Low-order multipole maps of cosmic microwave background anisotropy derived from WMAP

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ABSTRACT
Recent analyses of the Wilkinson Microwave Anisotropy Probe (WMAP) data have suggested that the low-order multipoles of the cosmic microwave background (CMB) anisotropy distribution show cosmologically interesting and unexpected morphologies and amplitudes. In this paper, we apply a power equalization (PE) filter to the high-latitude WMAP data in order to reconstruct these low-ℓ multipoles free from the largest Galactic foreground modelling uncertainties in the Galactic plane. The characteristic spatial distributions of the modes of order ℓ = 2, 3, 4, 5 have been determined as a function of frequency, sky coverage and two methods for foreground correction: using the template-based corrections of Bennett et al. and by a simple linear projection scheme assuming the spectral dependence of the foreground components. Although the derived multipole maps are statistically consistent with previous estimates from Tegmark, de Oliveira-Costa & Hamilton (TOH) and Efstathiou, our analyses suggest that the K and Ka frequency bands remain significantly contaminated by residual foreground emission for the WMAP Kp2 mask. However, the ℓ = 3, 4, 5 multipole maps for the Q, V and W channels indicate that, after foreground cleaning, these multipoles are dominated by the CMB anisotropy component. We confirm the TOH result that the octopole does indeed show structure in which its hot and cold spots are centred on a single plane in the sky, and show further that this is very stable with respect to the applied mask and foreground correction. The estimated quadrupole is much less stable showing non-negligible dependence on the Galactic foreground correction. Including these uncertainties is likely to weaken the statistical significance of the claimed alignment between the quadrupole and octopole. Nevertheless, these anisotropy patterns are also present in the COBE-DMR data and are unlikely to be associated with instrumental systematic artefacts.

Key words: cosmic microwave background – cosmology: observations.

1 INTRODUCTION
In the standard cosmological interpretation, the distribution and statistical properties of the cosmic microwave background (CMB) temperature anisotropies reflect the properties of the Universe approximately 300 000 yr after the big bang. The simplest inflationary models of the origin of the primordial density fluctuations predict statistically isotropic and Gaussian fluctuations for the CMB. Thus, in the conventional approach, where the temperature anisotropy field, ∆T(θ, φ), is expanded as a sum over the spherical harmonic basis on the sphere

$$\Delta T(\theta, \phi) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi),$$

the decomposition coefficients, a_{\ell m}, are independent, Gaussian distributed stochastic variables. Then, as a result of statistical isotropy, their variances C_\ell = \langle |a_{\ell m}|^2 \rangle depend only on \ell. The validity of these predictions can be constrained by direct observation of the CMB anisotropy.

The sum over m for a given \ell, \Delta T_\ell(\theta, \phi), is then a multipole map of order \ell and typically represents the CMB fluctuations on angular scale \sim 1/\ell. The lowest multipoles of the CMB anisotropy, in particular the dipole (\ell = 1), quadrupole (\ell = 2) and octopole (\ell = 3), are especially important for studying the homogeneity and

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isotropy of the Universe after the last scattering of the CMB photons (Maartens, Ellis & Stoeger 1996). The measurement of these low-order multipoles requires observations over a large fraction of the sky, as provided initially by the COBE-DMR experiment and more recently by the Wilkinson Microwave Anisotropy Probe (WMAP) satellite. The dipole is generally interpreted as being dominated by a kinetic effect related to the peculiar motion of the Earth with respect to the CMB rest frame, so cosmologically relevant studies usually consider multipoles only from the quadrupole upwards. The first measurement of the quadrupole by COBE-DMR suggested an intriguingly low value for its amplitude and led to vigorous discussion about the credibility of its estimation (Bennett et al. 1992; Smoot et al. 1992; Gould 1993; Stark 1993; Kogut et al. 1996). In particular, doubts centred on limitations imposed by systematic effects, instrumental noise and non-cosmological foreground contributions as a result of local astrophysical emission. Nevertheless, the low amplitude of the quadrupole was recently confirmed by high-precision measurements from the WMAP satellite (Bennett et al. 2003a) after the application of a refined analysis of Galactic foreground contamination.

Subsequently, Tegmark et al. (2003) (hereafter TOH) have pointed out that the quadrupole and octopole appear rather planar, with most of the hot and cold spots centred on a single plane in the sky. Moreover, the two planes appear rather aligned. Other studies have also demonstrated the presence of intriguing correlations (Copi, Huterer & Starkman 2004; Schwarz et al. 2004) and anomalies (Eriksen et al. 2004b; Larson & Wandelt 2004; Vielva et al. 2004) of the low-order WMAP multipoles. Some of these features could be explained by a specific non-standard topology for the Universe, although a broad class of finite universe models, including the Poincaré Dodecahedron model proposed by Luminet et al. (2003), was recently ruled out by Cornish et al. (2004). However, many of these previous analyses have determined the multipole amplitudes on the full sky anisotropy as measured by either the internal linear combination (ILC, Bennett et al. 2003b) or TOH map. Importantly, Eriksen et al. (2004a) have shown that residual Galactic foreground uncertainties for the ILC map in the Galactic plane can be at the level of approximately 50 μK, whilst the analysis of Slosar & Seljak (2004) strongly suggests that much of the detected alignment is indeed related to that part of the data most contaminated by (residual) Galactic foreground.

The goal of this paper is to estimate the low-order WMAP multipoles using a technique that allows the explicit determination of their reconstruction uncertainties, and subsequently to clarify whether the unexpected features of the quadrupole and octopole are significant. Some of these uncertainties are related to Galactic foreground emission. In what follows, we consider the estimation of CMB anisotropy based on the WMAP maps in the Galactic plane can be at the level of approximately 50 μK, whilst the analysis of Slosar & Seljak (2004) strongly suggests that much of the detected alignment is indeed related to that part of the data most contaminated by (residual) Galactic foreground.

The (true) sky signal \( x \) itself the sum of the true CMB anisotropy field \( x_{\text{CMB}} \) and the foreground contribution \( x_{\text{fg}} \) by

\[
y = Bx + n.
\]

where the matrix \( B \) encodes the effect of beam and pixel smoothing, while the additional term \( n \) represents the contribution from detector noise. The foreground contamination can initially be minimized by masking out those regions of the sky with strong foreground emission, e.g. as a result of the Galactic plane or point sources. The remainder of the map can then be corrected for the high-latitude Galactic (and extragalactic if necessary) emission. The vector \( d \) is then considered to denote a cut sky, foreground corrected map.

In the spherical harmonic, \( Y_{\ell m} \), space, the vector \( \hat{a} \) of decomposition coefficients of the map on the cut sky, \( \hat{a} \), is related to the corresponding vector on the full sky, \( a \), by

\[
\hat{a} = K \cdot B \cdot a + n_c,
\]

where \( n_c \) denotes the noise vector on the cut sky and \( B \) now consists of the appropriate harmonic coefficients of the beam window function. We will assume that the beam profile is azimuthally symmetric. Then \( B \) is a diagonal matrix dependent only on \( \ell \) and consisting of the weights \( w_{\ell, \text{beam}} \). To simplify notation, hereafter \( a \) will denote the vector multiplied by \( B \). \( K \) is referred to as the coupling matrix and its \( (i,j) \)th component is given by

\[
K_{(i,m),(j',m')} = \int_{\text{cut sky}} Y_{\ell m}^* (\hat{\theta}) Y_{\ell' m'} (\hat{\theta}) \, d\Omega_\theta.
\]

Our aim is to determine the full sky representation of the anisotropy written in terms of the coefficients \( a \) using the above relation. Unfortunately, we do not know the noise contribution to this specific set of observations, only a statistical description of the ensemble properties, and moreover the sky cut causes the coupling matrix \( K \) to be singular. This means that there is no information on those anisotropies that project on to modes in the spherical harmonic basis where the only structure lies within the sky cut. Hence, it is

\[ i = \ell^2 + \ell + m + 1. \]
impossible to reconstruct all of the modes from the \( \vec{a} \). However, for low-order multipoles, small sky cuts and a high signal-to-noise ratio, a good approximation is to simply truncate the vectors and coupling matrix at some value of the index \( i(\ell, m) \) then reconstruct the multipole coefficients by inverting the non-singular truncated matrix \( K \). This method was applied to the first-year WMAP data in Efstathiou (2004) and will hereafter be referred to as direct inversion.

