On helioseismic tests of basic physics

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ABSTRACT

An important goal of helioseismology is to provide information about the basic physics and parameters that determine the structure of the solar interior. Here we discuss the procedures applied in such analyses, using as an example attempts to obtain significant constraints on the value of Newton’s gravitational constant \( G \) from helioseismology. The analysis is based on complete direct and inverse helioseismic analysis of a set of accurate observed acoustic frequencies. We confirm, as found by previous investigations based on different approaches, that the actual level of precision of the helioseismic inferences does not allow us to constrain \( G \) with a precision better than that which can be reached with direct experimental measurements. The conclusion emphasizes the importance in helioseismic inferences of considering not only the accuracy with which solar oscillations are measured, but also the effect of uncertainties in other aspects of the model computation and helioseismic analysis.

Key words: Sun: fundamental parameters – Sun: helioseismology – Sun: interior – cosmological parameters.

1 INTRODUCTION

Extremely precise measurements of the solar oscillation frequencies and advances both in the inversion techniques resulting in the seismic structure of the Sun and in the standard solar models have demonstrated that the models are in very good overall agreement with the seismic inferences (Christensen-Dalsgaard et al. 1996; Bahcall, Pinsonneault & Basu 2001). As a result, the Sun has become a very effective laboratory for stellar and fundamental physics. Some examples are investigations of details of the equation of state (e.g. Basu, Díppen & Nayfonov 1999), the emission of axions (Schlattl, Weiss & Raffelt 1999), the accumulation of weakly interacting massive particles (WIMPs; Lopes, Silk & Hansen 2002), the screening of nuclear reaction rates (Fiorentini, Ricci & Villante 2001; Weiss, Flaskamp & Tsytovich 2001), and the overshooting from the convective envelope (Schlattl & Weiss 1999). In many of these cases the postulated or badly known physical effect is included into the solar model and the thus modified model compared to the seismic structure; this leads to upper limits to the magnitude of the effect investigated, above which the modified model becomes inconsistent with helioseismology.

The effects under consideration are generally subtle. Thus it is important to consider with care the potential systematic errors that might be introduced by the helioseismic procedures. Also, it must be kept in mind whether other uncertainties in solar modelling might mask the effects of the particular aspect that is being investigated. A general understanding, e.g. obtained from a simplified analytical analysis, of how solar structure and oscillation frequencies are affected by changes to the model physics is very helpful in this process.

The Sun is – from the point of stellar evolution theory – a very particular object because so many well-defined global quantities have been measured, such as radius and luminosity, with errors usually well below the 1 per cent level. Among those, the solar mass \( M_\odot \) is comparatively ill-determined. From Kepler’s third law

\[
GM_\odot = 1.327 \times 10^{26} \text{ cm}^3 \text{ s}^{-2},
\]

\( G \) being Newton’s gravitational constant, can be determined and is known to a relative accuracy of \( \approx 10^{-7} \) (IAU 1977). Actually, the quantity

\[
k^2 = GM_\odot = 0.017202098955 \pm 0 \text{ au}^3 \text{ d}^{-2}
\]

is the primary defined constant, expressing Kepler’s third law in solar units (astronomical units and mean solar day); the error in \( G \) arises from the determination of the astronomical unit in physical length (1 au = 149 597 871 475 ± 30 m; M. Standish, private communication). However, Newton’s constant \( G \) – as measured in terrestrial laboratories – is known to only \( \approx 10^{-9} \), and thus \( M_\odot \) is similarly uncertain.

Recently, the attention of astrophysicists has been caught by a few new experimental results that lie at the extreme edges of the error range of the standard CODATA (Mhor & Taylor 1999) value,
\[ G = (6.6723 \pm 0.0015) \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}, \] used so far. Given the success of using the Sun as a laboratory, the idea of looking into ‘solar’ constraints on the value of \( G \) is obvious, and Ricci & Villante (2002) and Lopes & Silk (2003) have indeed done so, with the conclusion that the difference between the various experimental values for \( G \) is too small to be detectable at present by the standard ‘solar laboratory’ approach. We note, however, that Demarque et al. (1994) and Guenther, Krauss & Demarque (1998) used helioseismology to constrain the possible variation with time of \( G \).

