Stochastic Fermi acceleration of subrelativistic electrons and its role in impulsive solar flares

Robert Selkowitz* and Eric G. Blackman

Department of Physics and Astronomy, and Laboratory for Laser Energetics, University of Rochester, Rochester NY, 14627, USA

ABSTRACT

We re-examine stochastic Fermi acceleration (STFA) in the low-energy (Newtonian) regime in the context of solar flares. The particle energization rate depends on a dispersive term and on a coherent gain term. The energy dependence of pitch angle scattering is important for determining the electron energy spectrum. For scattering by whistler wave turbulence, STFA produces a quasi-thermal spectrum. A second well-constrained scattering mechanism is needed for STFA to match the observed 10–100 keV non-thermal spectrum. We suggest that STFA most plausibly acts as phase one of a two-phase particle acceleration engine in impulsive flares: STFA can match the thermal spectrum below 10 keV, and possibly the power-law spectrum between 10 and 100 keV, given the proper pitch angle scattering. However, a second phase, such as shock acceleration at loop tops, is probably required to match the spectrum above the observed knee at 100 keV. Understanding this knee, if it survives further observations, is tricky.

Key words: acceleration of particles – Sun: flares – Sun: X-rays, gamma-rays.

1 INTRODUCTION

Fermi acceleration was first proposed as a mechanism for cosmic ray acceleration (Fermi 1949, 1954). In the original model, compressive perturbations in the Galactic magnetic field, associated with molecular clouds, reflect charged particles. If these clouds converge, the particles gain energy over time. If they diverge, the particles lose energy. Later, it was realized (Krymskii 1977; Axford, Leer & Skadron 1978; Bell 1978a; Blandford & Eichler 1987), that shock fronts are another site of Fermi acceleration. In the shock acceleration model, charged particles stream into magnetic perturbations in the post-shock region, reflect, and are scattered back across the shock by pre-shock Alfvén waves. Repeated reflections steadily accelerate particles to a power-law distribution. This has been studied extensively (e.g. Jones & Ellison 1991). If the shock thickness is determined by the ion gyroradius then ions are picked up out of the thermal population, but electrons must be injected at energies at or above the thermal energy by a factor of the ratio of proton to electron mass, \( \frac{m_p}{m_e} \), to incur power-law acceleration. The injection process is a critical outstanding problem for many applications of shock acceleration. Shock Fermi acceleration is commonly referred to as first-order Fermi acceleration because the sign of the energy gain after each cycle is positive and \( \frac{dE}{dt} \propto v_e/v \), where \( E \) and \( v \) are the electron energy and speed, and \( v_e \) is the velocity of the magnetic compression.

Fermi acceleration can also take place in a fully turbulent plasma. There, the mirroring sites are turbulent perturbations, typically fast mode magnetohydrodynamic waves, randomly distributed throughout the plasma (e.g. Achterberg 1984). Electrons encounter these perturbations such that there is a stochastic distribution of energy gaining and energy losing reflections. As is demonstrated below, there is a net dissipation of turbulent energy into high-energy electrons. Because the energy gaining and energy losing reflections are equal to first-order, stochastic Fermi acceleration (STFA) is often referred to as second-order Fermi acceleration and proceeds more slowly than the first-order process. For STFA, \( \frac{dE}{dt} \propto \left( \frac{v_e}{v} \right)^2 \).

STFA has been considered extensively as the acceleration mechanism in impulsive solar flares, (e.g. LaRosa et al. 1996). Observations of these flares show hard X-ray emission with a downward breaking power-law spectrum extending from 10 keV to at least 0.5 MeV with the break energy narrowly distributed around \( E_{br} = 100 \) keV and a thermal distribution at energies below 10 keV (Dulk, Kiplinger & Wingler 1992; Krucker & Lin 2002). The time structure of the emission shows distinct spikes of duration 1 s and typical energy \( 10^{26} \) erg (Aschwanden, Schwartz & Alt 1995). Non-thermal emission occurs principally at the foot-points of the soft X-ray loop, and to a lesser extent at a loop-top hard X-ray source (Masuda et al. 1996; Tsuneta 1996). Brown (1971) has shown that the emission at a dense target is consistent with bremsstrahlung radiation by electrons accelerated to a power-law energy distribution at some height above the target; in impulsive flares the acceleration site can be associated with the loop-top region, while the thick target is associated with the foot-points. We demonstrate that STFA is possibly responsible for the acceleration of electrons below 10 keV, while first-order Fermi processes at the loop-top fast shock may produce the highest energy electrons. In this picture STFA also provides power-law distributed electrons in the range \( 10 < E < 100 \) keV, thus satisfying the shock...
injection criterion and producing the observed spectral break at 100 keV. It is shown that in order to produce this spectrum the pitch angle scattering must obey the restriction that the scattering distance, $\lambda_P$, is inversely proportional to the energy of the scattered electron in the 10–100 keV range. Matching the knee is difficult, requiring either a cut-off in the secondary pitch scattering at 100 keV, or the sudden onset of yet another pitch scattering agent with a much shorter wavelength. While both are possible, these requirements provide a serious challenge to STFA models of electron acceleration.

It is important to note that the downward break is not observed in all impulsive solar flares. Indeed, Dulk et al. (1992) observed a spread in break energies and spectral indices. Some of their flares exhibited no discernible break, or even some upward breaks (angles). In this paper we address the fundamental process of STFA, and show how it can accommodate the downward breaking spectra observed in a subset of impulsive flares. The general form of the electron spectra we model is illustrated in Fig. 1. We assume a thick target bremsstrahlung emission model where the electron spectral index $\delta$ and the photon spectral index $\gamma$ are related by $\delta = \gamma + 1$ (Brown 1971; Tandberg-Hanssen & Emslie 1988). The data shown are mean values taken from the flares studied by Dulk et al. (1992): $E_{br} = 100$ keV, $\delta = 4$ below $E_{br}$, and $\delta = 5.25$ above $E_{br}$.

Furthermore, not all flares are observed to be dominated by electron acceleration. A recent flare observed by Hurford et al. (2003) clearly shows regions of emission that are dominated by X-ray emission from electrons and regions that are dominated by gyrosynchrotron emission from MeV/nucleon ions. Proton and ion emission appears to be primarily associated with large flare loops. Miller & Roberts (1995) and Miller, Emslie & Brown (2004) propose that this can be explained by a two stage process for ion acceleration. First, ions are accelerated via gyroresonance to speeds of roughly $v_A$ by Alfvén waves, and subsequently are accelerated preferentially over electrons by magnetosonic waves. They argue convincingly for the gyroresonant acceleration by Alfvén waves on the basis of relative ion abundances. However, it is unclear whether or not the second stage acceleration by magnetosonic modes must be resonant. It appears that the second stage is consistent with STFA. In any event, proton acceleration in long flare loops presents an interesting problem for acceleration models, in that super-Alfvénic protons must be preferentially accelerated over electrons, but electron acceleration must still be dominant in shorter loops. In this work, we presume that the loops are sufficiently short for the protons to remain sub-Alfvénic. Longer loops, and the effects of proton acceleration on the shaping of the observed bremsstrahlung X-ray spectra from high-energy electrons, require further study.

STFA is found to depend on two competing effects, which we refer to as the steady and diffusive acceleration rates. The steady rate represents the net acceleration of electrons due to the slight advantage of head-on or energy gaining reflections over catch-up or energy losing reflections. The diffusive term represents the spreading of the electron distribution function as a result of the stochastic nature of the reflections. Longair (1994) treats the two effects together using the Fokker–Planck equation. Likewise, Park & Petrosian (1996) discuss general solutions of a simplified Fokker–Planck equation for STFA. In follow-up work, Park, Petrosian & Schwartz (1997) apply their solution to solar flares. While they produce spectra consistent with observations, they do not discuss the physics of electron escape. Furthermore, they focus mainly on the regime above 100 keV. Some past treatments of STFA in impulsive flares focused exclusively on the diffusive term. LaRosa et al. (1996) derive the diffusive acceleration rate using simple physical arguments. Chandran (2003) derives the diffusive term using both phenomenological arguments and quasi-linear theory. The latter also includes Coulomb losses as a small correction; this is of note as the Coulomb-loss term is mathematically similar to a negative steady acceleration term. Herein we derive the steady term and compare it to the diffusive

Figure 1. The typical downward broken power law of an impulsive flare, as observed by Dulk et al. (1992). The left-hand panel shows the hard X-ray spectrum, with $E_{br} = 100$ keV, and spectral indices above and below $E_{br}$ of 4.25 and 3, respectively. The right-hand panel shows the electron spectrum in the emission region inferred from the given photon spectrum using a thick target bremsstrahlung model for the emission (Brown 1971; Tandberg-Hanssen & Emslie 1988). Again, $E_{br} = 100$ keV and the spectral indices above and below $E_{br}$ are 4 and 5.25.
term, finding the steady acceleration to be dominant in the non-relativistic regime for impulsive flares. As will be seen, our steady term differs slightly from that of some previous calculations such as that in Longair (1994) in its dependence on the turbulent magnetic fluctuation strength. This arises because we only average over the pitch angle phase space for which STFA operates in magnetic mirroring, whereas Longair (1994) averaged over all pitch angles in considering a more generic form of Fermi acceleration. A similar averaging over all pitch angles is performed in Skilling (1975) and Webb (1983).