Direct inversion is a special case of a more general technique for estimation of the true sky signal by a linear transformation of the data vector \( \hat{a} \):

\[
\hat{a} = F \cdot \hat{a},
\]

where matrix \( F \) acts as a filter and satisfies certain conditions imposed on the solution \( \hat{a} \). In this paper, we will consider both Wiener (1949) and power equalization (PE) filters, and utilize the detailed formalism developed by Górski (1997) for the application of these filters to CMB observations.

### 2.1 Wiener filter

The Wiener filter (Wiener 1949) is constructed from the requirement that the mean square deviation between the filtered signal, \( \hat{a} \), and the true signal, \( a, \epsilon = (\hat{a} - a)^T \cdot (\hat{a} - a) = tr((\hat{a} - a) \cdot (\hat{a} - a)^T) \), is minimized. Then we obtain

\[
F_\psi = S_{LL} \cdot (K \cdot S \cdot K^T + N)^{-1}
\]

(see, e.g. Bunn et al. 1994; Zaroubi et al. 1995) where \( S \equiv (a \cdot a^T) \) and \( N \equiv (n \cdot n^T) \) are covariance matrices of the signal and noise, respectively. One should notice that this filter practically annihilates non-cosmological modes can be removed from the analysis by assumption that their power spectrum in the matrix \( (K \cdot S \cdot K^T + N)^{-1} \) is very large. In practice, the monopole and dipole are such terms, because for differenting experiments such as WMAP the monopole may correspond to an unphysical quantity related to the map-making process, whereas the dipole is attributed to the motion of the Earth in the Universe. For negligible noise the filter is essentially equivalent to the direct inversion method and does not depend on the assumed signal covariance matrix. However, we would like to explicitly take into account the correlations induced by the cut between the lower order modes we are interested in reconstructing with the higher order modes present in the data. These correlations are non-negligible and play the role of noise in the Wiener filter.

For further consideration, it will be useful to introduce a new orthonormal basis of functions on the cut sky. A method of construction of such orthonormal functions, \( \psi \), was proposed by Górski (1994).

In this new basis, the vector of decomposition coefficients of the cut sky map, \( c \), is related to vector \( a \) by

\[
e = L^T \cdot a + n_\psi,
\]

where \( L \) is the matrix derived by the Choleski decomposition of the coupling matrix and \( n_\psi \) is the vector of noise coefficients in the \( \psi \) basis. Because \( L \) is a lower triangular matrix, \( c \) can be decomposed into a part dependent on both lower \( a_\ell \) and higher \( a_{\ell' \ell} \) order modes

\[
c_\ell = L_{\ell \ell}^T \cdot a_\ell + L_{\ell \ell'}^T \cdot a_{\ell'} + n_{\psi, \ell}
\]

and a part dependent only on higher order modes

\[
c_{\ell'} = L_{\ell' \ell''}^T \cdot a_{\ell''} + n_{\psi, \ell'}
\]

The subscripts \( \ell \) and \( \ell' \) denote the range of indices \( i = 1, \ldots, (\ell_{rec} + 1)^2 \) and \( i = (\ell_{rec} + 1)^2 + 1, \ldots, (\ell_{max} + 1)^2 \) corresponding to those modes that are to be reconstructed with the filter (multipoles in the range from 1 to \( \ell_{rec} \)) and the remainder of the modes used in the analysis (multipoles in the range from \( \ell_{rec} + 1 \) to \( \ell_{max} \)), respectively. It should be noticed that only the first part of the vector \( c \) contains information about lower order modes \( a_\ell \), so to estimate them we will use only this part of the vector. This is a significant difference as compared to the vector \( \hat{a} \) where information about the lower order modes is also contained in the second part, \( \hat{a}_{\ell''} \).

Taking \( c_\ell \) as a data vector, estimation of the lower order modes will be given by \( \hat{a}_\ell = F \cdot c_\ell \). For the Wiener filter we have

\[
F_\psi = S_{LL} \cdot L_{LL} \cdot C^{-1},
\]

where \( C = L_{LL}^T \cdot S_{LL} \cdot L_{LL} + L_{HL} \cdot S_{HL} \cdot L_{HL} + N_\psi \) and \( S_{LL} \) and \( S_{HL} \) are the appropriate covariance matrices of the CMB modes corresponding to the lower and higher multipoles, respectively. \( N_\psi \) is the noise covariance matrix in the \( \psi \)-basis.

### 2.2 Power equalization filter

The PE filter is defined by the requirement that \( F \) is chosen such that

\[
⟨\hat{a} \cdot a^T⟩ = S_{LL}, \quad \text{i.e. } F_{PE} \cdot C_{PE} \cdot F_{PE}^T = S_{LL}.
\]

Using the Choleski decomposition of the relevant matrices, we can construct the PE filter as follows:

\[
S^{-1}_{LL} = L_{S^{-1}_{LL}} \cdot L_{S^{-1}_{LL}}^T, \quad C_{PE} = L_{C_{PE}}^T \cdot C_{PE} \cdot L_{C_{PE}},
\]

and as a result we obtain an upper triangular matrix,

\[
F_{PE} = L_{C_{PE}}^T \cdot L_{C_{PE}}^{-1},
\]

where the matrices \( S_{LL} \) and \( C \) are the same as in the Wiener filter case.

The condition (11) means that on average over many applications the PE filter renders filtered data whose statistically most likely power spectrum matches that of the underlying signal. Because the PE filter is an upper triangular matrix, the non-cosmological monopole and dipole terms can be explicitly eliminated from the analysis. Then, the restored modes (with \( i > 4 \)) are not contaminated by the monopole and dipole contributions.

### 2.3 Errors

The errors of estimation are determined on the basis of the diagonal terms of the covariance matrix, defined as \( M = ⟨(a_\ell - \hat{a}_\ell)(a_\ell - \hat{a}_\ell)^T⟩ \). Thus, we have

\[
M = \left(F \cdot L_{LL}^T - I\right) \cdot S_{LL} \cdot \left(F \cdot L_{LL}^T - I\right)^T + F \cdot \left(L_{HL}^T \cdot S_{HL} \cdot L_{HL}^T + F^T + F \cdot N_\psi \cdot F^T\right),
\]

where \( I \) is the identity matrix. This is a sum of three components: the first is a result of correlations between the recovered low-order multipoles, the second is induced by correlations with higher order multipoles and the last is generated by noise.

### 2.4 General considerations

In order to construct the filters, we have to assume the form of both the signal and noise covariance matrices. We will make the usual assumption that both the CMB and noise components are Gaussian stochastic variables. The variances for individual modes of the CMB anisotropy, \( C_\ell \), are uniquely expressed as

\[\text{CMB}\]
integrals over the power spectrum of the matter density fluctuations and depend only on $\ell$ as a result of the statistical isotropy of the CMB temperature field. The rms noise level in pixel $p$, $\sigma(p)_{\text{noise}} = \sigma_o/\sqrt{N_{\text{det}}(p)}$, depends on quantities specific to a given instrument and scan strategy: the number of observations in pixel, $N_{\text{det}}(p)$, and the rms noise per observation, $\sigma_o$. Dependence on the assumed power spectrum might seem to be a disadvantage of these filtering methods. However, we will see that in the case of the WMAP data, the multipole estimation does not depend significantly on the assumed power spectrum.

3 DATA

The WMAP satellite observes the sky in five frequency bands denoted $K, Ka, Q, V$ and $W$, centred on the frequencies of 22.8, 33.0, 40.7, 60.8 and 93.5 GHz, respectively. The maps $3^3$ are pixelized in the HEALPix$^4$ scheme (Górski, Hivon & Wandelt 1998) with a resolution parameter $N_{\text{side}} = 512$, corresponding to $3 \times 10^7$ pixels with a pixel size of $\sim 7$ arcmin. The maps are corrected for the Galactic foreground using three templates$^5$ [a radio survey at 408 MHz (Haslam et al. 1981), a Hα map (Finkbeiner 2003a) and a dust map based on the combined COBE-DIRBE and IRAS data (Finkbeiner, Davis & Schlegel 1999)] to trace the synchrotron, free–free and dust emission. Bennett et al. (2003b) have concluded that the combination of these templates, each scaled to the frequencies in question by a single factor derived for the high-latitude sky (i.e. making no allowances for spectral variations in the foreground behaviour), is sufficient to account for almost all of the foreground emission including the anomalous dust-correlated component (irrespective of its physical origin). Although the WMAP team did undertake an analysis of the data based on the maximum entropy method (MEM), which allowed for spectral index variations of the foreground components on a pixel-by-pixel basis, the resultant CMB map was deemed inappropriate for use as a result of complex noise properties. For all cosmological data analysis, the WMAP team have corrected the data using the template method.