The goal of the present paper is to investigate in some detail the steps involved in obtaining helioseismic constraints on solar interior physics, by considering the potential for constraining \( G \) based on helioseismic inferences. To do so, in Section 2 we consider how the helioseismic analysis is applied to test the properties of these models. In particular, we note that the use of a ‘seismic solar model’, while a good indication, at least in principle might involve systematic errors of some significance to the very subtle effects that are being investigated here. In Section 3 we analyse the effect of assuming different values of \( G \) in computing the solar model. In Section 4 we present results of computations of models differing only in the assumed value of \( G \), and carry out helioseismic analysis of a set of accurate observed acoustic frequencies, using these models as a reference, with the purpose of trying to discriminate among them. A discussion of the results, and the conclusions of the present study, are presented in Sections 5 and 6.

### 2 HELIOSEISMIC APPROACH

As is common in helioseismic analyses, we consider the oscillations as being adiabatic. We introduce dimensionless homology-scaled structure variables \((q, x, \hat{P}, \hat{\rho}, \hat{c})\) by

\[
\begin{aligned}
q &= Mq, & r &= Rx, \\
P &= \frac{GM^2}{R^2} \hat{P}, & \hat{P} &= M \hat{P}, \\
\hat{c} &= \frac{GM}{R} \hat{c}. 
\end{aligned}
\]

Here \( r \) is distance to the centre, \( m \) is mass within a sphere of radius \( r \), \( P \) is the pressure, \( \rho \) is the density, \( R \) is the total solar model radius, \( M \) is the total solar model mass, and \( c \) is the sound speed.

In the adiabatic case, it is easy to show that the oscillation equations can be expressed solely in terms of these homology-scaled structure variables and the adiabatic exponent \( \Gamma_1 \), the oscillation frequency being expressed in terms of the dimensionless frequency \( \sigma \), defined by

\[
\omega^2 = \frac{GM}{R^3} \sigma^2,
\]

where \( \omega \) is the dimensional angular frequency. Thus, to the extent that homology is satisfied and \( GM \) is fixed, the frequencies have no direct dependence on \( G \); this underscores the fact that constraining \( G \) is a subtle process.

#### 2.1 Inversion procedure

An inversion for solar structure is usually based on the linearization of the equations of stellar oscillations around a known reference model, under the assumption of hydrostatic equilibrium. This results in integral equations that can be analysed by means of an inverse procedure to determine the corrections that have to be imposed on the reference model in order to obtain the observed oscillation frequencies. Obviously, the results of the inversion depend both on the mode selection \( i \equiv (n, l) \) of radial order \( n \) and harmonic degree \( l \) and on the observational errors that characterize the mode set \( (i = 1, \ldots, N) \) to be inverted.

The above discussion suggests that we express the observed frequencies in dimensionless form, according to equation (2), and consider differences \( \delta \sigma_i = \sigma_i^{\text{obs}} - \sigma_i^{\text{mod}} \) between the observed and the model values. The relative differences \( \delta \sigma_i / \sigma_i \) between the frequencies of the Sun and the model are related to the differences at fixed \( r \) \((\delta_r \hat{c}^2 / \hat{c}^2, \delta \hat{\rho} / \hat{\rho})\) in scaled sound speed \( \hat{c} \) and density \( \hat{\rho} \) between the structure of the Sun and the reference model through the following integral equation (e.g. Dziembowski, Pamyatnykh & Sienkiewicz 1990):

\[
\frac{\delta \sigma_i}{\sigma_i} = \int_0^1 K_{\hat{c}, \sigma}^i(x) \frac{\delta \hat{c}^2}{\hat{c}^2}(x) \, dx \\
+ \int_0^1 K_{\hat{\rho}, \sigma}^i(x) \frac{\delta \hat{\rho}}{\hat{\rho}}(x) \, dx + E_i \hat{F}_{\text{surf}} + \epsilon_i, 
\]

(3)

where \( \epsilon_i \) are the observational errors in the data, which are assumed to be independent and Gaussian-distributed with zero mean and variances \( \sigma_i^2 \). Furthermore, since the mass and radius have been taken out in the homology scaling, the density difference obviously satisfies

\[
0 = \int_0^1 x^2 \frac{\delta \hat{\rho}}{\hat{\rho}} \, dx,
\]