First, we review the basic Fermi process using a test particle approach. We then derive an expression for the mean acceleration of electrons in a turbulent plasma via STFA and compare this to the diffusive STFA derived by LaRosa et al. (1996). Finally, we discuss the trapping of electrons in the turbulent accelerating region and show that at non-relativistic energies, the electron spectrum depends strongly on the energy dependence of pitch angle scattering. For scattering by whistler wave turbulence, the emerging spectrum is quasi-thermal. In order to produce the 10–100 keV power law in solar flares, we show that there must be an additional source of pitch angle scattering that has a length-scale inversely dependent on electron energy; this mechanism is tightly constrained. The existence of such a pitch angle scattering mechanism has yet to be determined. This brings to the fore the most pressing difficulty with STFA models of electron acceleration; the pitch angle scattering requirements are stringent and might not be possible to meet.

2 THE FERMI ACCELERATION PROCESS

Consider a particle of charge $q$ travelling with gyroradius $r_G$ in a magnetic field of strength $B$ (Fermi 1949, 1954; Spitzer 1956). The charge follows a helical path, orbiting the field line whilst also moving parallel (or antiparallel) to the field line. Taking the condition for circular motion and the Lorentz force

$$F = qvB = \frac{mv^2}{r_G}, \tag{1}$$

where $q$ is the electron charge, and applying conservation laws for angular momentum and kinetic energy ($r_Gv_\perp$ and $v^2$ are constant) yields

$$\frac{qL}{mE} = \frac{\sin^2 \theta}{B}, \tag{2}$$

where $L$ is the angular momentum of the charge, $E$ the kinetic energy, and $v \sin \theta = v_\perp$ relates the total velocity to the component perpendicular to the field line. The pitch angle, $\theta$, is the angle between the field line and the velocity vector. As the charge enters a region of increasing $B$, such as a magnetic compression, the pitch angle evolves according to

$$\frac{\sin^2 \theta}{B + \delta B} = \frac{\sin^2 \theta}{B}, \tag{3}$$

where $\delta B$ is the increase in field strength and $\theta_1$ is the pitch angle at field strength $B + \delta B$. When $\sin \theta_1 = 1$, the charge cannot penetrate further into the compression, and so it reflects. This process, known as magnetic mirroring, is commonly used to confine laboratory plasmas (Dendy 1990). It follows immediately that mirroring will not occur at a given compression unless the initial pitch angle satisfies

$$\sin^2 \theta \geq \frac{B}{B + \delta B}. \tag{4}$$

Fermi (1949, 1954) showed that moving magnetic mirrors, in particular molecular clouds, can accelerate charges. In the frame of reference of the cloud (primed), mirroring results in only a change in the sign of $v_\perp$, the component of the initial velocity of the charge parallel to the field line in the rest frame of the compression. Let us work for the moment in the limit where the compression speed and the particle speed are both $\ll c$. Transforming to the laboratory frame, $\delta v_\parallel = \pm 2v_\perp$, where $v_\perp$ is the drift velocity of the cloud. The positive (negative) sign is for head-on (catch-up) reflections between the charge and cloud. Catch-up reflections are defined as those where the components of the compression and charge velocities parallel to the field line have the same sign. Head on reflections are those where the parallel components have opposite signs. The net change in energy from a reflection is given by

$$\delta E_\parallel = \frac{m}{2}(v^2_f - v^2_i) = \frac{m}{2}[2\delta v_\parallel v_\parallel \cos(\theta) + (\delta v)^2]$$

$$= (2m)(\pm v_i v_\parallel + v^2_\perp), \tag{5}$$

where $v_f$ and $v_i$ are the speeds after and before reflection, $m$ is the mass of the charge, and $v_\parallel = v_\parallel \cos(\theta)$. Head-on reflections result in a positive energy change, while catch-up reflections can result in a negative energy change when $v_\parallel > v_i$. In both types of reflection, there is the positive term proportional to $v^2_\perp$.

One can repeat this derivation using Lorentz transformations in place of the Galilean transformations to generalize this result to particles of any energy scattered by compressions that are still restricted to non-relativistic velocities. For a derivation see Longair (1994); we simply cite the result:

$$\delta E_{BR} = 2E \left( \pm \frac{v_i v_\parallel}{c^2} + \frac{v^2_\perp}{c^2} \right). \tag{6}$$

where $c$ is the speed of light in vacuum. Over time, charges trapped between converging magnetic compressions are only subject to head-on reflections, and are accelerated to higher energies.

Note that the change in momentum of a Fermi accelerated electron is solely in the component parallel to the mean field. This corresponds to an increase in the pitch angle of the electron. As an extreme example, consider an electron of initial energy $E_0$ with pitch angle in the mean field $B$ approaching $\pi/2$. Upon doubling the energy of the electron via Fermi acceleration, the pitch angle in the mean field is reduced to $\pi/4$. Clearly, acceleration to high energy must be accompanied by some additional scattering agent that isotropizes electron pitch angles on a short time-scale, otherwise Fermi acceleration shuts off after small accelerations as pitch angles evolve out of the range given by equation (4). The well-known problem of pitch angle scattering remains largely unsolved (e.g. Melrose 1974; Achterberg 1981; LaRosa et al. 1996).

Fermi acceleration is distinct from the transit time damping (TTD) treated, for example, in Miller, Larosa & Moore (1996). TTD is the magnetic analogue of Landau damping. In TTD, electrons (or ions) that are near gyroresonance with waves of wave-number $k$ are pushed towards the resonance by field gradients in the wave that alter the parallel component of the velocity. Gradients in the electron velocity spectrum result in a net damping or enhancement of the waves. In the presence of a spectrum of waves, an electron can drift from resonance at $k$ to $k \pm \delta k$ and so forth, eventually reaching high energies. Fermi acceleration, however, is a non-resonant interaction. Electrons will mirror at a compression regardless of energy provided that the pitch angle is sufficiently large. TTD is often referred to as resonant Fermi acceleration because the two processes rely on similar physics. Table 1 lists the relevant length and time-scales for STFA in impulsive flares.
3 STOCHASTIC FERMI ACCELERATION

We now consider the behaviour of charges in a turbulent magnetic plasma where magnetosonic modes provide the sites of magnetic mirroring. This scenario differs from first-order Fermi acceleration by shocks in two ways. (i) Consecutive mirroring events are not coherent, but rather stochastically distributed between head-on and catch-up. (ii) The turbulent cascade governs the acceleration efficiency; the system picks out a scale where acceleration competes with the cascade. In the solar corona plasma, $v_c$, the velocity of the magnetic compressions, is the phase speed of the magnetosonic modes, which is roughly the Alfvén speed for the fast mode and the speed of sound ($c_s$) for the slow mode. Typically, thermal electrons in the corona are super-Alfvénic and non-relativistic, $v_A \ll v_0 \ll c$, and $\beta \sim 0.05$; we will solve the STFA problem in this regime.

3.1 Determination of the steady acceleration rate

Recall that the energy gain from a typical reflection, given in equation (5), is $\delta E_s = 2m(\pm v || v_c + v_c^2)$. We define three parameters: $R$, the total rate of reflections; $R_\perp$, the rate of head-on reflections; and $R_\parallel$, the rate of catch-up reflections. The relation $R = R_\perp + R_\parallel$ is automatically satisfied by this definition as all reflections must be of either the head-on or catch-up type. This allows us to write the approximate acceleration rate as the sum of a coherent term and an incoherent term:

$$\left(\frac{dE}{dt}\right)_s = 2m \left[ (R_\perp - R_\parallel) v || v_c + (R_\perp + R_\parallel) v_c^2 \right],$$

(7)

where the subscript $S$ is used to distinguish our derived acceleration rate from that of LaRosa et al. (1996). The first term, proportional to $(R_\perp - R_\parallel)$ represents the mean acceleration due to the offset in the rates of the two types of reflection. The second term, proportional to $(R_\perp + R_\parallel) = R$, the total reflection rate, represents the coherent term. This expression gives a full description of the mean acceleration of charges by the non-relativistic STFA process. To evaluate it in a particular plasma requires the determination of the head-on and catch-up reflection rates $R_\perp$ and $R_\parallel$.