We adopt the coefficients from Bennett et al. (2003b), which were derived by fitting the templates to the data for the Kp2 sky coverage, and then subtract the appropriately scaled data from the WMAP sky maps.$^3$ For the $Q, V$ and $W$ bands five coefficients were determined: an amplitude for each of the synchrotron and free–free components at 41 GHz (and scaled to the other frequencies by assuming spectral indices of $-2.7$ and $-2.15$, respectively) and three for the dust, one at each frequency. The anomalous dust-correlated emission was thus effectively absorbed into the dust template scaling (which also accounts for the thermal contribution). For the $K$ and $Ka$ channels, the coefficients from table 3 of Bennett et al. (2003b) were used. The $Q$-, $V$- and $W$-band maps are then combined using inverse-noise-variance weights to create the corrected co-added map.

We utilize several masks in the work that follows: the Kp2 mask (eliminating $\sim 15$ per cent of the sky), the Kp0 mask (eliminating $\sim 23$ per cent of the sky), an extended Galactic mask and mask 30.

The extended Galactic mask (hereafter denoted 20+) is the same as that used for the COBE-DMR data analysis and is defined as the region with latitude $|b| > 20^\circ$ plus custom cut-outs in the vicinity of Orion and Ophiuchus (Bennett et al. 1996). The broadest mask 30 is defined as the region with latitude $|b| > 30^\circ$. The latter masks were additionally modified by excluding the point source mask included in the Kp2 and Kp0 masks, so that they finally eliminate $\sim 38$ and $\sim 50$ per cent of the sky, respectively. It should be noted here that changing the Galactic cut applied to the data could, in principle, require modulation of the scaling coefficients used for the templates, particularly if the foregrounds do demonstrate strong variations in spectral index with Galactic latitude. However, we proceed with the simple foreground model specified by the WMAP team and used by them (and elsewhere in the literature) irrespective of the applied mask.

4 CHOICE OF $\ell_{\text{max}}$ AND $\ell_{\text{rec}}$

The choice of $\ell_{\text{max}}$ and $\ell_{\text{rec}}$ is based on Monte Carlo (MC) simulations of the WMAP data, consisting of both realizations of the CMB anisotropy and noise. The CMB anisotropy maps were generated as random realizations of a Gaussian field with an angular power spectrum corresponding to the best-fitting ΛCDM cosmological model with running spectral index$^6$ (Bennett et al. 2003a; Hinshaw et al. 2003; Spergel et al. 2003), convolved with the channel specific beam window function $w_{\text{beam}}^2$. The noise maps were produced from Gaussian random numbers with pixel-dependent variance corresponding to the properties of the corrected co-added map. The simulated maps were studied after masking by the Kp2 mask.

The upper limit for possible values of $\ell_{\text{max}}$ and $\ell_{\text{rec}}$ is imposed by the feasibility of the Choleski decomposition of the coupling matrix. For a given cut, the coupling matrix becomes numerically singular for multipoles larger than $\ell_{\text{Choleski}}$ at which point the Choleski decomposition fails. For the Kp2 mask, this multipole is found at $\ell_{\text{Choleski}} \sim 70$. The studies in this paper are focused mostly on estimation of the lowest multipoles ($2 \leq \ell \leq 10$) and particularly on those modes that have been shown to have potentially anomalous properties ($2 \leq \ell \leq 5$). Thus it is sufficient to include only multipoles up to $\ell_{\text{max}} = 30$. Increasing $\ell_{\text{max}}$ to 35 or 40 did not appreciably change the estimated values of these multipoles. This is a direct consequence of the decreasing amplitude of higher multipoles, scaling roughly as $\ell^{-1}$. In order to establish a suitable value for $\ell_{\text{rec}}$, 1000 MC simulations were generated and the PE filter computed with $\ell_{\text{max}} = 30$ applied to the simulated maps masked by the Kp2 mask. The power spectrum of the difference between the true and restored multipole coefficients over the range $2 \leq \ell \leq 5$,

$$c_{\ell}^{\text{error}} = \frac{\sum_{\ell = r + 1}^{\ell_{\text{max}}} (a_{\ell}^{\text{output}} - a_{\ell}^{\text{input}})^2}{2\ell + 1}.$$

is shown in Fig. 1 as a function of $\ell_{\text{rec}}$. Notice that for $\ell_{\text{rec}} = \ell_{\text{max}} = 30$, the errors are the largest. In this case, the PE filter does not adequately take into account the aliasing of higher order modes into the range $\ell \leq \ell_{\text{rec}}$. Significant errors are seen also for lower values of $\ell_{\text{rec}}$, caused by correlations of the multipoles $\ell = 2, \ldots, 5$ with modes of order $\ell_{\text{rec}} + 1, \ell_{\text{rec}} + 2$, which for low values of $\ell_{\text{rec}}$ have significant amplitudes. In what follows, we adopt $\ell_{\text{rec}} = 10$; although the flat power spectrum of the uncertainties over the range $10 \leq \ell_{\text{rec}} \leq 25$ allows a larger value for $\ell_{\text{rec}}$ to be selected.

$^3$ Available at http://lambda.gsfc.nasa.gov

$^4$ http://www.eso.org/science/healpix

$^5$ In fact, the LAMBDA web provides a set of foreground corrected maps generated with this recipe. However, a problem was discovered (see http://lambda.gsfc.nasa.gov/product/map/IMaps_cleaned.cfm) in that the templates used in this process were convolved with a too low $\ell_{\text{max}}$ (the maximum multipole component), resulting in ringing around strong point sources. Although we do not expect this to compromise our results on large angular scales, we have, nevertheless, generated our own template corrections.

$^6$ Available at http://lambda.gsfc.nasa.gov
over 10,000 WMAP by the Wiener and PE filtering. In addition, we show the averages $C_\ell$ denote just denition method of Efstathiou (2003) adopts two parameters without later in the paper we will consider the broader Galactic cut 30 for which the PE filter fails when too large a value for $\ell_{\text{rec}}$ is used.

It should be noted that other methods for the determination of the low multipoles also require the choice of some parameters to facilitate the reconstruction method. In particular, the direct inversion method of Efstathiou (2003) adopts two parameters without justification: the order of the multipole at which the coupling matrix is truncated and the FWHM of the beam for the initial smoothing of the data.

**5 COMPARISON OF THE WIENER AND PE FILTERS**

Fig. 2 provides a comparison of the WMAP power spectra obtained by the Wiener and PE filtering. In addition, we show the averages over 10,000 WMAP map simulations for the power spectra of the restored maps. In both cases the Kp2 mask and filters with $\ell_{\text{rec}} = 10, \ell_{\text{max}} = 30$ were applied.

The Wiener and PE filters differ mainly in the recovered power of the multipoles: there is a systematic suppression of power with increasing $\ell$ for maps reconstructed by the Wiener filtering. Conversely, the definition of the PE filter ensures an unbiased estimate for the multipole amplitudes. However, the PE filter does not give an optimal estimation in the minimum variance sense, i.e. an estimate that has the least variance between the true and restored data, which is the case for the Wiener filter case. Nevertheless, we decided to use the PE filter in the remainder of the paper because for the lowest multipoles and Kp2 mask the results do not differ significantly from those derived by the optimal Wiener filter and the unbiased nature of the estimator is considered to be more important. This is particularly relevant with increasingly aggressive sky cuts where the Wiener filter demonstrates even more suppression of power.