(4)

which is formally similar to equations (3), with a vanishing ‘frequency difference’. The kernels \( K_{\hat{c}, \sigma}^i \) and \( K_{\hat{\rho}, \sigma}^i \) are calculated, for each mode \( i \), from the equilibrium model quantities and oscillation eigenfunctions; as for the dimensionless oscillation equations, the kernels do not depend explicitly on \( G, M \) and \( R \). Non-adiabatic effects, and other errors in modelling the surface layers that can give rise to frequency shifts, have been taken into account by including an arbitrary function of frequency \( F_{\text{surf}} \) normalized by the mode inertia \( E_i \) in the variational formulation, as suggested by Dziembowski et al. (1990).

Equation (3), which forms the basis for the linearized structure inversion problem, involves three unknown functions: \( \delta_r \hat{c}^2 / \hat{c}^2, \delta \hat{\rho} / \hat{\rho} \) and \( F_{\text{surf}} \). However, the number of unknown functions can be reduced to one by adopting the method of optimally localized averages.

This method aims at solving equation (3), together with the constraint in equation (4), by estimating a localized weighted average of the unknown quantity, \( \delta_r \hat{c}^2 / \hat{c}^2 \) or \( \delta \hat{\rho} / \hat{\rho} \), at selected target radii \( x_0 \), by means of linear combinations of the data \( \delta \sigma_i / \sigma_i \) with coefficients \( \alpha_i(x_0) \), chosen so as to localize the solution while suppressing the contributions from the other terms. The solutions are characterized by the so-called averaging kernel and by the cross-term kernel, which in the case of the sound-speed inversion are expressed, respectively, by

\[
K_{\hat{c}, \sigma}^i(x_0, x) = \sum_{i=1}^N \alpha_i(x_0) K_{\hat{c}, \sigma}^i(x) 
\]

(5)

and

\[
K_{\hat{\rho}, \sigma}^i(x_0, x) = \sum_{i=1}^N \alpha_i(x_0) K_{\hat{\rho}, \sigma}^i(x). 
\]

(6)

The cross-term kernel measures the influence of the contribution from \( \delta \hat{\rho} / \hat{\rho} \) to the inferred \( \delta_r \hat{c}^2 / \hat{c}^2 \). Analogous relations can be obtained for the density inversion, so that the averaging kernel is expressed as a linear combination of the mode kernels \( K_{\hat{c}, \sigma}^i \), while
the cross-term, which represents the contribution of the sound speed to the inferred density, is given as a linear combination of the $K^\delta_{r,\rho}$.

We have solved equation (3) using the subtractive optimally localized averages (SOLA) method (Pijpers & Thompson 1992, 1994). Details on the inversion method were provided by Rabello-Soares, Basu & Christensen-Dalsgaard (1999).

2.2 Seismic solar model

As a result of the inversion procedure, we obtain estimates of the differences in $r^2$ and $\bar{r}$ between the Sun and the model:

$$\frac{\delta r^2}{r^2}(x) = \int_0^1 K^\delta r_{\rho}(x, x) \frac{\delta r^2}{r^2}(x) \, dx$$

(7)

and

$$\frac{\delta \bar{r}}{\bar{r}}(x) = \int_0^1 K^\delta \rho_{\rho}(x, x) \frac{\delta \bar{r}}{\bar{r}}(x) \, dx.$$  

(8)

From these, the solar internal density and sound speed can be estimated as

$$c^2_{int}(r) = c^2_{int}(r) \left[ 1 + K^\delta r_{\rho}(r/R) \right]$$

(9)

and

$$\rho_{int}(r) = \rho_{int}(r) \left[ 1 - \frac{G}{\rho} + \left( \frac{\delta G}{\rho} \right) (r/R) \right]$$

(10)

where $c^2_{int}(r)$ and $\rho_{int}(r)$ define the structure of the reference model; as usual, we have assumed that $GM$ and $R$ are fixed. It is common practice to regard $c^2_{int}(r)$ and $\rho_{int}(r)$ as defining a seismic solar model and, for example, to test aspects of model modifications by comparing the sound speed of the model with $c^2_{int}(r)$, as was done, for example, by Lopes & Silk (2003) (see also e.g. Neuforge-Verheecke et al. 2001; Turck-Chièze et al. 2001; Winnick et al. 2002).