In order to obtain $R_\perp$ and $R_\parallel$, consider the path a charge takes to encounter a mirror. Note that there is a distinction between an encounter and a mirroring because of the pitch angle condition for reflection. We take the fraction of encounters that reflect to be $F$ and assume that this fraction is the same for both head-on and catch-up encounters: $F = F_\perp = F_\parallel$. This assumption is often taken in the regime where $v \gg v_c$ (LaRosa et al. 1996). While this assumption is not strictly true, the effects of relaxing it are negligible. In Appendix B we repeat this calculation without assuming $F_\perp = F_\parallel$. There is a well-defined mean distance between encounters, $\lambda_T$ and a relative velocity between the particle and the compression $v_\perp = v_\parallel \pm v_c$. The small difference in the head-on and catch-up speeds is responsible for the offset in rates. For each type of encounter, the mean separation is $2\lambda_T$ and the rate of reflections of any sort is specified by the general relation $R = F v_\parallel \lambda_T$, with $v_\parallel$ the relative velocity between the particle and compression, yielding

$$R_\perp = F \frac{v_\parallel + v_c}{2\lambda_T},$$

$$R_\parallel = F \frac{v_\parallel - v_c}{2\lambda_T},$$

(8)

The offset in rates is thus

$$R_\perp - R_\parallel = F \frac{v_c}{\lambda_T},$$

(9)

and we can rewrite the average acceleration rate as

$$\left(\frac{dE}{dt}\right)_s = 2mF \frac{v_\parallel v_c^2}{\lambda_T} + 2mF \frac{v_\parallel v_c^2}{\lambda_T} = 4mF \frac{v_\parallel v_c^2}{\lambda_T},$$

(10)

with the associated acceleration time-scale $\tau_S = E/\langle dE/dt \rangle_S$. We thus see that the acceleration due to the offset in head-on and catch-up reflection rates and the acceleration from the coherent term in $v_c$ are equal.

To simplify the derivation of the electron spectrum, we recast equation (10) as

$$\left(\frac{dE}{dt}\right)_s = 4mv_c^2 R = \frac{dE}{dM} \frac{dM}{dt},$$

(11)

where we used equation (8) and define $M$ as the total number of reflections experienced by an electron, $dM/dt = R$, and $dE/dM = 4mv_c^2$.

The quantity $dE/dM$, the mean acceleration of an electron per reflection, will be of particular use when examining electron escape in Section 4.

3.2 Comparison to the diffusive acceleration rate

A different approach was taken by LaRosa et al. (1996). They set $R_\perp = R_\parallel$ in the $v \gg v_c$ regime and studied the diffusion of particles through energy space via random walk. The starting point in their calculation of the electron acceleration was the time-scale for the e-folding of the energy of a charged particle in the turbulent plasma

$$\frac{1}{\tau_D} = \frac{1}{E} \left(\frac{dE}{dt}\right)_D = \frac{F}{N \delta t},$$

(13)

where $N$ is the number of mirroring required to double the energy of the particle and $\delta t$ is the time between encounters. They set
\( R_+ - R_- = 0 \), and also dropped the last term in \( v_n^2 \) in equation (6). From equations (7) and (10) it is clear that if one of two assumptions is valid the other must also apply, and \( (dE/dt)_S = 0 \) in that limit. Under these assumptions \( \delta E_+ \) and \( \delta E_- \) are equal in magnitude, and from the standard solution of an evenly weighted random walk \( N = (E/\delta E)^2 \). The acceleration rate is then

\[
\frac{dE}{dt} = \frac{E}{\tau_0} = \frac{\delta E^2_f}{\lambda_\tau} v_0 = \frac{8F \cdot m \cdot v_n^2 \cdot v_0^3}{\lambda_\tau} \cdot \frac{\delta B}{B}, \tag{14}
\]

where \( v_0 \) is the total initial speed and the subscript D is used to denote LaRosa et al.'s diffusive acceleration rate.

To complete the calculation of the acceleration rates, we must obtain \( F \) and average over pitch angles. The minimum accessible pitch angle for reflection is related to \( F \) by

\[
F = \cos(\theta_{\text{max}}) = \left( \frac{\delta B}{B} \right)^{1/2}, \tag{15}
\]

where we have applied the reflection condition from equation (4). In the regime where the compression ratio \( \delta B/B \ll 1 \), which will apply to the plasma of interest, and taking the assumption that pitch angles are isotropic gives \( \langle \cos(\theta) \rangle = \cos(\theta_{\text{max}})/2 \) and \( \langle \cos^3(\theta) \rangle = \cos^3(\theta_{\text{max}})/4 \). We now write the averaged acceleration rates as

\[
\frac{dE}{dt} = 4mF \cdot \left( \frac{v_0^2}{\lambda_\tau} \right) = \frac{2m}{v_0^2} \cdot \left( \frac{\delta B}{B} \right), \tag{16}
\]

\[
\frac{dE}{dt} = \frac{8F \cdot m \cdot v_n^2 \cdot v_0^3}{\lambda_\tau} \left( \frac{\delta B}{B} \right)^2. \tag{17}
\]

What do these two acceleration rates represent? \( (dE/dt)_S \) is the steady growth of the mean kinetic energy due to the drain of turbulence by the combined effects of the slightly non-zero \( (R_+ - R_-) \) and the coherent \( v_n^2 \) term: a shift of the mean electron energy to higher energy. On the other hand, \( (dE/dt)_D \) represents the diffusion of energies away from the mean via random walk: a spreading of the distribution. The relative importance of the two is fixed by their ratio

\[
\xi = \frac{(dE/dt)_S}{(dE/dt)_D} = \frac{B}{\delta B}. \tag{18}
\]

The combined result of the action of both processes on an initially narrow Gaussian energy distribution is shown in Fig. 2, where we have chosen \( \xi = 65 \), and assume that electrons do not escape. Thus we can examine the evolution of electron energy spectra solely due to the influence of the two acceleration rates. As \( \xi \) is increased, the steady (mean growth) term becomes increasingly dominant over the diffusive (distribution widening) term. To understand STFA in a particular plasma, both acceleration rates must be calculated. In the event that the diffusive rate is very small compared with the steady growth rate, it can be ignored. The steady growth rate is always faster than the diffusive growth rate for \( a > 1 \). As will be shown later, \( \xi \sim 100 \) in flare plasmas, and the diffusive term is negligible.

It is very important to note that our result differs from the standard for Fermi acceleration, in which both the diffusive and steady terms depend on the same power of \( \delta B/B \). This difference arises as a result of the averaging over pitch angles. To correctly obtain \( \langle \Delta E \rangle \), one must only average over those encounters that result in a reflection. For traditional STFA, the range of pitch angles that reflect is ultimately determined by the turbulent magnetic field strength. One factor of \( \cos(\theta) \) in the expression to be averaged, results in one factor of \( (\delta B/B)^{1/2} \) in the acceleration rate. In other treatments, such as those of Longair (1994), Webb (1983) and Skilling (1975), the acceleration mechanism is assumed to act at all pitch angles. In this case, the averaging over \( \cos(\theta) \), while maintaining the assumption of pitch angle isotropy, yields a numerical value with no \( \delta B \) dependence. In this regime, the diffusive and steady acceleration terms have the same relative strength at all levels of turbulence.

### 3.3 Specification of the turbulent cascade

This leaves \( (\delta B/B) \) as the remaining parameter to be determined. It is related to the turbulent length-scale \( \lambda_\tau \) through the cascade law. Magnetohydrodynamic (MHD) turbulence proceeds by the shredding of like sized eddies and the subsequent formation of smaller eddies; energy input into eddies on a large (outer) length-scale, \( \lambda_\tau \), cascades rapidly to smaller length-scales on the eddy turnover time, \( \tau_\text{ed} \), and finally dissipates at \( \lambda_\tau \), the dissipation scale. If the cascade obeys Komolgorov’s steady-state assumption then the energy flow through all length-scales is constant and independent of the scale. This results in an inertial range between the scales \( \lambda_\tau \) and \( \lambda_\tau \), where the turbulent energy density has a power-law dependence on eddy size. The draining of turbulence at eddy size \( \lambda_\tau \) is usually determined by a microphysical process, such as resistivity. When STFA is active, the turbulence can instead be drained by pumping energy into electrons. This sets another condition for STFA to proceed in a plasma: there must be some \( \lambda_\text{ed} \) that is greater than the resistive length-scale at which the STFA time-scale is shorter than the cascade time, otherwise the turbulence will drain at the resistive scale before STFA can produce an appreciable electron acceleration.