**6 COMPARISON OF THE DIRECT INVERSION AND THE PE FILTERING METHODS**

10,000 simulations of the WMAP data were again used to compare the accuracy of reconstruction of the low-order multipoles by the direct inversion and the PE filtering methods. The reconstructions were performed for the Kp2 mask, and for a PE filter constructed with $\ell_{\text{rec}} = 10$ and $\ell_{\text{max}} = 30$. For the direct inversion method, the coupling matrix was evaluated after truncation at either $\ell_{\text{rec}} = 10$ or $\ell_{\text{rec}} = 30$. In both cases, following Efstathiou (2003) the input map was smoothed by a Gaussian beam with FWHM = 7°. Smoothing improves the accuracy of the direct inversion method reconstructions, because it damps higher order modes. The input and output (reconstructed) coefficients $a_{\ell m}$ for the multipoles $\ell = 2, \ldots, 5$ are compared in Figs 3 and 4. Only those modes with $m = \ell$ are shown because, as will be shown in Section 8.2, they (and the corresponding modes with $m = -\ell$) have the largest reconstruction uncertainties as a result of the removal of a large fraction of their power by the cut. Thus, they are the most sensitive to the method used in their estimation. Nevertheless, comparison of the other modes gives qualitatively similar results.

Fig. 3 shows that the PE filter recovers the quadrupole and octupole amplitudes with a similar accuracy to the direct inversion method. However, for higher order modes, the PE filter performs better. This does not appear to be connected to the use of $\ell_{\text{rec}} = 10$ for the direct inversion method; increasing this value to $\ell_{\text{rec}} = 30$ yields even worse reconstructions (Fig. 4). The higher accuracy of the PE filter reconstructions is a consequence of explicitly accounting for the aliasing of higher order multipoles into the lower multipole range in the filter construction.

**7 APPLICATION TO THE WMAP DATA**

The PE filter was applied to WMAP maps with $\ell_{\text{max}} = 30$ and $\ell_{\text{rec}} = 10$. We are mostly interested in the analysis of the lowest multipoles $\ell = 2, \ldots, 5$ because these are the most sensitive to the cut and foreground emission. The monopole and dipole were not restored because they do not provide any relevant cosmological information. In order to construct the filter matrix, the WMAP best-fitting $\Lambda$ CDM model ($\Omega_\Lambda = 0.73, \Omega_m = 0.27$) with a running primordial spectral was used. This choice is based on the fact that the spectrum provides a better fit to the WMAP data on large angular scales compared with a simple power-law spectrum. However, tests with a power-law spectrum yield similar results. The noise covariance matrices were constructed in the usual fashion, with pixel variances determined...
Figure 3. Comparison of the accuracy of estimation of the $a_{\ell,\ell}$ modes using the direct inversion and PE filtering reconstruction methods, determined from simulations with the Kp2 mask. The abscissae give the input values of harmonic coefficients used to generate the simulated skies. The ordinates give the output values from the direct inversion (left column) and PE filtering (right column) methods. The coupling matrix was truncated at $\ell_{rec} = 10$ in the direct inversion method.

The power spectrum will provide a useful complement to the studies of the multipole maps. We will use an estimator of the power spectrum given by

$$\hat{C}_\ell = \frac{\sum_m |\hat{a}_{\ell m}|^2}{2\ell + 1},$$

where $\hat{a}_{\ell m}$ denotes the restored harmonic coefficients. The definition of the PE filter ensures that the estimator is unbiased. Given the error covariance matrix $\mathbf{M}$, one can estimate the uncertainty in the quantity $C_\ell$ by

$$\sigma^2_\ell = \frac{2}{2\ell + 1} \sum_{m, m'} M_{mm'} (2\langle \hat{a}_m \hat{a}_m^* \rangle - M_{mm}).$$

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$$\sigma^2_\ell = \frac{2}{2\ell + 1} \sum_{m, m'} M_{mm'} (2\langle \hat{a}_m \hat{a}_m^* \rangle - M_{mm}).$$
where the indices $i,j$ in $\ell$-ordering are expressed by $i = \ell^2 + \ell + m + 1$. This estimation of the uncertainty takes into account only errors induced by the sky cut and noise; it does not take into account cosmic variance related errors that enter through the assumption of a cosmological power spectrum to build the PE filter. Table 1 indicates the reconstruction accuracy of the method.

7.1 The multipoles maps

Maps of the recovered multipoles computed from the WMAP corrected co-added map are compared to the multipoles of the cleaned TOH and ILC maps in Fig. 5. Because the details of the noise model are essentially negligible for the estimation of low-order multipoles, we have also used the PE filter as computed for the corrected co-added map to analyse the TOH and ILC maps. The comparison confirms the previous results of estimations of the multipole amplitudes. Small discrepancies are seen only at $\ell = 2$, where the hotspots are closer to the Galactic equator and the amplitude is smaller for our results. The power spectrum of the maps is shown in Fig. 6 and summarized in Table 2. The cut sky analysis of Hinshaw et al. (2003) shows substantially less power at $\ell = 3, 5$ and 7 than for the other estimates. Efstathiou (2003) has claimed that this is a consequence...
Table 1. Rms difference between the true and estimated power spectra $C_{\ell}^{\text{true}}$ (in $\mu$K$^2$) and the correlation coefficients $r_{\ell\ell}$ of the $\ell$-order, $m=\ell$ modes estimated from a set of 10000 simulations of the WMAP data analysed after application of the Kp2 mask. In the third column, the signal to error ratio is shown where $\hat{C}_{\ell}$ is the estimated power spectrum of the WMAP multipoles maps given by PE filtering.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$C_{\ell}^{\text{true}}$</th>
<th>$\hat{C}<em>{\ell}/C</em>{\ell}^{\text{true}}$</th>
<th>$r_{\ell\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>69</td>
<td>2.8</td>
<td>0.99</td>
</tr>
<tr>
<td>3</td>
<td>41</td>
<td>12</td>
<td>0.98</td>
</tr>
<tr>
<td>4</td>
<td>26</td>
<td>9.6</td>
<td>0.97</td>
</tr>
<tr>
<td>5</td>
<td>19</td>
<td>17</td>
<td>0.96</td>
</tr>
<tr>
<td>6</td>
<td>14</td>
<td>5.2</td>
<td>0.94</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>11</td>
<td>0.92</td>
</tr>
<tr>
<td>8</td>
<td>9.0</td>
<td>6.3</td>
<td>0.90</td>
</tr>
<tr>
<td>9</td>
<td>7.6</td>
<td>7.6</td>
<td>0.87</td>
</tr>
<tr>
<td>10</td>
<td>6.2</td>
<td>7.3</td>
<td>0.84</td>
</tr>
</tbody>
</table>

of the quadratic nature of this estimator. Our linear estimator results, and in particular the ILC results in Table 2, are in good agreement with those of Efstathiou (2003).

7.2 Dependence on the assumed angular power spectrum

Because construction of the filters requires some assumption about the CMB anisotropy power spectrum $C_{\ell}$, one should investigate how sensitive the amplitudes of the restored multipoles are to this choice and thus quantify the uncertainties in the amplitudes related to the assumption. To answer these questions, we have applied filters constructed using two other power spectra: the canonical one described above but with the square root of the cosmic variance either added or subtracted. In Fig. 7, we see that the power spectrum of the resultant maps is changed only weakly by these perturbations, up to few per cent. For the WMAP analysis presented here, the PE filtering method gives robust estimates of the amplitudes of the low-order multipoles, essentially independent of the assumed input power spectrum.

7.3 Dependence on frequency

Comparison of the multipoles maps determined from the foreground corrected WMAP data frequency-by-frequency allows us to determine the extent to which the derived structures are cosmological and the degree of residual foreground contamination. In Fig. 8, it is seen that the foreground cleaning of the $K$-band map gives the worst results. The multipole maps differ substantially from the rest of the maps showing an excess of the power near the Galactic plane. Nevertheless, apart from the quadrupole, there is a qualitative similarity in the distribution of hot and cold spots relative to the other frequency channels. In particular, the octopole is planar with approximately the same preferred axis, along which the power is suppressed, for all frequencies. The quadrupole is also planar, although changes of
Small deviations between the Q (95 per cent confidence level).

Running spectral index. The grey band indicates the cosmic variance errors K.

Angular power spectrum of the Figure 6.