It should be noted, however, that the interpretation in terms of a seismic solar model requires some care. As is evident from equations (7)–(10), what is determined from the inversion unavoidably involves averages over averaging kernels of finite extent. Thus a comparison of a solar model with the resulting $c^2_{int}(r)$, neglecting the finite resolution, results in possible systematic errors. This is the reason why, in our analysis, we carry out inversions using each of the modified models as reference. Although this again involves averages, these at least are well-defined averages of the differences between the solar and the model structure. In this way we obtain a consistent estimate of the effects of the model modification under investigation.

3 EFFECTS ON SOLAR MODELS OF VARYING G

In Section 4 we present results based on computing models with different values of $G$, corresponding to the range of recent measurements. However, it is instructive to carry out a simplified analysis of the expected effects.

The effects of changing $G$ can be roughly estimated from homology arguments (e.g. Kippenhahn & Weigert 1990). There are various statements in the literature concerning the dependence of the stellar structure equations on $G$. As also noted by Ricci & Villante (2002), these effects are described most clearly in terms of homology-scaled variables, already introduced in equation (1). These already indicate that, according to strict homology, a change in $G$ that leaves $GM$ unchanged will have no effect on the sound speed. In addition, we introduce the homology-scaled variable $\tilde{T}$ by

$$T = \frac{GMm_a}{k_B \tilde{T}}. \quad (11)$$

Here $T$ is the temperature, $m_a$ is the atomic mass unit, and $k_B$ is Boltzmann’s constant. $\tilde{T}$ is introduced such that $T \approx \tilde{T}/\bar{\rho}$, where $\mu$ is the mean molecular weight, if the equation of state is approximated by the ideal gas law.

Then the two mechanical equations of stellar structure become

$$\frac{\partial x}{\partial q} = \frac{1}{4\pi x^2}$$

(12)

and

$$\frac{\partial \bar{p}}{\partial q} = \frac{-q}{4\pi x^3},$$

(13)

where obviously $G$ and $M$ do not appear explicitly. The energy transport equation can be written as

$$\frac{\partial \tilde{T}}{\partial q} = \frac{-q \tilde{T}}{4\pi x^3} \nabla.$$  

(14)

Here, except in a very thin near-surface boundary layer, $\nabla = \ln T/\ln P$ is approximately adiabatic in the convection zone, $\nabla \approx \nabla_{ad} = (\partial \ln T/\partial \ln P)_{ad}$. In radiative regions,

$$\nabla = \nabla_{rad} = \frac{3k \nu L}{16\pi a c T^2} \left( \frac{1}{Gm_a} \right).$$

(15)

Here $\kappa$ is the opacity, $a$ is the radiation-density constant, $\tilde{c}$ is the speed of light and $L$ is the local luminosity, which obviously is determined by the energy generation rate $\epsilon$.

Analysis of these thermal equations is complicated by the complex dependence of $\kappa$ and $\epsilon$ on the physical conditions. However, a rough idea about the behaviour can be obtained by assuming power-law approximations to the opacity $\kappa \approx \kappa_0 \nu^\delta$ and energy generation, for which the pp-chain approximation $\epsilon \sim \rho T^4$ is used. Teller (1948) took $\delta = 1$ and $\nu = 3$; this leads to a scaling for luminosity as $L \sim G^3 M^4 \nu^4$, involving a strong dependence of luminosity on $G$. The dependence on the molecular weight has been added by us following Kippenhahn & Weigert (1990). Degl’Innocenti et al. (1996), using $\delta = 0.5$ and $\nu = 1.75$ for the opacity scaling, obtained $L \sim G^3 M^4 \nu^2$ for constant $\mu$. This is indeed confirmed by their numerical results, yielding $L \sim G^{5.6} M^{1.7}$; however, the fact that in the present case the product $GM$ is taken to be constant already implies a much weaker sensitivity to changes in $G$ in the case of the Sun.