There are three major MHD wave-modes: Alfvén waves, and the fast and slow magnetosonic waves. Alfvén waves are purely transverse, and thus do not compress the magnetic field; they cannot participate in STFA. Both the fast and slow modes are compressive, and are, in principle, capable of Fermi acceleration. It has been argued that in low-\( \beta \) plasmas such as the solar corona, the slow mode is rapidly dissipated via Landau damping (Achterberg 1981). However, more recent studies of MHD turbulence indicate that the cascade time for Goldreich–Sridhar turbulence (hereafter GS) is significantly shorter than the electron damping time, and slow mode damping by electrons can be ignored in turbulent flare plasmas (Lithwick & Goldreich 2001). A key difference in the two analyses is that Lithwick & Goldreich (2001) treat MHD turbulence as inherently anisotropic, whereas Achterberg (1981) assumes isotropy. Furthermore, Maron (private communication) has shown in simulations, which neglect damping, that the slow mode may be driven with much higher total energy content than the fast mode at low \( \beta \). A definitive resolution of the issue is beyond the scope of this paper. However, we should note that up to this point, the calculation is independent of the choice of wave mode. However, there is one significant difference between the two: slow modes propagate at roughly the speed of sound \( c_s \) while fast modes propagate at roughly the Alfvén speed \( v_A \). It will be shown that due to the influence of pitch angle scattering, STFA is probably dominated by the fast mode.

MHD turbulence is in general anisotropic; the direction of any large-scale mean magnetic field defines a preferred axis. Also, even if the turbulence is isotropic on large scales, smaller scales may see the larger-scale turbulent structures as an effective mean field. Goldreich & Sridhar (1997) modified the Komolgorov assumption for the cascade of slow and Alfvén modes of MHD turbulence under the condition that the turbulence is anisotropic with scale \( \lambda_\parallel \) along the field line and \( \lambda_\perp \) perpendicular to the field line. The two directions are found to obey different cascade laws, with \( \lambda_\parallel \) cascading more weakly than \( \lambda_\perp \). The parallel direction is of more importance to STFA, as it represents the distance along the field line between...
Figure 2. The evolution of a sample electron energy distribution with an initially Gaussian velocity distribution. The peak of the distribution is evolved from the flare thermal energy 0.2 keV to the post-STFA mean energy at 16 keV. (a) The initial distribution function. (b) The same distribution after being evolved only by the steady process. (c) The distribution evolved through both the steady and diffusive processes. Note that the relative width of the electron energy distribution, $\Delta E/E_m$, decreases with increasing mean energy $E_m$.

The GS power law for the parallel scale is (Goldreich & Sridhar 1997; Lithwick & Goldreich 2001)

$$\frac{\delta B}{B} = \left(\frac{\lambda_T}{L_T}\right)^{1/2},$$

where $B$ is the mean magnetic field strength, and $\delta B$ is the turbulent field strength at parallel length-scale $\lambda_T$.

The exact power law of MHD turbulence remains the subject of some debate, so for now we assume a general power law of the form

$$\frac{\delta B}{B} = \left(\frac{\lambda_T}{L_T}\right)^{1/a},$$

where $a > 1$ is an arbitrary index. Using the turbulent power spectrum and substituting for $F$ from equation (15) we can rewrite the acceleration rates as

$$\left(\frac{dE}{dt}\right)_S = \frac{2m}{\lambda_T} v_A^2 v \left(\frac{\lambda_T}{L_T}\right)^{1/a} = \frac{2}{L} m v_A^2 \left(\frac{\lambda_T}{L_T}\right)^{(1/a)-1}$$

so that $\tau_S = \frac{1}{2} \frac{v L_T}{v_A} \left(\frac{\lambda_T}{L_T}\right)^{1-(1/a)}$.

$$\left(\frac{dE}{dt}\right)_D = \frac{2m}{\lambda_T} v_A^2 v \left(\frac{\lambda_T}{L_T}\right)^{2/a} = \frac{m}{L} v_A^2 \left(\frac{\lambda_T}{L_T}\right)^{(2/a)-1}$$

so that $\tau_D = \frac{1}{2} \frac{v L_T}{v_A} \left(\frac{\lambda_T}{L_T}\right)^{1-(2/a)}$.

For a typical turbulent cascade, where $\lambda_T < L_T$ and $a > 0$, $\tau_S < \tau_D$. 

3.4 $\lambda_{SF}$ and the role of pitch angle scattering

There remains a final step in determining the particular dissipation scale $\lambda_{SF}$ at which the energy drain takes place. In order for STFA to overcome the cascade of MHD turbulence, it must drain energy at a rate equal to the input rate at the outer scale. If turbulent compressions are shredded and cascade faster than electrons can draw out energy via reflections, then the cascade continues to smaller length-scales. At smaller scales, STFA is more rapid. STFA becomes competitive with the cascade at a scale determined by $(dE/dt)_T = dE_T/dt$. $(dE_T/dt)$ is acceleration rate of electrons and $dE_T/dt$ is the cascade rate of turbulent energy, $n m_e v^3_{A}/L_T$. The STFA scale $\lambda_{SF}$ is different for acceleration by the two compressive MHD modes.

For slow mode turbulence, $v_c = v_p$, and the balance is

$$\frac{2n m_e v^3_{A}}{L_T} \left( \frac{\lambda_T}{L_T} \right)^{(1-a)-1} = \frac{n m_e}{L_T} v^3_{A}.$$  

Solving for $\lambda_T/L_T$ and associating this particular $\lambda_T$ with $\lambda_{SF}$ gives

$$\frac{\lambda_{SF}}{L_T} = \left( \frac{1}{2 \ m_e} \right)^{(1-a)} \left( \frac{v_{A}}{v} \right)^{(1-a)}.$$  

In solar flares and a GS turbulent cascade ($a = 2$), $v_0 = 1.2 \times 10^8$, $c_s = 3 \times 10^7$ and $v_A = 1 \times 10^8$ cm s$^{-1}$ (LaRosa et al. 1996), this gives $\lambda_{SF}/L_T \sim 10^{-4}$. We have taken the total electron velocity to be the mean velocity of the thermal background plasma. The cascade will proceed down the inertial range to this length-scale where STFA then acts as the microphysical damping agent, rapidly draining the energy from turbulence into particles.

In the case of the fast mode, where $v_c = v_A$, the rate balance is

$$\frac{2n m_e v^3_{A}}{L_T} \left( \frac{\lambda_T}{L_T} \right)^{(1-a)-1} = \frac{n m_e}{L_T} v^3_{A}.$$  

and the STFA length-scale is then given by

$$\frac{\lambda_{SF}}{L_T} = \left( \frac{1}{2 \ m_e} \right)^{(1-a)}.$$  

For solar flare conditions, and a GS cascade ($a = 2$), $\lambda_{SF}/L_T = 10^{-4}$. We have tacitly assumed that the length-scale for pitch angle isotropization is roughly equal to $\lambda_T$. If it is not, the acceleration rate is retarded significantly, and STFA can be shut off. To understand this we must further explore the role of pitch angle scattering.

As discussed above, pitch angle scattering is necessary during acceleration to maintain a population of electrons that satisfy the pitch angle condition for reflection. The strength of the pitch angle scattering strongly regulates the rate of acceleration. We consider three cases: $\lambda_p \gg \lambda_{SF}$, $\lambda_p \ll \lambda_{SF}$ and $\lambda_p \sim \lambda_{SF}$ where eddy $\lambda_p$ is the typical distance over which pitch angles are isotropized. In the first case, $\lambda_p \gg \lambda_{SF}$, electrons reflect a few times and quickly leave the pitch angle range in which they can reflect. They must then stream a distance of the order of $\lambda_p$ before they can scatter again. Thus the rate of reflections and the acceleration rate are both decreased by a factor of $\lambda_{SF}/\lambda_p$. As the acceleration rate and cascade rate are not in balance, the cascade continues down to smaller scales $\lambda_T < \lambda_{SF}$. The nominal acceleration rate (equation 22) is proportional to $\lambda_T^{1/2}$, while the retardation factor is proportional to $\lambda_T$. The combined effect is a net acceleration rate that is proportional to $\lambda_T^{1/2}$: smaller-scale turbulence is actually less efficient as an accelerator. As a result, STFA never turns on in this regime. In the second case, $\lambda_p \ll \lambda_{SF}$, the pitch angle scattering is far more rapid than acceleration. As pitch scattering can take an electron through $\mu = 0$, very strongly pitch scattered electrons traverse the plasma by random walking in steps of length $\lambda_p$, again reducing the rate of reflection, this time by a factor of $(\lambda_p/\lambda_{SF})^2$. Unlike the previous case, this is not a problem for STFA; the retarding factor tends towards unity as the cascade continues to scales $\lambda_T < \lambda_{SF}$. The net acceleration rate is proportional to $\lambda_T^{1/2}$, and STFA turns on as the cascade proceeds to a sufficiently small scale. In case three, where $\lambda_p \sim \lambda_{SF}$, pitch angle scattering and reflections proceed at the same rate. Thus, electrons are capable of streaming freely from compression to compression, while they maintain a nearly isotropic pitch angle distribution. This is the simplest pitch angle scattering regime for STFA.