Does not remove all the contamination for the rized in Table 2 clearly show that the applied foreground correction (despite the fact that the Kp2 mask is derived from considerations the strong signals near to the Galactic plane at those frequencies bands. This suggests that the Kp2 cut is not sufficient to remove from the Hinshaw et al. (2003) cut sky

Power spectrum amplitudes \( \Delta T^2_\ell = (\ell + 1) C_\ell / 2\pi \) (in \( \mu K^2 \)) for the foreground corrected WMAP frequency maps and related combinations estimated by application of the PE filter for the Kp2 sky coverage. \( Q + V + W \) denotes the corrected co-added map and \( * \) denotes estimation on the full sky. The direct inversion method was applied to the corrected co-added map for the Kp2 mask. Uncertainties do not take into account cosmic variance.

<table>
<thead>
<tr>
<th></th>
<th>( \Delta T^2_2 )</th>
<th>( \Delta T^2_3 )</th>
<th>( \Delta T^2_4 )</th>
<th>( \Delta T^2_5 )</th>
<th>( \Delta T^2_6 )</th>
<th>( \Delta T^2_7 )</th>
<th>( \Delta T^2_8 )</th>
<th>( \Delta T^2_9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>963 ± 79</td>
<td>1710 ± 160</td>
<td>3290 ± 220</td>
<td>2070 ± 130</td>
<td>710 ± 110</td>
<td>1060 ± 120</td>
<td>1320 ± 120</td>
<td>1330 ± 150</td>
</tr>
<tr>
<td>( Kp2 )</td>
<td>176 ± 26</td>
<td>1060 ± 120</td>
<td>1190 ± 110</td>
<td>1550 ± 110</td>
<td>312 ± 55</td>
<td>960 ± 100</td>
<td>786 ± 84</td>
<td>920 ± 130</td>
</tr>
<tr>
<td>( Q )</td>
<td>158 ± 32</td>
<td>990 ± 110</td>
<td>875 ± 80</td>
<td>1610 ± 120</td>
<td>459 ± 77</td>
<td>1100 ± 110</td>
<td>683 ± 75</td>
<td>880 ± 130</td>
</tr>
<tr>
<td>( V )</td>
<td>206 ± 40</td>
<td>930 ± 100</td>
<td>755 ± 63</td>
<td>1600 ± 120</td>
<td>523 ± 79</td>
<td>1070 ± 110</td>
<td>644 ± 69</td>
<td>800 ± 130</td>
</tr>
<tr>
<td>( W )</td>
<td>219 ± 42</td>
<td>910 ± 100</td>
<td>804 ± 67</td>
<td>1580 ± 130</td>
<td>553 ± 79</td>
<td>1110 ± 120</td>
<td>645 ± 69</td>
<td>780 ± 130</td>
</tr>
<tr>
<td>( Q + V + W )</td>
<td>182 ± 36</td>
<td>950 ± 110</td>
<td>804 ± 70</td>
<td>1600 ± 120</td>
<td>491 ± 78</td>
<td>1090 ± 110</td>
<td>656 ± 71</td>
<td>830 ± 130</td>
</tr>
<tr>
<td>Direct inversion</td>
<td>322 ± 36</td>
<td>873 ± 96</td>
<td>1570 ± 120</td>
<td>1370 ± 110</td>
<td>560 ± 78</td>
<td>870 ± 100</td>
<td>1037 ± 99</td>
<td>950 ± 140</td>
</tr>
<tr>
<td>ILC</td>
<td>248 ± 45</td>
<td>950 ± 110</td>
<td>750 ± 61</td>
<td>1620 ± 130</td>
<td>615 ± 85</td>
<td>1120 ± 120</td>
<td>619 ± 66</td>
<td>790 ± 130</td>
</tr>
<tr>
<td>TOH</td>
<td>279 ± 49</td>
<td>870 ± 98</td>
<td>613 ± 49</td>
<td>1420 ± 110</td>
<td>586 ± 89</td>
<td>1040 ± 110</td>
<td>631 ± 63</td>
<td>860 ± 140</td>
</tr>
<tr>
<td>TOH*</td>
<td>195 ± 40</td>
<td>1050 ± 110</td>
<td>834 ± 66</td>
<td>1670 ± 130</td>
<td>606 ± 81</td>
<td>1290 ± 140</td>
<td>660 ± 68</td>
<td>700 ± 130</td>
</tr>
<tr>
<td>TOH**</td>
<td>202 ± 41</td>
<td>866 ± 99</td>
<td>651 ± 50</td>
<td>1350 ± 110</td>
<td>608 ± 90</td>
<td>1010 ± 110</td>
<td>681 ± 72</td>
<td>710 ± 120</td>
</tr>
</tbody>
</table>

Table 2. Power spectrum amplitudes \( \Delta T^2_\ell = (\ell + 1) C_\ell / 2\pi \) (in \( \mu K^2 \)) for the foreground corrected WMAP frequency maps and related combinations estimated by application of the PE filter for the Kp2 sky coverage. \( Q + V + W \) denotes the corrected co-added map and \( * \) denotes estimation on the full sky. The direct inversion method was applied to the corrected co-added map for the Kp2 mask. Uncertainties do not take into account cosmic variance.
Figure 8. Maps of the $\ell = 2, \ldots, 5$ WMAP multipoles determined from the different frequency bands.
Figure 9. The angular power spectrum of the WMAP data for $K$-band (filled circles), $Ka$-band (open circles), $Q$-band (filled diamonds), $V$-band (open diamonds) and $W$-band (triangle) maps obtained by PE filtering for the Kp2 mask. The smooth curve shows the WMAP team best-fitting $\Lambda$CDM model ($\Omega_m = 0.73$ and $\Omega_k = 0.27$) with running spectral index. The grey band indicates the cosmic variance errors (95 per cent confidence level).

Because we have such a situation for the quadrupole estimated for the cut 30, we do not quote the corresponding uncertainties in the table.

The general trend for increasing uncertainties with increasing sky cut is easily seen in Fig. 12 where linear regressions, analogous to that in Section 6, is shown for various masks. It should be noticed that uncertainties in the estimation of the multipoles $\ell = 4, 5$ is especially large for the cut 30. The true and reconstructed coefficients are nearly uncorrelated. Slightly better correlations are seen for modes with $m \neq -\ell, \ell$, though the trend is still maintained. The poor correlation between input and recovered coefficients implies that for $\ell > 5$ their estimation is no longer credible. Indeed, the large uncertainties associated with the widest cut 30 helps to explain why the appearance of the estimated WMAP multipoles is different from that estimated with narrower masks. Nevertheless, the multipole amplitudes remain statistically consistent over the range of masks at approximately the $2\sigma$ level.

8 UNCERTAINTIES OF THE MULTIPOLe Maps

One of the merits of the PE filtering method is the possibility to directly estimate the uncertainties in the derived results and the preservation of the well-defined noise properties of the original maps. This is a clear advantage of the method over the MEM applied by the WMAP team (Bennett et al. 2003b), which results in anisotropy maps with computationally complex noise properties, which are largely inappropriate for most CMB analyses. Indeed, it is generally true that the noise evaluation for linear methods are computationally more feasible than for non-linear methods such as MEM. Here, we derive errors both in pixel and spherical harmonic space.

