR THE TIDAL FOURIER COEFFICIENTS

As we discussed above, the Fourier coefficients of the dimensionless tidal forcing amplitude $\Phi_{m,k}$ can be expressed in terms of two real quantities, $\phi_{m,k} = \frac{1}{2} (\alpha_{m,k} - \beta_{m,k})$, where

$$\alpha_{m,k} = \frac{1}{\pi} \int_0^{2\pi} d\tau \frac{\cos[m \Phi(\tau)] \cos(k \tau)}{D(\tau)^3},$$

$$\beta_{m,k} = \frac{1}{\pi} \int_0^{2\pi} d\tau \frac{\sin[m \Phi(\tau)] \sin(k \tau)}{D(\tau)^3}. \quad \text{(A1)}$$

Furthermore, it is sufficient to consider only the case of $m = 0$, 2 and positive $k$, and for this case we approximately have $\alpha_{2,k} \approx 2 \alpha_{0,k}$. Accordingly we find approximate expressions only for $\alpha_{2,k}$ and $\alpha_{0,k}$ hereafter.

The integrals given in (A1) can be represented in a more useful form with respect to the eccentric anomaly $\xi$ such that

$$\alpha_{2,k} = \frac{1}{\pi} \int_0^{2\pi} \frac{R(e, \cos \xi)}{(1 - e \cos \xi)^2} \cos(k \tau),$$

$$\alpha_{0,k} = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{(1 - e \cos \xi)^2} \cos(k \tau), \quad \text{(A2)}$$

where

$$R(e, \cos \xi) = 2e^2 - 1 - 2e \cos \xi + (2 - e^2) \cos^2 \xi. \quad \text{(A3)}$$

The integrals (A2) cannot be evaluated analytically. Hence, in order to obtain an analytic representation, some reasonable approximations are needed. Often the integrands are developed in powers of the eccentricity $e$ and integrated term by term. However, the resulting series converge very slowly and are not convenient for analytic calculations involving orbits with even a moderate eccentricity $e \sim 0.5$. Here we use another approach based on the fact that the integrals are mainly determined by the region of integration near periastron $\xi \approx 0$. Accordingly, we expand the quantities $R(e, \cos \xi)$, $D(\xi - e \sin \xi)$ in power series in $\xi$. Substituting these series in (A2) and truncating them, we get integrals that can be easily evaluated in principle. This approximation gives good results in the limit of highly eccentric orbits $e \to 1$, and also for eccentricities as small as 0.2 and sufficiently large values of $k$.

First, let us discuss the evaluation of $\alpha_{2,k}$. In order to obtain an approximate value of $\alpha_{2,k}$, we expand $\tau(\xi)$ and $D(\xi)$ in powers of $\xi$ keeping terms up to third order. Then we may write

$$\tau \approx \sqrt{\frac{2(1 - e^2)}{e}} \left( x + \frac{x^3}{3} \right), \quad \text{(A4)}$$

$$D \approx (1 - e)(1 + x^2), \quad \text{(A5)}$$

with $x$ being a scaled form of $\xi$ defined such that

$$x = \sqrt{\frac{e}{2(1 - e)}}. \quad \text{(A6)}$$

For an accurate approximation to $\alpha_{2,k}$, the quantity $R(e, \cos \xi)$ must be expanded up to at least fourth order so that

$$R(e, \cos \xi) \approx (1 - e)^2 \left[ 1 - \frac{2(2 + e)}{e} x^2 + \frac{8 - 4e^2}{3e^3} x^4 \right]. \quad \text{(A7)}$$

Substituting equations (A4)–(A7) into equation (A2) we get

$$\alpha_{2,k} \approx \frac{1}{\pi(1 - e)^3} \int_0^{2\pi} \frac{d\tau}{(1 + x^2)^3} \frac{\cos(k \tau)}{D(\tau)^3} \left[ 1 - \frac{2(2 + e)}{e} x^2 + \frac{8 - 4e^2}{3e^3} x^4 \right]. \quad \text{(A8)}$$

Finally, we need to express $x$ in terms of $\tau$. For that, formally, one should solve the cubic equation (A4), but this would lead to significant technical difficulties. Instead, we again make the assumption that the integral is mainly determined by the values of the integrand at small values of $x$ and therefore $\tau$. This enables us to make use of the following approximate solution of equation (A4), which is correct to fifth order in these quantities and which also gives a linear relation between $x$ and $\tau$ for large $x$ in the form

$$x \approx \tilde{x} = \frac{\tilde{x}^3}{3(1 + \tilde{x}^2)^2}, \quad \text{(A9)}$$

where we adopt the rescaled time

$$\tilde{\tau} = \frac{\tau}{\lambda}, \quad \lambda = \sqrt{\frac{2(1 - e^2)}{e}}, \quad \text{(A10)}$$