The identity of the accelerating wave-mode is now easy to determine. In Section 4.1 we show that whistler wave turbulence is a plausible source of pitch angle scattering, at least for the lower-energy quasi-thermal component of the spectrum. At 3 keV, $\lambda_{SF}/L_T$ is roughly $10^{-4}$. This places slow mode turbulence ($\lambda_{SF}/L_T \sim 10^{-6}$) well in the first regime. Slow modes do not participate in STFA in these flares. Fast modes, however, have $\lambda_{SF}/L_T \sim 10^{-4}$ and therefore are in the nearly ideal range for acceleration. Furthermore, both $\lambda_{SF}$ and $\lambda_p$ grow linearly with electron energy, so as electrons undergo STFA by fast modes in the presence of whistler wave turbulence, they remain in the same pitch angle scattering regime throughout.

4 THE POST-ACCELERATION SPECTRUM

The simplest case of STFA is the steady state, where we assume that electrons are injected into a turbulent region at energy $E_0$ at a rate equal to that at which accelerated electrons escape. The turbulent energy supply is continuously replenished at a large scale. We are concerned with the energy spectrum, $N(E)$, of electrons escaping the region. Note that this is, in general, different from the spectrum of the electrons within the turbulent region. We define $N_t(E)$ to be the total number of electrons reaching energy at least $E$ before escaping, such that

$$N(E) \propto -dN_t(E)/dE.$$  

Initially, we consider the case of strongly relativistic electrons; a full derivation of this regime is presented in Appendix A.

To appreciate the calculational differences between the non-relativistic regime of interest to solar flares and the more commonly studied relativistic regime, we begin with the latter. Following the approach used by Bell (1978a) for Shock Fermi acceleration, and writing $dN_t/dM = -p_{esc} N_t$, where $p_{esc}$ is the mean probability of an electron escaping from the acceleration region, gives

$$dN_t/dE = -dN_t/dM \frac{dM}{dE} = -p_{esc} N_t \frac{dM}{dE} = -p_{esc} \frac{N_t}{\alpha E},$$  

where $\alpha E$ is the right-hand side we have for the moment taken the strongly relativistic limit: $dE/dM = \alpha E$ and assumed $p_{esc}$ to be constant. This treatment of the highly relativistic limit follows Fermi (1949). One can solve for $N_t(E)$ by separating variables, integrating both sides and inverting the logarithms, resulting in the familiar power law (see, e.g., Fermi 1949; Jones 1994; Longair 1994)

$$N_t(E) = N \left( \frac{E}{E_0} \right)^{-p_{esc}/\alpha},$$  

where $N$ is the total number of electrons. From equation (26), one obtains

$$N(E) = N_0 \left( \frac{E}{E_0} \right)^{-\left(1 + p_{esc}/\alpha \right)},$$  

where $N_0$ dE is the number density of escaped electrons with $E = E_0$. Note that the logarithmic integrals in both $N_t$ and $E$ are vital to producing the power law.
For STFA by fast mode waves, the computation is more complicated because $p_{\text{esc}}$ is energy dependent. We solve for a general $p_{\text{esc}}$ in the non-relativistic regime, leaving the specification of the trapping for later discussion. In the non-relativistic regime, the acceleration rate is not proportional to the kinetic energy as it is in the strongly relativistic regime. Instead, one has $dE/dM = 4mv_{A}^2$. We thus write, using equation (11),

$$\frac{dN}{dE} = -\frac{N_{t} p_{\text{esc}}}{4mv_{A}^2},$$  

(30)

which can be rewritten as

$$\frac{dN}{N_{t}} = \frac{p_{\text{esc}}}{4mv_{A}^2} dE.$$

(31)

We have assumed that $dN_{t}/dM = p_{\text{esc}}N_{t}$. This is reasonable as long as the electrons can be treated as statistically independent and collisionless. In this case, it is reasonable to assume that $p_{\text{esc}}$ carries no inherent dependence on $N_{t}$ and the escape rate is simply given by the product of the number of electrons in the volume and the mean escape probability. Taking $p_{\text{esc}} = p_{0}E/E_{0})^{-1}$ allows us to solve for a particularly interesting $N(E)$. We see immediately that

$$\frac{dN}{N_{t}} = -\frac{E_{0} p_{0}}{4mv_{A}^2} dE$$

(32)

and $N(E)$ is again a power-law energy distribution

$$N(E) = N_{0} \left( \frac{E}{E_{0}} \right)^{-(1+\delta)},$$

(33)

where $\delta = p_{0}E/E_{0}$. However, in any other case, STFA does not produce a simple power law. The importance of the trapping mechanism is now clear; the combined energy dependence of the acceleration and escape must be $E^{-1}$ to produce a power-law spectrum. Such a spectrum relies on the coincidental logarithmic integrals over both $N$ and $E$.

### 4.1 Calculation of $p_{\text{esc}}$

Let us now calculate $p_{\text{esc}}$ for non-relativistic electrons within the turbulent volume, and consequently the energy spectrum of electrons. For simplicity, let us take the turbulent region to be rectangular, with the long axis, $z$, parallel to the direction of the bulk flow, with $z = 0$ and $L_{T}$ fixed to the downstream and upstream boundaries of the region, respectively. $L_{T}$ is taken to be the extent of the region of turbulent flow, which is presumed to be the entire distance between the reconnection sheet and the top of the soft X-ray loop. This distance is typically of size $10^{10}$ cm for solar flares (Tsuneta 1996). The largest eddy size in the turbulence, $L_{F}$ is set by the width of the outflow, typically $10^{8}$ cm. Thus the turbulent volume consists of a number of cells, each of which flows downward from the reconnection point towards the loop top. An electron escapes the acceleration region only when it reaches the X-ray loop at the base of the turbulent region (Fig. 3). These individual cells may be associated with single bursts or fragments of X-ray emission, and thus are responsible for the temporal structure of impulsive flares. In order to escape the region with energy $E(M)$, an electron must stream from its location in the region at some height $z$ to the boundary at $z = 0$ after the $M$th reflection without further reflection. We will assume that the electrons are contained in the region in the $x$-$y$ plane by gyration around large-scale field lines. To further simplify the problem, we shall assume that the electron density remains uniform throughout the turbulent region. We also neglect the bulk flow speed, $v_{f} = 8 \times 10^{7}$ cm s$^{-1}$ (Tsuneta 1996) as the length of the downflow region is roughly $10^{10}$ cm. This gives a flow time from the reconnection region to the loop top of 100 s. The acceleration process is fixed to the much shorter 1 s time-scale by the temporal size of the observed energy release fragments and the MHD eddy turnover time. Thus, bulk flow into the flare loop is not likely to be a dominant process in cutting off the acceleration.

Take the mean $z$-component of the distance streamed between reflections to be $\lambda_{z}$; $\lambda_{z}$ carries an energy dependence inherited from the energy dependence of the pitch scattering. The probability of escaping at $z = 0$ after the $M$th reflection from a point at height $z$ is given by

$$p_{\text{esc}}(z) = \frac{1}{2} e^{-\zeta/\lambda_{z}},$$

(34)

and the mean escape probability of electrons distributed uniformly across the length of the region is

$$p_{\text{esc}} = \frac{1}{L_{T}} \int_{0}^{L_{T}} p_{\text{esc}}(z) dz = \frac{\lambda_{p}}{L_{T}} (1 - e^{-2E/E_{0}}),$$

(35)

where we have taken $\lambda_{z} = \lambda_{p}/2$ from the isotropy in pitch angles, with $\lambda_{p}$ the pitch angle scattering length-scale. To obtain the spectrum of solar flare electrons requires specification of the pitch angle scattering.

Both Miller et al. (1996) and LaRosa et al. (1996) assume resonant scattering, and suggest that the scattering agent above 1 keV is resonant with lower hybrid (LH) turbulence, or circularly polarized electromagnetic waves, such as whistler waves. Below 1 keV, Coulomb interactions are thought to be sufficiently strong to isotropize the electrons. It should be noted that Melrose (1974) sets the threshold for the whistler mode resonance at 25 keV for flare plasmas, while Miller & Steinacker (1992) argue that the resonances extend down to 1 keV. We assume the latter. In a recent study, Luo, Wei & Feng (2003) considered whether or not LH wave turbulence is the primary mode of electron acceleration in solar flares. They concluded that the pitch angle scattering is too inefficient to maintain isotropy. Thus, we assume that LH wave turbulence cannot supply sufficient pitch angle scattering to sustain STFA either. Whistler waves are more promising.