<table>
<thead>
<tr>
<th>WMAP multipoles</th>
</tr>
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<tr>
<td>$\ell = 2$</td>
</tr>
<tr>
<td>mask Kp2</td>
</tr>
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</table>

Figure 10. Maps of the $\ell = 2, \ldots, 5$ WMAP multipoles for different masks: Kp2, Kp0, 20+ and 30.
8.1 The maps of the standard deviation

To study the spatial distribution of the uncertainties of the restored multipole maps, we use the simulations of the WMAP corrected co-added map as described in Section 4. The map

\[
\Delta^2(p)_{\text{error}} = \sum_{\ell, m} \langle a_{\ell m}^{\text{error}} a_{\ell m}^{\text{error}} \rangle Y_{\ell m}^* Y_{\ell m}(p) \tag{18}
\]

shows the spatial distribution of the squared standard deviation of the difference between the restored and true multipoles maps. \(a_{\ell m}^{\text{error}}\) are the corresponding decomposition coefficients of the difference maps and terms \(\langle a_{\ell m}^{\text{error}} a_{\ell m}^{\text{error}} \rangle\) are averaged over 10000 simulations. Using the identity

\[
Y_{\ell_1 m_1}(p)Y_{\ell_2 m_2}(p) = \sum_{\ell, m} \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell + 1)}{4\pi}} Y_{\ell\ell_1 m_1 m_2}^{\ell_1\ell_2 m_1 m_2} Y_{\ell m}(p),
\]

where \((\ell_1, \ell_2, \ell)\) is the Wigner 3j symbol, the expression (18) can be written as

\[
\Delta^2(p)_{\text{error}} = \sum_{\ell, m} b_{\ell m} Y_{\ell m}(p),
\]

where \(b_{\ell m}\) are the appropriate coefficients. The \(\Delta^2_{\text{error}}\) map is composed of modes \(0 \leq \ell \leq 2\ell_1\), because \(\ell\) obeys the triangular condition \(\ell = \ell_1 + \ell_2, \ell_1 + \ell_2 - 1, \ldots, (\ell_1 - \ell_2)\) and \(\ell_1 = \ell_2\).

The \(\Delta^2_{\text{error}}\) maps computed for the Kp2 mask are shown in Fig. 13 in logarithmic scale. As expected, as a result of the lack of data in the cut part of the sky the reconstructed multipoles have the largest uncertainties in this region, especially close to the Orion and Ophiuchus exclusion zones. The multipoles \(\ell = 2\) and 3 have the smallest uncertainties inside the cut. This is a consequence of the fact that the multipole maps characterize the CMB anisotropy on angular scales \(< 1/\ell\). If this is bigger than the angular width, \(\Delta \theta\), of the mask then the excised region of the multipole map can be easily extrapolated from the data outside of the cut. Otherwise, there will be fluctuations with typical angular size smaller than \(\Delta \theta\), which will be completely contained within the cut and their reconstruction from the data will be much less accurate. The statistical isotropy assumption used in our approach is then very helpful to reduce uncertainties in these cases. For the Kp2 mask, the widest part, cutout toward the Galactic centre, has width \(\Delta \theta \approx 40^\circ\).

The \(\Delta^2_{\text{error}}\) map corresponding to the sum of the multipoles \(\ell = 2, \ldots, 5\) shows much bigger uncertainties than for the corresponding individual multipoles. This is a result of the additional contributions from the correlations between errors in the multipoles. These correlations are easily seen in the covariance matrix \(\mathbf{M}\).

8.2 The covariance matrix

The map of the standard deviation does not show all of the information about the errors. The covariance matrix \(\mathbf{M}\) shows that the uncertainties are correlated in the spherical harmonic space.

As mentioned in Section 2, the covariance matrix, \(\mathbf{M}\), consists of three components: the noise component and, induced purely by the cut, contributions from correlations between the restored multipoles and correlations with higher order multipoles. In Fig. 14, we show that part of the covariance matrix corresponding to the multipoles \(\ell = 2, \ldots, 5\) for the PE filtering of the WMAP corrected co-added map, together with the corresponding submatrices for the separate components. The largest contribution arises from correlations with higher order modes. The correlations between the low-order reconstructed multipoles are weaker although still non-negligible. The smallest contribution comes from the noise component, which is plotted multiplied by a factor of 50 to render it visible. The approximately axisymmetric cut causes stronger correlations between modes with \(\ell\) value different by 2. Notice that the modes with the biggest reconstruction uncertainties are for \(m = -\ell\) and \(m = \ell\), which is a consequence of removing a large fraction of the power of these modes, mainly concentrated in the Galactic plane, by the cut.

9 FOREGROUND SEPARATION

In this section, we will reconsider the sensitivity of the estimated CMB multipoles to the Galactic foregrounds. To achieve this, we derive a CMB anisotropy map by the spectral decomposition of multifrequency WMAP data by making assumptions about the frequency dependence of the foreground components. The advantage...
of this method is that, unlike with the template fits of Bennett et al. (2003b), we assume nothing about the morphological distribution of the foreground emission. Thus, foregrounds that are not well traced by the existing templates, but with similar spectra, e.g. the free–free haze proposed by Finkbeiner (2003a), are taken into account. The disadvantage is that, to separate the components properly, we need to know their spectral dependence. Given that our intention is not to provide a definitive foreground separation analysis, we have considered a broad range of spectral index values to aggressively test the dependence of the reconstructed low-order multipoles on any foreground assumptions. Nevertheless, it should be recognized that our assumptions may lead to worse foreground subtraction than in the WMAP analysis, although this is unlikely given that the range of spectral properties adopted here are well motivated by previous results in the literature.

We consider a method following Bennett et al. (1992) and the formalism from Dodelson & Stebbins (1994). We suppose now that we observe the sky in \( m \) different frequency channels, so that at each point \( r \) on the sky we measure \( m \) different temperatures \( d_i(r) \), \( i = 1, \ldots, m \). Let us assume also that we know the temperature...
Maps of the standard deviation, Kp2 mask

Figure 13. The $\Delta^2_{\text{true}}$ maps (see text) corresponding to the sum of the errors for the restored $\ell = 2, \ldots, 5$ WMAP multipoles (centre) and for each multipole separately. The maps are shown in logarithmic scale.

spectra $f_j(\nu)$ of all $n$ components $x_j(r)$, $j = 1, \ldots, n$. Then we can write

$$d(r) = F \cdot x(r) + n(r),$$

where the vector $n(r)$ corresponds to the instrumental noise in the various channels and $F$ (not to be confused with the filter matrix used in Section 2) is the $m \times n$ frequency response matrix given by

$$F_{ij} \equiv \int_0^\infty w_i(\nu) f_j(\nu) d\nu,$$

$w_i(\nu)$ being the frequency response (bandwidth) of the $i$th channel. Here, we will approximate $w_i(\nu)$ by Dirac’s delta $w_i(\nu) = \delta(\nu - v_i)$ where $v_i$ corresponds to the mean frequency in the $i$th channel, so that $F_{ij} = f_j(v_i)$. If the number of frequencies, $m$, equals the number of components $n$ and noise is negligible, the separation of the components can be obtained by inverting the frequency response matrix,

$$x(r) = F^{-1} \cdot d(r).$$

For the CMB component, if temperature is expressed in thermodynamic units, $F_{ij} = 1$ over all frequencies as a result of the blackbody nature of the spectrum. For the range of frequencies of interest here, the foreground spectra may be reasonably well modelled by a power law in antenna temperature, $f_j(\nu) \propto \nu^{\beta_j}$, where the spectral index $\beta_j$ depends on the component. To express the spectral dependence in thermodynamic units, the function $f_j(\nu)$ must include the frequency dependent antenna to thermodynamic temperature conversion factor, $p(\nu)$. Thus, finally the functions take the form $f_j(\nu) = p(\nu/v_{0,j})^{\beta_j}$, where $v_{0,j}$ is the reference frequency for component $j$ at which the foreground amplitude is normalized.

For the WMAP data, sky maps are available at five frequencies: $K$, $Ka$, $Q$, $V$ and $W$ bands ($i = 1, \ldots, 5$, respectively). These may be interpreted as consisting of five components: the CMB anisotropy, Galactic thermal dust, synchrotron, free–free and anomalous dust emissions ($j = 1, \ldots, 5$, respectively).

The synchrotron, free–free and anomalous dust components will be searched at $K$-band frequency $v_{0,s} = v_{0,ff} = v_{0,a} = 23$ GHz and the thermal dust component at $W$-band frequency $v_{0,d} = 94$ GHz. It should be recognized that the Galactic emission does in general have spectral properties that depend on position on the sky (especially in the Galactic plane), however we will neglect these variations and use a fixed index over the high-latitude parts of the sky we are interested in. This is no different to the assumption that the template fits can be described by single scale factors over the same region. Even in the case of the WMAP ILC method, the high-latitude emission (as defined by the Kp2 mask) was treated as a single contiguous region over which the foregrounds were assumed to exhibit unvarying spectral behaviour. Only within the Galactic

Note that, before proceeding with the separation, these were smoothed to a common angular resolution of $1^\circ$.7

Low-order multipole maps of CMB anisotropy

Figure 14. Decomposition of the covariance matrix for the PE filtering of the WMAP data into specific contributions is shown. The total covariance matrix (upper left figure) is compared with the components induced by: correlations between restored multipoles (upper right figure), correlations of restored multipoles with higher order multipoles (bottom left figure) and noise correlations (bottom right figure). Only that part of the matrices corresponding to the range ℓ = 2, ..., 5 are shown. The noise component is multiplied by 50 to render it visible. Maximal correlation is at most ∼14 µK² and minimal down to ∼−11 µK². The modes are ordered such that index i is given by $i = ℓ^2 + ℓ + m + 1$.