Expressing $\nabla_{rad}$ in terms of homology-scaled variables and using the ideal gas approximation for $\tilde{T}$, we obtain

$$\nabla_{rad} = \frac{3k_0 R^{1-\beta}}{16\pi a c (GM)^{1+\beta}} \left( \frac{k_B}{m_a} \right)^{4+\nu} \times \frac{L}{\mu^{4+\nu} (G)^{1+\beta}}.$$  

(16)

The surface boundary conditions are often defined by determining the photospheric pressure, at the point where $T = T_{eff}$, the effective temperature, from a more or less sophisticated atmospheric model. In the present case where $GM$, $L$ and $R$ are fixed, the atmospheric solution is also essentially fixed, apart from the effect of the change in composition required to calibrate the model. It follows that the dimensional photospheric pressure $P_{ph}$ will change little with changing $G$; consequently, the homology-scaled photospheric pressure is expected to change approximately proportional to $G$. 

From this analysis, it is evident that the effects on the seismic properties of solar models due to a change in $G$ at fixed $GM$ are subtle. The change in density would cause a change in the rate of nuclear reactions and hence in the present composition profile of the Sun, essentially reflecting the fact that simple homology provides an incomplete description of the changes to the model. Also, the recalibration to obtain the proper luminosity changes the initial composition. To leading order, there is no effect on the sound speed on which the frequencies principally depend.

The solar surface luminosity, $L_\odot$, has to be independent of the numerical value for $G$. The composition is calibrated such that the surface luminosity is unchanged. This is achieved by modifying the initial helium abundance, and hence the molecular weight. Obviously a relative change in $G$ of the order of $10^{-3}$ will result in a relative change in $\mu$ of the same order. Since the temperature does not change, to leading order, we expect that the internal distribution $L_r$ of luminosity is also approximately unchanged. According to equation (16) this is satisfied if the composition is modified to keep $\mu^{1+\omega}G^{1+\delta}$ fixed. (A more complete description would take into account also the changes in the energy generation rate; however, because of the strong dependence of $\epsilon$ on $T$, this has little effect on the result.) Assuming this calibration, the homology-scaled equations are independent of the value of $G$, and so, therefore, is the expected scaled structure. Thus the sound speed would be unaffected, whereas according to equation (1) the density would change proportionally to $M \propto G^{-1}$.

However, homology is not the whole story, since the variation in the composition also affects the sound speed. In particular, the change in the envelope composition modifies the sound speed near the surface, due to the change in $\Gamma_1$. Anyway, these are second-order effects only. Since sound speed is the property to which the oscillation frequencies are most sensitive, this evidently makes a helioseismic test of $G$ far less sensitive. Nevertheless, we note that helioseismic inversion can also provide inferences of the density, which, according to the homology argument, scales as $M$ and hence would be directly sensitive to a change in $G$.

To make these statements more precise, we note that helioseismic analyses are typically carried out in terms of differences, denoted by $\delta$, between the Sun and the model at fixed $r$ (which, given that $R$ is kept the same, corresponds to differences at fixed $x$). At constant $GM$ it obviously follows from equations (1) that

$$\frac{\delta c^2}{c^2} = \delta \frac{c^2}{c^2},$$

$$\frac{\delta P}{P} = -\delta \frac{G}{G},$$

$$\frac{\delta \rho}{\rho} = \delta \frac{\rho}{\rho} = -\delta \frac{G}{G}.$$  \hspace{1cm} (17)

Under strict homology the scaled variables are unaffected by the change; hence $\delta \frac{c^2}{c^2} = 0$ and $\delta \frac{P}{P} = \delta \frac{\rho}{\rho} = -\delta \frac{G}{G}$.