Melrose (1974) associates the frequency of pitch angle scattering $v$ with the pitch angle diffusion coefficient in the quasi-linear equation. Thus, $\nu/v$ is the characteristic time-scale for effective pitch angle isotropization of the electron distribution. It should be noted that, in general, some small anisotropy is likely to remain in the distribution, and that this anisotropy could be responsible for the generation of the whistler waves. However, the source of these waves is still uncertain. From Melrose (1974), we have that

$$v = \frac{\omega_{p}^{2} x (\omega_{p})}{\gamma_{e} \Omega_{e} n_{e} m_{e} c^{2}},$$

(36)

where $\gamma_{e} \sim 1$ is the Lorentz factor of the electron, $\Omega_{e}$ is the electron gyrofrequency, $\omega_{p}$ is the plasma oscillation frequency, and

$$\epsilon(x) = \frac{m_{e}}{2} n_{e} v_{A}^{2} \frac{\delta B}{B} = \frac{\delta B}{B} \frac{\lambda_{p}}{L_{T}} \frac{\nu^{2}}{\lambda_{p}},$$

(37)

is the energy density of the turbulence at the resonant wavelength. This allows us to rewrite equation (36) as

$$v = 5 \times 10^{7} B_{100} n_{10}^{-1/2} \left( \frac{\lambda_{p}}{L_{T}} \right)^{1/2},$$

(38)

where we have used $v_{A}^{2} = B^{2}/4\pi\sigma_{e} n_{e}$, $\Omega_{e} = 1.8 \times 10^{9} B_{100}$, $\omega_{p} = 5.7 \times 10^{8} n_{10}^{1/2}$ and the dimensionless parameters $B_{100} = B/100 G$ and $n_{10} = n_{e}/10^{10}$ cm$^{-3}$.
We must compare \( \nu \) to the growth time for pitch angle anisotropy due to STFA. Bearing in mind that for STFA, \( d\nu / dt = 0 \),

\[
\frac{dE}{dr} = m v_i \frac{dv_i}{dr}
\]

and the pitch angle evolves according to

\[
\frac{d(\cos \theta)}{dr} = \frac{1}{m v^2 \cos \theta} \frac{dE}{dr}
\]

Substituting in from equation (21) for \( dE/dr \) and assuming a GS cascade \((a = 2)\), gives

\[
v_{SF} = \frac{d(\cos \theta)}{dr} = \frac{2}{L_T} \left( \frac{\lambda_T}{L_T} \right)^{-1/2} \frac{\lambda^2}{v \cos \theta} \frac{1}{v}\]

For impulsive flares, \( E_0 = 0.3 \text{ keV} \), \( B_{100} = 2 \), and \( n_{10} = 1 \). Taking \( \lambda_T = \lambda_{SF} \) results in \( \nu = 1 \times 10^6 \text{ s}^{-1} \) and

\[
v_{SF} = 2 \times 10^5 \left( \frac{E_0}{E} \right)^{1/2} \text{ cm}^{-1}.
\]

where \( v_{SF} \) is evaluated at the threshold pitch angle for reflection. To maintain pitch angle isotropy, scattering by whistler waves must occur on a time-scale shorter than pitch angle evolution by STFA. Thus, as long as \( v_{SF} < \nu \), isotropy can be maintained. This condition is met for all \( E > E_0 \). Whistler modes, if present, are capable of providing sufficient pitch angle scattering to maintain isotropy.

In addition to maintaining pitch angle isotropy, the scattering mechanism must also operate at a length-scale that traps electrons in the volume; \( \lambda_{wh} \ll L_T = 10^8 \text{ cm} \) must be satisfied, or else electrons rapidly leave the acceleration region and STFA shuts off. We obtain the pitch scattering length-scale for whistler waves, \( \lambda_{wh} \),

\[
\lambda_{wh} = \frac{\nu}{v} = 2 \times 10^3 \left( \frac{E}{E_0} \right)^{1/2} \text{ cm}.
\]

The required condition is satisfied for energies below 100 keV.

### 4.2 The electron spectrum and constraints on the secondary pitch angle scattering

In order to obtain the spectrum of the escaped electrons, we can now substitute the functional form for \( p_{esc} \) from (35) into (31)

\[
\frac{dN}{dE} = \frac{p_{esc}}{4m v^2} \frac{1}{4m v^2} \left( 1 - e^{-L_F/v} \right) \frac{1}{4m v^2} dE.
\]

Next, by choosing \( \lambda_p = \lambda_{wh} \), we use \( \lambda_p / L_F = AE^{1/2} \), with \( E \) in units of keV, and rearranging, obtain

\[
N(E) = \frac{dN}{dE} = AE^{1/2} \left( 1 - e^{-A E^{1/2}} \right) \frac{1}{4m v^2} N_i.
\]

The resulting spectrum, \( N(E) \), is plotted in Fig. 4. This is consistent with the thermal component to the flare spectrum observed using the Reuven Ramaty high energy solar spectroscopic imager (RHESSI) (Krucker & Lin 2002).

In addition to the thermal component, RHESSI observations show a clear power-law region extending from roughly 10 keV up to at least 50 keV, above which the data are uncertain, but consistent with a continuing power law. Previous observations using the ISEE3/ICE instrument also show a power law throughout the range of the instrument, 25-300 keV; the spectrum typically breaks downward at 100 keV (Dulk et al. 1992). More recent observations with RHESSI could push the low-energy threshold for the power law as high as 35 keV (Holman et al. 2003). The spectral index below the break is \( \sim 3 \), while above the break it is \( \sim 4 \). Elsewhere (Blackman 1997; Selkowitz & Blackman in preparation), we discuss first-order acceleration at the loop-top fast shock. Fast shocks are well known to accelerate super-thermal particles to power-law energy spectra, even in the non-relativistic limit (Bell 1978a). However, in order to be accelerated, electrons must satisfy the requirement that \( E \gg (m_e / m_e) v_i^2 = 10 \text{ keV} \) in solar flare plasmas, where \( v_i = 10^4 \text{ cm} \text{ s}^{-1} \) is the inflow speed of the plasma at the shock (Blackman & Field 1994). This places the injection energy at roughly 100 keV. The correspondence of the shock injection energy and the observed break energy is noteworthy. A possible mechanism to reproduce the observations is for STFA to produce a power-law spectrum in the 10–100 keV regime, which then satisfies the injection criterion.
for loop-top fast shocks. STFA by magnetosonic turbulence in the presence of whistler wave turbulence pitch scattering is insufficient to produce the power-law component. However, it is possible that a second pitch scattering agent exists that produces the power law in the 10–100 keV range. We examine the constraints imposed on this scattering.

To produce a power-law spectrum, non-relativistic STFA requires $p_{m0} \propto E^{-1}$. From equation (35), we see that this is true only if $\lambda_p / \lambda_r = C_1 / \Gamma > 1$ and the exponential term is small. Here $\Gamma$ is a constant parameter that fixes the strength of the pitch scattering. While the physics of the pitch angle scattering is not well understood, this constrains the scattering mechanisms available to STFA. We define $\lambda_C = L_p / \Gamma = 2 \times 10^7 (E_0 / E) \, \text{cm}$ to be the pitch scattering length-scale of the constrained mechanism, and the electron spectrum is given by

$$N(E) = N_0 \left( \frac{E}{E_0} \right)^{-\frac{\Gamma + 1}{4 \Gamma}}.$$  \hspace{1cm} (46)

$\Gamma$ is constrained by the observed X-ray spectral index of $\gamma = 3$. It is a standard prediction of flare models (Brown 1971; Kiplinger et al. 1984; Stepanov & Tsap 2002) that the spectrum of electrons accelerated above the loop top is steeper than the spectrum of the thick target bremsstrahlung X-rays emitted at the foot-points in solar flares. We assume that to match the RHESSI X-ray data requires an index $\sim 4$ for the electrons, or

$$\Gamma = 12m_v^2 \lambda^2 = 0.072 \, \text{keV} = 0.17E_0.$$ \hspace{1cm} (47)

The exponential term in $p_{m0}$ is indeed small as $2E / \Gamma = 270$ at $E = 10 \, \text{keV}$. The electron spectral index could conceivably be as high as 6, in which case $\Gamma = 0.28E_0$, and the exponential can still be safely neglected.

It is insufficient to merely produce the proper power law. The scattering agent must also be able to reproduce the transition from thermal to power-law spectrum at the correct energy, $E_c$ (Fig. 5). We recall $\lambda_{wh}$ to be the length-scale of pitch angle scattering associated with whistler wave turbulence. In impulsive flares, $\lambda_p = \lambda_{wh}$ below $E_c$. Above $E_c$, $\lambda_p = \lambda_C$. To obtain $E_c$ one sets $\lambda_{wh} = \lambda_C$. $E_c = 23 \, \text{keV}$, which is consistent with the observed threshold of $\sim 10 \, \text{keV}$.