Using equation (A9) to substitute for $x$ in the integral (A8) we get

$$\alpha_{2,k} \approx \frac{2}{\pi \lambda e} \left[ J_0 - \frac{2(2 + e)}{e} J_1 + \frac{8 - 4e^2}{3e^3} J_2 \right], \quad \text{(A11)}$$

where

$$J_i = \int_{-\infty}^{\infty} \frac{dt}{(1 + \tilde{x}^2)^{i + 1}} \frac{\cos(k \tilde{x})}{(1 + 2\tilde{x}^2 + \frac{1}{4} \tilde{x}^4)^i}. \quad \text{(A12)}$$

Here, again making use of the assumption that the integral is mainly determined in a small interval near $\tilde{x} = 0$, and therefore to a good approximation is independent of the interval over which it is taken, as long as that is big enough, we have changed the limits of integration to $\pm \infty$. 

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Table A1. The coefficients $a_n^i$ determining (together with $b_n^i$) the values of the integrals $J_i$ given by equation (A12). Each row is labelled by the value of $i$, and each column by the value of $n$ (see equation A13).

<table>
<thead>
<tr>
<th>$a_n^i$</th>
<th>$n = 0$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>$3.57 \times 10^{-2}$</td>
<td>$2.05 \times 10^{-2}$</td>
<td>$2.91 \times 10^{-2}$</td>
<td>$1.34 \times 10^{-3}$</td>
<td>$1.83 \times 10^{-5}$</td>
</tr>
<tr>
<td>1</td>
<td>$6.76 \times 10^{-2}$</td>
<td>$-1.68 \times 10^{-2}$</td>
<td>$-1.13 \times 10^{-2}$</td>
<td>$-9.74 \times 10^{-4}$</td>
<td>$-2 \times 10^{-5}$</td>
</tr>
<tr>
<td>2</td>
<td>$-9.97 \times 10^{-3}$</td>
<td>$-2.31 \times 10^{-2}$</td>
<td>$2.115 \times 10^{-5}$</td>
<td>$5.25 \times 10^{-4}$</td>
<td>$2.18 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table A2. Same as Table A1, but for the coefficients $b_n^i$.

<table>
<thead>
<tr>
<th>$b_n^i$</th>
<th>$n = 0$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>$-4.77 \times 10^{-2}$</td>
<td>$-5.17 \times 10^{-2}$</td>
<td>$-6.12 \times 10^{-3}$</td>
<td>$1.33 \times 10^{-2}$</td>
<td>$5.64 \times 10^{-3}$</td>
</tr>
<tr>
<td>1</td>
<td>$-9.25 \times 10^{-3}$</td>
<td>$4.54 \times 10^{-2}$</td>
<td>$7.04 \times 10^{-2}$</td>
<td>$2.29 \times 10^{-2}$</td>
<td>$-1.08 \times 10^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td>$5.84 \times 10^{-2}$</td>
<td>$1.52 \times 10^{-2}$</td>
<td>$6.405 \times 10^{-2}$</td>
<td>$-1.36 \times 10^{-1}$</td>
<td>$2.05 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Note that we have also neglected terms proportional to $\tau^6$ in the denominator of the integrals (A12). These terms may easily be taken into account in principle but do not change the results significantly. The integrals (A12)) can be easily evaluated using the theory of functions of a complex variable. They are determined by two poles of fifth order in the complex $\tilde{\tau}$ upper half-plane located at $\tilde{\tau} = i\sqrt{3/2 + \sqrt{6}}$. The $J_i$ so found can be represented in the form

$$J_i = e^{-y_+ \lambda k} \sum_{n=0}^{n=4} a_n^i (\lambda k)^n + e^{-y_- \lambda k} \sum_{n=0}^{n=4} b_n^i (\lambda k)^n,$$  \hspace{1cm} (A13)

where $y_\pm = \sqrt{3/2 + \sqrt{6}}$, and the coefficients $a_n^i$ and $b_n^i$ are given in Tables A1 and A2, respectively.

We evaluate $a_{0,k}$ by exactly the same method. Doing this we get

$$a_{0,k} \approx \frac{2}{\pi \lambda e} I,$$  \hspace{1cm} (A14)

where

$$I = \int_{-\infty}^{\infty} d\tilde{\tau} \frac{(1 + \tilde{\tau}^2)^3 \cos (\lambda k \tilde{\tau})}{(1 + 2\tilde{\tau}^2 + \frac{1}{3} \tilde{\tau}^4)^{3/2}}.$$  \hspace{1cm} (A15)

The integral (A15) is determined by the behaviour at two poles of third order located at $\tilde{\tau} = iy_\pm$ in the upper complex half-plane and can be expressed as

$$I = e^{-y_+ \lambda k} \sum_{n=0}^{n=2} c_n (\lambda k)^n + e^{-y_- \lambda k} \sum_{n=0}^{n=2} d_n (\lambda k)^n,$$  \hspace{1cm} (A16)

where the coefficients $c_n$ and $d_n$ are given in Table A3.

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