For the thermal dust component the spectral index generally lies in the range 1.6 ≲ βd ≲ 2.5 (Dupac et al. 2002). Three values were selected: βd = 1.7 suggested by Finkbeiner et al. (1999); βd = 2.2, which is the mean value of the index determined from the MEM-fitting dust component derived from the WMAP data by Bennett et al. (2003b); and the upper limit of the range βd = 2.5.

The synchrotron spectral index is negative and again three values were selected: βs = −3.3, which approximately corresponds to the mean spectral index between the Ka and Q bands of the WMAP MEM-fitting synchrotron component; βs = −3.1, which is a better fit to the Galactic halo emission; and βs = −2.9 more appropriate for the Galactic plane.

The spectral dependence of the so-called anomalous dust component was studied by Banday et al. (2003). Over the frequency range 19–90 GHz, the anomalous dust has a thermal dust-like morphology but a synchrotron-like spectrum with an index βa ≈ −2.5. Unfortunately, the similarity of the latter two spectra results in the frequency response matrix becoming poorly conditioned and the clean separation of the components is not possible. To sidestep this issue, we adopted the spinning dust model spectrum proposed by (Draine & Lazarian 1998; hereafter DL98). In particular, we consider a three-component model of the spinning dust containing contributions from dust found in the cold neutral medium (CNM), warm neutral medium (WNM) or warm ionized medium (WIM). The preferred DL98 model has these components in the ratio 0.43, 0.43 and 0.14, respectively. It should be noted that the model predictions are fundamentally uncertain because changes in the grain composition can affect the turnover frequency, the spectral normalization and so forth. In addition, variations in the mixture of contributions from the three phases of the interstellar medium can affect the integrated spectrum, making our fit a very poor approximation.

We have tested the sensitivity of the final CMB multipole reconstruction to the assumptions by replacing the preferred model mix by either the CNM, WNM or WIM spectra, respectively. The results are remarkably stable to these variations, the angle between the quadrupole and octopole axes changing by less than 3°. Furthermore, de Oliveira-Costa et al. (2004b) find spectral behaviour in data from 10–90 GHz in reasonable agreement with the predictions of DL98 and we persist with these assumptions.

Nevertheless, further degeneracy in the components caused by the similarity of the free–free spectral βff ≈ −2.15 and the synchrotron indices renders it impossible to perform a robust separation of all five components. We have to eliminate the free–free or synchrotron component from the linear system. Because the former is better known (in the sense that there are less variations in the spectral index over the sky) and of lesser contribution to the foreground signal, we adopt a free–free template to describe its spatial morphology and subtract it from the five temperature maps, $d(r)$, with the correlation coefficients derived by Benetti et al. (2003b; as described in Section 3). However, a major uncertainty when using

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8 Detailed spectra are available at http://www.astro.princeton.edu/~draine/dust/dust.mwave.spin.html
the Hα template is the absorption of the Hα by foreground dust. To study the sensitivity of the lowest multipoles to the dust absorption correction, we will use two templates of the free–free emission: the Hα template given by Finkbeiner (2003a; it was used by Bennett et al. 2003b) uncorrected for dust absorption and the Hα template corrected for dust absorption given by Dickinson, Davies & Davis (2003). Unfortunately, dust may not be the only problem in removing free–free emission. Finkbeiner (2003b) has suggested that the Hα map does not trace all of the free–free emission and proposed the existence of a so-called free–free haze, which has the same spectral dependence as free–free component. The influence of the free–free haze on the estimation of the lowest multipoles will be studied using maps corrected and uncorrected for this component.

After elimination of the free–free component, an overdetermined linear system of five equations and four unknowns remains to be solved. This is performed using the singular value decomposition (SVD) method, because it gives the minimum variance solution. Then, the PE filter was applied to the derived CMB component maps. Residuals associated with poorly subtracted Galactic plane emission were circumvented by application of the Kp2 mask.

The influence of the dust absorption and free–free haze corrections on the lowest WMAP multipoles is shown in Fig. 15. The dust and synchrotron spectral indices were fixed here as $\beta_d = 2.2$ and $\beta_s = -3.1$, respectively. One can see that the quadrupole is especially sensitive to these corrections. The dust absorption correction shifts the hotter spots of the quadrupole toward the Galactic poles and the free–free haze correction shifts the spots toward the Galactic plane. Curiously, when both corrections are applied, the quadrupole remains largely unchanged. Notable changes are seen also for the $\ell = 4$ multipole, but the results for $\ell = 3$ and 5 are very stable with respect to the corrections.

The dependence of the $\ell = 2$ multipole on the thermal dust and synchrotron spectral indices is shown in Fig. 16. We do not show analogous figures for $\ell = 3, 4, 5$ because they remain unperturbed ($\ell = 3, 5$) or show minimal changes ($\ell = 4$) with respect to the spectral indices. It should be noted that an increase in the synchrotron spectral index tends to shift the hotspots of the quadrupole toward the Galactic poles and enhance its amplitude. Conversely, increasing the dust spectral index shifts the spots toward the Galactic plane and decreases their amplitude. Thus, for the flattest dust $\beta_d = 1.7$ and synchrotron $\beta_s = -2.9$ indices, the quadrupole hotspots are at the Galactic poles. Similar changes, though much smaller, are also seen for multipole $\ell = 4$.

The stability of the multipoles $\ell = 3$ and $\ell = 5$ with respect to the spectral indices, combined with the earlier results related to the removal of the free–free template, is compelling evidence in
favour of the claim that the planarity of the octopole is not caused by the Galactic foregrounds. Conversely, the strong dependence of the quadrupole amplitude and morphology on the spectral properties of the Galactic foreground emission implies that the latter must be known in some detail in order to recover the quadrupole anisotropy with some degree of precision. Indeed, this will also have important consequences for the statistical significance of its alignment with the octopole. This issue is studied in next section.

10 THE QUADRUPOLE AND OCTOPOLE PREFERRED AXES

As initially noticed by de Oliveira-Costa et al. (2004a), and seen here in Fig. 5, the quadrupole and octopole are rather planar, with most of their hot and cold spots placed on a single plane in the sky. de Oliveira-Costa et al. (2004a) use a wave function formalism to quantify a preferred axis for the multipoles and study the significance of their alignment. However, their analysis was based on the full sky TOH map, which may contain residual contamination in the Galactic plane region. We repeat the analysis here on the multipoles derived from the cut sky analysis (thus avoiding this problem), and quantify the additional statistical uncertainties introduced by the mask.

Given the covariance matrix $M$, we can estimate uncertainties of the preferred axes coordinates on the sky. For each of the masks, Kp2, Kp0 and 20+, the covariance matrix $M$ was used to perform 100 constrained realizations of the WMAP quadrupole and octopole. Then, using the same technique as de Oliveira-Costa et al. (2004a), the preferred axis for each of the multipoles was found. The uncertainties in the axis coordinates are summarized in Table 4. Notice that the multipole axes coordinates substantially vary for different masks. The most changeable is the quadrupole, particularly in latitude, whereas the octopole axis coordinates are much more robust. Table 5 indicates the frequency dependence of the preferred axis for the quadrupole and octopole. One should notice that the best foreground corrected octopole maps (those of the $Q$, $V$ and $W$ bands) are stable and prefer approximately the same axis: $(l, b) \sim (308^\circ, 63^\circ)$ with a deviation of order $2^\circ$. The quadrupole axis is more poorly determined. Significant differences between the $Q$, $V$, $W$ and $K$, $K_a$ frequency bands suggest the presence of residual foregrounds at the lowest frequencies.