There is one more important constraint imposed by the observations: the knee at 100 keV. Dulk et al. (1992) demonstrate a distinct downward break in the power-law spectrum at roughly $E_b = 100 \, \text{keV}$. The break energy varies somewhat from flare to flare, but is consistently observed in all of the impulsive flares in their sample. Unlike downward breaks, upward breaks are easily explained by the meshing of two acceleration mechanisms, as the shallow component that dominates above the break emerges naturally from beneath the steeper power law that dominates below. For upward breaks, $E_b$ is the naturally occurring crossover point. The matching problem is much more difficult for knees in the absence of significant cooling on time-scales of interest. As the steep component is above the break energy and the shallow component below it, both must be truncated at the break energy. If either one extended beyond the break, then that one would over-run the other, and there would be no break at all. The most natural solution for a knee is a single acceleration mechanism that undergoes some transition at the break energy. One such example is the knee found by Bell (1978b) in the spectra of shock accelerated electrons at roughly 1 GeV. This knee results from the transition from the non-relativistic to the relativistic regime. There is no apparent natural transition for STFA of electrons at 100 keV. However, there is a well-defined low-energy cut-off for a power-law spectrum at 100 keV, the shock injection energy. The shock injection threshold is not only at the right energy, but is also a variable cut-off, depending on the ion temperature and local magnetic field strength, consistent with the variability in the observed $E_b$.

Bell (1978b) has shown that shock acceleration does not change the spectrum of electrons if the pre-shock spectrum is shallower than the post-shock spectrum that is obtained from a steep pre-shock spectrum. Shock Fermi acceleration cannot steepen a power-law spectrum; it can only make it shallower. This is another difficulty for knee matching. The STFA spectrum must have a sharp cut-off at $E_b$ in order to match the knee. This may not be an impossible condition to meet, especially as $E_b$ may be greater than the injection energy, not precisely equal to it.

One natural cut-off occurs when $\lambda_C = \lambda_p$; acceleration will shut off when the cascade reaches the resistive scale. For slow modes in impulsive flares, $\lambda_p = 10^3 \, \text{cm}$ (LaRosa et al. 1996; Lithwick & Goldreich 2001; Chandran 2003), which for $\Gamma = 0.073 \, \text{keV}$ gives a cut-off energy of $7 \times 10^7 \, \text{keV}$, which is both too high and too far outside of the non-relativistic regime. A possible solution is that the constrained pitch scattering has a maximum resonant threshold at $E_b$. Another possible solution is that yet another very strong pitch scattering mechanism has a threshold energy of $E_b$ and a length-scale $< \lambda_p$. Both of these solutions are presently ad hoc. This underscores the need for a more thorough understanding of pitch angle scattering in astrophysical plasmas. It also illustrates the limitations of STFA as an acceleration mechanism in solar flares; if STFA alone were to account for the spectrum from 10 to 100 keV, the tight constraints on the pitch angle scattering mechanism that we have identified are required.

5 SUMMARY AND DISCUSSION

STFA in the non-relativistic limit behaves differently from highly relativistic STFA. At the core of these differences is the energy dependence of the electron velocity at low energies. Thus, unlike the relativistic case, both the rate of reflections and the probability of escaping the acceleration region at an energy $E$ vary. Using a test
particle approach, we have examined this behaviour and derived the spectrum of post-acceleration electrons in a plasma under impulsive solar flare conditions.

For traditional STFA, where there is a minimum pitch angle constraint that determines whether an individual encounter results in reflection, it is seen that the steady acceleration rate can dominate over the diffusive acceleration rate. This arises from the averaging over pitch angles to evaluate $\Delta E$. Some previous treatments of the generalized Fermi acceleration problem do not have such reflection conditions, and thus do not retain factors of the turbulent field strength, $B/\mu$, when averaging. In those treatments, (such as Skilling 1975; Webb 1983; Longair 1994) the steady and diffusive terms are typically seen to be of the same order. For some processes this is appropriate, however, non-resonant STFA is not one of them. Thus, the phase space conditions for scattering by the acceleration mechanism can play a very significant role, even in cases where pitch angle isotropy is maintained.

The nature of the pitch angle scattering turns out to be the dominant factor in determining electron escape, and therefore the shape of the spectrum. We find that whistler wave turbulence, which is well studied in solar flares (Melrose 1974; Miller & Steinacker 1992), is an excellent source of pitch angle scattering, which allows STFA to produce a quasi-thermal electron distribution that peaks at $E \approx 5$ keV. This matches the lowest energy portion of the observed X-ray emission very well. However, to produce the power-law spectrum observed in the range $\sim 10$–100 keV by STFA requires at least a second scattering mechanism. Matching the spectral index and the transition energy from a quasi-thermal to a power-law spectrum requires an undetermined scattering mechanism that satisfies $\lambda_c / L_F = \Gamma / E$ with $\Gamma = 0.073$ keV, and naturally becomes the dominant pitch angle scatterer at roughly 20 keV.

If the constrained pitch angle scattering mechanism is discovered, it implies that the acceleration of electrons in solar flares is at least a two stage process. The first stage, STFA in the downflow region, produces both the quasi-thermal spectrum below $\sim 10$ keV and the lower half of the power-law spectrum up to 100 keV. To produce the highest energy electrons, and the spectral break at $E = 100$ keV requires a second acceleration mechanism at the top of the soft X-ray loop. We are further exploring the possibility that first-order acceleration at a weak fast shock, formed as the downflow impacts the top of the closed flare loop, is responsible for electron acceleration to the highest observed energies. Acceleration at fast shocks is known to have an injection energy of roughly 100 keV, and varies with temperature. This coincides with the break in the power-law spectrum at 100 keV, and is consistent with the variability observed by Dulk et al. (1992) in $E_X$.

Recently, Chandran (2003) concluded, using quasi-linear theory, that STFA for slow modes is not viable in the 10–100 keV regime. While we also find slow modes to be ineffective, differences between our paper and Chandran (2003) must be kept in mind. Chandran (2003) assumed that $dp/d\lambda \propto p$ for STFA. While this is true in the strongly relativistic limit for STFA, we do not assume that this is true in the lower-energy regime (see equation 22). Secondly, unlike Chandran (2003) we do not assume herein that $P_{esc}$ has to be energy independent. These two assumptions play a significant role in shaping electron spectra.

Another concern that can be raised concerning the effectiveness of STFA as the electron acceleration engine in impulsive flares is the total energetics of the process. Since STFA, as developed above, is only efficient in a short length-scale regime where one also has $\delta B / B \ll 1$ it might seem that only a small fraction of the released flare energy is available for electron acceleration. This is not the case. The total energy contained in single turbulent cell is given by $\frac{1}{2} m_p v^2 n L^3 \sim 10^{26}$ erg, where $n = 10^{10}$ cm$^{-3}$ is the electron number density in the flare plasma. The energy in a single turbulent cell is similar to the energy contained in one X-ray emission fragment. Although only a fraction of the energy in one turbulent cell is ever at $\lambda_e$ at one time, it does all cascade down to $\lambda_p$ over an eddy turnover time. Thus, while $\delta B / B$ is always small, the energy throughput can still be high enough to accelerate the electrons. The similarity in total energy between a single turbulent cell and an individual impulsive X-ray fragment strongly suggests that the two are related.

Miller et al. (1997) estimates that as much as $\approx 94$ per cent of the magnetic energy in a flare is available in the turbulence, which is sufficient to produce the high-energy electrons inferred from the observed X-rays, but raises concerns about the efficiency of STFA, particularly in competition with other sources of dissipation. While we have not fully studied other dissipation mechanisms that might compete with STFA for this energy, three significant ones can be ruled out: proton acceleration by STFA, Landau damping and resistive dissipation of the turbulence. The latter two have already been discussed. Proton acceleration is a significant concern as Fermi acceleration of protons and heavy ions was in fact the very problem Fermi intended to solve. Thermal protons in coronal flare plasmas are sub-Alfvénic and thus cannot meet the condition for mirroring (LaRosa et al. 1996; Blackman 1999). However, Miller & Roberts (1995) argue convincingly that gyroresonant interaction of protons and heavy ions with Alfvén waves can accelerate them to velocities above $v_A$ on a relatively short time-scale. Within their model, the ions then are accelerated by compressive magnetosonic waves at the expense of electron acceleration. Recent RHESSI observations (Hurford et al. 2003) indicate that the emission signatures of ions and electrons are spatially separated, with the ion emission associated with longer loops. Miller et al. (2004) concluded that these observations are consistent with the gyroresonance model of ion acceleration; as the loops grow longer, protons are more likely to reach supra-Alfvénic speeds, and thus can be accelerated by the magnetosonic waves. This second phase of acceleration need not be gyroresonant. While it appears promising, further study is required to determine whether or not STFA models can accommodate these results.