In fact, for changes to the solar structure to be completely described in terms of homology, one must assume that the physical conditions are of the same form throughout the star, with radiative transport controlling the temperature gradient. This is manifestly not satisfied in the convection zone. Here, on the other hand, the properties of the model are simplified by the fact that the stratification is essentially adiabatic, with an adiabatic gradient that may be assumed to be roughly constant in most of the convection zone. As discussed in Appendix A, this leads to a relatively simple description of the model changes in the convection zone. It is found that, as also predicted by simple homology, the sound speed is unaffected by the change of $G$. The change in $\rho$ can be written as

$$\frac{\delta \rho}{\rho} \simeq -\frac{\delta G}{G} - C \frac{\delta r_{cz}}{H_{r_{cz}}},$$  \hspace{1cm} (18)

where $\delta r_{cz}$ is the change of the radius of the base of the convection zone, $H_{r_{cz}}$ is the pressure scaleheight at this point, and $C$ is a constant of the order of unity. Here the first term is as predicted from homology; however, our numerical calculations (see Section 4.1) show that the second term is of similar magnitude.

4 RESULTS

4.1 Solar models with varying $G$

A comprehensive list of recent estimations of the value of Newton’s gravitational constant such as obtained by laboratory experiments was provided by Lopes & Silk (2003). In particular, the three experiments by Luo et al. (1999), Gundlach & Merkowitz (2000) and Quinn et al. (2001) revived interest in $G$, since they yielded values differing (relatively) by $-0.0004, 0.0004$ and $0.0005$ from the CODATA98 recommendation (Mhor & Taylor 1999), and in particular do not lie within the mutual error bars. They are within the large CODATA98 error range of $\pm 1.5 \times 10^{-3}$, which is, however, only the consequence of including the Michaelis, Haars & Augustin (1996) result of $G = 6.715 \times 10^{-8} \text{cm}^3 \text{g}^{-1} \text{s}^{-2}$, which deviates from the CODATA value by as much as 0.65 per cent, with a quoted uncertainty of only 8.3 $\times 10^{-5}$.

For the purpose of this investigation, we have assumed the reference values $(GM)_{\odot} = 1.32712 \times 10^{30} \text{cm}^2 \text{s}^{-2}$ and the gravitational constant $G_0 = 6.6723 \times 10^{-8} \text{cm}^3 \text{g}^{-1} \text{s}^{-2}$. Consequently, the mass of the Sun has been assumed to be $M_\odot = 1.989 \times 10^{33} \text{g}$.

Different, fully calibrated solar models were calculated using several values for Newton’s gravitational constant $G$ (denoted as $G'$), while $G' M' = G_0 M_\odot$ has been kept constant. As a consequence, the mass $M'$ differs for each model considered. While Lopes & Silk (2003) considered only the cases with different measured values for $G$, Ricci & Villante (2002) used values of $G' = G_0(1 \pm 0.01)$, which are extreme variations in $G_0$. Given the various determinations of Newton’s constant, we followed Ricci & Villante (2002) and applied typical modifications. In particular, we used $G' = G_0(1 \pm 0.001)$, i.e. an order of magnitude smaller variation, consistent with the differences found in the most recent experiments, and one extreme case of $G' = 1.0065 G_0$ representing the Michaelis et al. (1996) result as the most extreme experimental case. All models were calibrated, to within a relative error of $2 \times 10^{-6}$, to a photospheric radius of $R_\odot = 6.95508 \times 10^{10} \text{cm}$, and a surface luminosity of $L_\odot = 3.84600 \times 10^{33} \text{erg s}^{-1}$ at an age of 4.57 Gyr.

As our standard solar model, we used the latest Garching Solar Model (GARSOM5), a follow-up model of GARSOM4, which was described by Schlattl (1999). It agrees well with those by Christensen-Dalsgaard et al. (1996) and Bahcall, Basu & Pinsonneault (1998) and also employs the standard physical input, which is that of most standard solar models at this epoch. In particular, particle diffusion for hydrogen, helium and metals is fully taken into account. Inversion procedures have shown an overall excellent agreement of order 0.002 between the sound-speed profiles of the Sun and that of this model, with a major discrepancy in the tachocline region between 0.6 $\lesssim r/R_\odot \lesssim 0.7$, where the relative difference in squared sound speed $c^2$ reaches 5 $\times 10^{-3}$, outside the conservative error range of the seismic model.