As was shown in Section 9, the quadrupole of the CMB map derived by the linear projection depends significantly on templates used to eliminate the free–free emission and assumptions concerning the spectral dependence of the dust and synchrotron emission. The coordinates of the quadrupole and octopole preferred axes for variants of the free–free model emission are shown in Table 6. Again, the octopole is much more stable than the quadrupole. Correction for the putative free–free haze emission shifts the quadrupole axis toward the Galactic poles (larger latitude) and the dust absorption correction toward the Galactic plane (smaller latitude).
respectively. Significant uncertainties of estimating the spectral indices of the dust and synchrotron emission. For all studied Galactic coordinates of the quadrupole axes for various spectral indices it is $(\ell, \beta)$. The quadrupole axis, as in previous studies, is much less stable (see Table 7).

In summary of this section, it is worth emphasizing that the above results show a much better determination of the octopole and its preferred axis than for the quadrupole. Significant uncertainties of estimating the quadrupole preferred axis and its strong dependence on the details of templates used to eliminate free–free emission and foreground spectral indices must increase the low probability of the WMAP quadrupole–octopole alignment estimated by de Oliveira-Costa et al. (2004a). The importance of quantifying errors as a result of the foreground subtraction should be clear.

### 11 APPLICATION TO THE COBE-DMR DATA

As a useful cross-check, in particular against systematic effects, we investigate whether the lowest order multipole maps derived from COBE-DMR are consistent with the WMAP results. We treated the COBE-DMR data in similar way to the WMAP data. First, we corrected the maps for the Galactic foreground using the 20-μK and the Galactic emission templates as was described in Górski et al. (1996) and then co-added the 53- and 90-GHz maps using inverse-noise-variance weights. The 31.5-GHz data were not used because under such a weighting scheme the contribution of these least sensitive maps is minimal. The PE filter was constructed using the best fit to the data power spectrum parameterized by (Bond & Efstathiou 1987; Fabbi, Lucchin & Matarrese 1987)

$$C_{l} \equiv \langle a_{l,m}^{2} \rangle = \frac{4\pi}{\Gamma \left(\ell + \frac{n-3}{2}\right)} \frac{\Gamma \left(\frac{n}{2}\right)}{\Gamma \left(\ell + \frac{n}{2}\right)} \frac{\Gamma \left(\frac{3+n}{2}\right)}{\Gamma \left(\frac{3+n}{2}\right)},$$

where $\Gamma$ denotes the gamma function. The rms quadrupole normalisation $Q_{\text{rms-PS}}$ and spectral index $n$ were estimated in Górski et al. (1996). $Q_{\text{rms-PS}} = 15.3^{+2.7}_{-2.8}$ μK, $n = 1.2 \pm 0.3$, and we used the values of the parameters that maximize the likelihood function. The noise covariance matrix was estimated analogously to the WMAP matrix. $\ell_{\text{rec}}$ and $\ell_{\text{max}}$ were the same as for the WMAP PE filter.

The COBE-DMR and WMAP multipoles maps for the same 20-μK mask are compared in Fig. 17. One sees that the maps are similar. The COBE-DMR octopole appears to be planar like the WMAP octopole and prefers roughly the same axis, in the direction of $(277^\circ, 63^\circ)$. The hotspots of the COBE-DMR quadrupole are shifted more toward the Galactic poles than the spots of the WMAP quadrupole, which may reflect differences in the foreground removal.

### 12 SUMMARY

In this paper, we have re-examined the nature of the low-order multipoles derived from the WMAP data. Previous work has suggested that these modes demonstrate unusual properties, but such results have been derived from full sky analyses of maps of the CMB anisotropy derived by methods that are susceptible to residual Galactic foregrounds (Eriksen et al. 2004a), particularly in the Galactic plane. By application of a PE filter on the cut sky we circumvent these difficulties and have estimated the amplitudes of the $\ell = 2–5$ multipoles and their dependence on frequency, sky cut and foreground correction. Because the construction of the filter takes into account correlations of the lower order modes with higher order modes, it significantly improves the accuracy of multipole estimation in comparison with the direct inversion method used by Efstathiou (2003).

Our studies of the dependence on frequency of the multipole amplitudes showed that:

(i) the Kp2 mask is insufficient to remove strong foreground contamination near the Galactic plane for the $K$ and $K_a$ frequency bands, even for the foreground corrected WMAP data;

(ii) small deviations between the $\ell = 3, 4, 5$ multipole determined from the $Q, V$ and $W$ frequency bands indicates that, after foreground contamination near the Galactic plane for the $K$ and $K_a$ frequency bands, even for the foreground corrected WMAP data;

The octopole axis is also stable and does not change for different spectral indices of the dust and synchrotron emission. For all studied indices it is $(l, b) = (299^\circ, 60^\circ)$. The quadrupole axis, as in previous cases, is much less stable (see Table 7).
cleaning, these multipoles are dominated by the CMB anisotropy component outside the Kp2 mask;

(iii) the octopole preferred axis, approximately \((l, b) \sim (308^\circ, 63^\circ)\), shows deviations of order 2\(^\circ\) for the \(Q\), \(V\) and \(W\) frequency bands;

(iv) noticeable changes of the quadrupole suggest a non-negligible residual foreground emission.

That the quadrupole (and to a lesser extent \(\ell = 4\)) may be compromised by such residuals is supported by the changes observed after the application of increasing sky cuts. It also appears that the estimation of the quadrupole is sensitive to the dust absorption correction of the free–free template and free–free haze correction. This may provide yet further independent confirmation of significant contamination of the quadrupole by residual Galactic foregrounds.

Further tests of the sensitivity of the multipoles to the spectral dependence of the foreground emission have utilized a linear projection technique for foreground removal. Studies of the dependence of the multipole amplitudes and morphologies on the dust and synchrotron spectral indices showed that:

(i) multipoles \(\ell = 3, 4, 5\) are independent of the indices and do not vary with respect to them;

(ii) an increase of the synchrotron spectral index shifts hotter spots toward the Galactic poles and enhances amplitude of the quadrupole;

(iii) an increase of the dust spectral index shifts hotter spots toward the Galactic plane and decreases amplitude of the quadrupole.

In conclusion, it remains premature to make definitive statements in relation to the quadrupole. Large uncertainties of the quadrupole...
preferred axis should substantially increase the very low probabilities of the WMAP quadrupole–octopole alignment estimated by de Oliveira-Costa et al. (2004a) and other quadrupole correlations studied by Schwarz et al. (2004). To similar conclusion came also Slosar & Seljak (2004) who showed that much of the alignment between quadrupole and octopole is the result of foreground contamination. Nevertheless, estimates of $\ell = 3, 4, 5$ are quite robust and consistent with previous values. The observed anisotropy structures are also present for in COBE-DMR data and are therefore unlikely to be associated with instrumental systematic artefacts.

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APPENDIX A: COVARIANCE MATRIX

The posterior probability is a multivariate Gaussian of the form

$$P(r \mid c) \propto \exp \left(-\frac{(\mathbf{c} - \hat{\mathbf{c}})^T \cdot \mathbf{M}^{-1} \cdot (\mathbf{c} - \hat{\mathbf{c}})}{2} \right) / \sqrt{\det \mathbf{M}},$$

(A1)

which has its maximum value at the estimate $\hat{\mathbf{c}}$ of the signal vector and $\mathbf{M}$ is the covariance matrix defined by $\mathbf{M} \equiv \langle (\mathbf{c} - \hat{\mathbf{c}})(\mathbf{c} - \hat{\mathbf{c}})^T \rangle$. The diagonal terms of the covariance matrix $\mathbf{M}$ are the squared errors $\Delta c_i$ of the estimate $\hat{\mathbf{c}}$ and non-diagonal elements of the matrix represent correlations between errors. However, to study correlations a more convenient representation of the correlation matrix is used: the covariance matrix normalized by its diagonal terms, $\mathbf{M}_{ij} / \sqrt{\mathbf{M}_{ii} \mathbf{M}_{jj}} (i, j \in [5, \ldots, (\ell_{rec} + 1)^2])$. Then, the coefficient of correlation between uncertainty in $a_i$ and uncertainty in $\hat{a}_i$ is a number between $-1$ and 1.

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