The strong dependence of the post-acceleration electron spectrum on the pitch angle scattering agent is both a positive and a negative feature. It leaves STFA considerable flexibility in matching various characteristics of solar flare X-rays that fall outside the simple scenario studied in this paper. For example, Lin et al. (1981) first observed a superhot component in a solar flare, which has since been supported by RHESSI observations (Krucker, Hurford & Lin 2003). This thermal, or nearly thermal, spectral component is seen at energies of up to 35 keV. Within our STFA framework, the superhot emission can easily be explained by an enhancement of pitch angle scattering at lower energies, either by increased whistler wave turbulence, or some other scattering agent. While this flexibility naturally allows for the wide range of flare characteristics observed, it does not yet definitively solve the flare acceleration problem. Instead, it shifts the focus exclusively to a well constrained, but largely unspecified, array of pitch angle scattering mechanisms. This is the single greatest obstacle to STFA models of acceleration.

In short, STFA can naturally account for the thermal spectrum below 10 keV, and somewhat less naturally for the non-thermal spectrum between 10 and 100 keV. We have shown that $\lambda_e$ must depend inversely on particle energy, in contrast to that of pitch angle scattering by whistler waves below $\sim 10$ keV, which is proportional to the particle energy. Above 100 keV, shock acceleration is a natural
A stochastic Fermi acceleration
881

possibility; the needed injection of super-thermal electrons may be
provided by STFA operating at energies below \( E_{\text{tr}} \). The knee at 100
keV remains the most difficult spectral feature to accommodate, and
we have explained the difficult requirements to pitch angle scattering
that this demands.

ACKNOWLEDGMENTS

EB thanks B. Chandran for discussions. We thank J. Maron for shar-
ing the results of his simulations and acknowledge support from
DOE grant DE-FG02-00ER54600 and the Laboratory for Laser
Energetics at the University of Rochester. RS acknowledges sup-
port from the DOE Horton Fellowship. We also acknowledge the
insightful critique and comments of the reviewer.

REFERENCES

B’igarska Akademiia na Naukte, Sofia, p. 132–137, 11, 132
Fermi E., 1949, Phys. Rev., 75, 1169
L105
Sov. Phys. – Dokl., 22, 327, 328
Press, Cambridge

APPENDIX A: THE POWER LAW SPECTRUM
OF RELATIVISTIC STFA

Note that the spectrum we obtain for STFA is different from the
power-law result of Jones (1994). This is a matter of regime; we
discussed in the text the acceleration of non-relativistic particles in
a region of non-relativistic turbulence, here we show that when the
particles are relativistic, a power-law spectrum emerges.

Recall that equation (5) for fully relativistic electrons in a region
of non-relativistic turbulence is given by equation (6)

\[
\delta E = 2E \left( \pm \frac{v^2}{c^2} \right),
\]

where \( E \) is the total energy, kinetic plus rest, of the electron before
reflection. If we now take the low velocity limit, \( v \ll c \) where \( E = mc^2 \), the expression reduces to (5). The relative velocity between
the compression and electron for head-on and catch-up type interactions
is still given by (8) so the steady acceleration rate is given by

\[
\frac{dE}{dt} = 4E \left( \frac{v^2}{c^2} \right). \quad (A1)
\]

Alternatively, we can find the mean acceleration per reflection by
multiplying equation (A1) by \( R^{-1} \)

\[
\frac{dE}{dt} = 4E \left( \frac{v^2}{c^2} \right). \quad (A2)
\]

In the highly relativistic limit, \( E \) is just the kinetic energy, and we
recover the familiar result (Jones 1994) that \( dE/dt \propto E \). This propor-
tionality is expected to produce a power law. We derive the power-
law spectrum for STFA of highly relativistic electrons by following
the approach of Bell (1978a) and assume that \( p_{\text{esc}} \) is a constant in
flare plasma, independent of electron energy. We start by integrat-
ing \( dE/dt \) to obtain \( E(l) \)

\[
E = \frac{1}{A} \ln \left( \frac{E}{E_0} \right),
\]

\[
A = 4E_0^2 \left( \frac{v^2}{c^2} \right)^2. \quad (A3)
\]

The probability of an electron remaining in the acceleration region
for at least \( l \) reflections is given by

\[
P(l+) = (1 - p_{\text{esc}})^l. \quad (A4)
\]

Taking the logarithm and substituting in for \( l \) from equation (A4),
gives

\[
\ln P(E+) = l \ln (1 - p_{\text{esc}}) = \frac{1}{A} \ln \left( \frac{E}{E_0} \right) \ln (1 - p_{\text{esc}})
\]

\[
= \ln \left( \frac{E}{E_0} \right)^{(1 - p_{\text{esc}}^{-1})^{-1}}. \quad (A5)
\]

where we used the approximation \( \ln (1 - p_{\text{esc}}) = -p_{\text{esc}} \) for \( p_{\text{esc}} \ll 1 \) in obtaining the expression to the right of the final equals sign.
Differenitimating with respect to \( E \), results in

\[
P(E) = E_0 \left( \frac{E}{E_0} \right)^{(1 - p_{\text{esc}}^{-1})^{-1}}, \quad (A6)
\]

where \( P(E) \) is the unnormalized probability of a post-acceleration
electron having energy \( E \). In the limit, where \( p_{\text{esc}} \) is


Downloaded from https://academic.oup.com/mnras/article-abstract/354/3/870/993652 by guest on 24 January 2019
extremely small, the relativistic STFA spectrum has a power-law index of \( \sim 1 \). In a plasma where \( p_{\text{esc}} \sim A \), the power-law index can grow larger, and the index is very sensitive to \( p_{\text{esc}} \). In the third regime, where \( p_{\text{esc}} \gg A \), electrons stream out of the turbulent volume quickly, do not experience much acceleration, and have a very steep power-law energy distribution with virtually no very high-energy electrons (\( E \gg E_0 \)).

**APPENDIX B: DERIVATION OF STEADY ACCELERATION RATE WITH \( F_+ \neq F_- \)**

In Section 3.1 we derived the steady acceleration rate for electrons in a low-\( \beta \) turbulent magnetic plasma. This derivation was contingent on the assumption that \( F_+ = F_- = F \), which is not strictly valid. Blackman (1999) calculates \( F \) for Fermi acceleration. By resetting the limits of the integral in his equation (12), and renormalizing for the smaller phase space, one arrives at

\[
F_\pm = \cos \phi_m \left( \pm \frac{v_A}{v} + \left[ 1 - \left( \frac{v_A}{v} \right)^2 \left( 1 - \cos^2 \phi_m \right) \right]^{1/2} \right),
\]

where \( \cos \phi_m \) is the minimum pitch angle at which an electron will reflect and \( v_A/v \) is the ratio of the Alfvén speed to the electron speed. We rename these quantities \( A \) and \( B \), respectively; both are small quantities. By taking a series expansion of equation (B1) and truncating it at second order in \( B \), it can be simplified to

\[
F_\pm = A \left( 1 \pm B - \frac{1}{2} B^2 \right).
\]

Recall that from (8),

\[
R_\pm = F_\pm \left( \frac{v \pm v_A}{2\lambda} \right) = A \left( 1 \pm B - \frac{1}{2} B^2 \right) \left( A \pm B \right) \frac{v}{2\lambda}.
\]

From this one easily obtains

\[
R = (R_+ + R_-) \frac{v}{\lambda} \left[ A^2 \left( 1 - \frac{1}{2} B^2 \right) + AB \right].
\]

and

\[
(R_+ - R_-) \frac{v}{\lambda} \left[ A^2 B + AB \left( 1 - \frac{1}{2} B^2 \right) \right].
\]

This gives us all of the ingredients for calculating the steady acceleration from (7)

\[
\left( \frac{dE}{dt} \right)_S = 2m \left( (R_+ - R_-)v_A + (R_+ + R_-)v_A^2 \right).
\]

The resulting acceleration rate is

\[
\left( \frac{dE}{dt} \right)_S = \frac{2mv^3}{\lambda} \left( 2A + A^2 - \frac{1}{2} B - \frac{1}{2} AB^2 + B^2 \right),
\]

where we have added the additional subscript \( b \) to indicate the distinction from the previously calculated rate. The steady acceleration rate found in (10) from the assumption \( F_+ = F_- = F \) is

\[
\left( \frac{dE}{dt} \right)_S = \frac{4mv^3}{\lambda} A^2 B^2.
\]

Note that provided \( A > 4B \) this is the largest term in (B6). Indeed, for coronal flare plasma, \( B \sim 0.1 \) at electron energy \( E_0 \) and decreases with increasing energy while \( A \sim 0.1 \) as well at \( E_0 \), but is largely insensitive to electron energy. At the onset of the power-law regime, \( E = 10 \text{ keV} = 30E_0 \) and \( B \sim 0.01 \); all terms of the order of \( B^3 \) or higher can be neglected, as can the term in \( A^3 \). Thus we can safely use the assumption \( F_+ = F_- = F \) in this regime, and (11) is reasonable.

This paper has been typeset from a \TeX/LaTeX file prepared by the author.