GARSOM5 differs from the previous one only in that the solar models were started from the zero-age instead of the pre-main
sequence, and the latest version of the OPAL equation of state (EOS; Rogers, Swenson & Iglesias 1996) of 2001 has been implemented. The differences as compared to the 1996 version of the EOS are the improvement of the activity expansion method for repulsive interactions (Rogers 2001), a relativistic treatment of electrons, and the inclusion of HeH$^+$ and He$^2+$ in addition to H$_2$, H$^+_2$ and H$^-$. GAR-SOM5 in fact shows a slightly degraded agreement with the seismic results of the 1996 version of the EOS (Rogers, Swenson & Iglesias 1996) of 2001 has been implemented. Table 1 contains initial and final solar quantities for this and all other models used in this paper. Note that $r_{CM}$ is the mixing length parameter used in the Canuto–Mazzitelli convection theory (Canuto & Mazzitelli 1992).

The computed changes in the models, resulting from varying G, are illustrated in Fig. 1, which shows the relative differences in squared sound speed and density between the GARSOM5 models with different $G'$ values and the model that assumes standard $G_0$.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Influence of modified $G$ on solar model sound speed. In the upper panel are shown the relative differences in squared sound speed, in the sense (modified model) minus (standard model) for two values of G (dashed, $G' = 1.001 G_0$; solid, $G' = 0.999 G_0$). The lower panel shows the changes in density relative to the standard model.

### 4.2 Comparison of frequency differences

A direct way to test a solar model is to consider differences between observed frequencies and those calculated for the theoretical model. Since $G'$ is affecting mostly the sound speed in the central parts of the Sun (see Fig. 1), it is interesting to consider predictions of the frequencies of low-degree acoustic modes which are able to penetrate towards the centre. The present analysis has been carried out by using the oscillation data set published by Basu et al. (1997), which includes modes with $l < 100$ and in particular very accurate frequencies of low harmonic degree, which allow us to resolve the solar core. We used the adiabatic oscillation code of Christensen-Dalsgaard (see Christensen-Dalsgaard & Berthomieu 1991) to calculate the p-mode theoretical eigenfrequencies for each model.

The frequency differences between observed values and those computed for the solar models are shown in Fig. 2 for modes with $l < 6$. The differences are very similar for models with $G' = G_0$ and $G' = 0.999 G_0$. The differences in frequencies appear to be

![Figure 2](https://example.com/figure2.png)

**Figure 2.** Relative differences between the observed frequencies of Basu et al. (1997), for $l < 6$, and theoretical frequencies for several models considered, plotted as a function of frequency; the differences are shown in the sense (Sun) minus (model).
respectively: by two parameters, known as large and small frequency separations, calculated for several models with different modified $G'$ values (indicated by different symbols), plotted as a function of frequency. Left panel: for the data set by Basu et al. (1997). Right panel: for the most recent data from the BiSON network (Chaplin et al. 2002).

An interesting way to conduct such seismic analysis is to consider the asymptotic properties of the oscillation modes. Theory predicts that the cyclic frequencies $\nu_{n,l}$ are characterized by radial order $n$, at low harmonic degree $l$ should satisfy a simple asymptotic approximation (Vandakurov 1967; Tassoul 1980). According to this, the oscillation spectrum is characterized by two parameters, known as large and small frequency separations, respectively:

$$\Delta_l(\nu) := \nu_{n,l} - \nu_{n-1,l}, \quad (19)$$

$$\delta_l(\nu) := \nu_{n,l} - \nu_{n-1,l+2}, \quad (20)$$

In particular, the small frequency separation is sensitive to the chemical composition gradient in central regions of the star and hence to its evolutionary state (e.g. Gough 1983). Fig. 3 shows the differences between the small frequency separations obtained for the observed set of frequencies and the theoretical frequencies calculated for the models that use different values of $G'$. The right panel, based on the most recent observations obtained by the BiSON network, provides more details at lower frequencies. The differences are very similar for the models with $G' = G_0$ and with $G' = 0.999 G_0$, while slightly smaller differences, in absolute value, are obtained for the model with $G' = 1.0065 G_0$, particularly at low frequencies. However, it is evident that no clear conclusion can be obtained from these results, concerning the most appropriate value of $G$.

### 4.3 Inversion for the sound speed and the density

To obtain a more detailed test of the models we have carried out inversion for sound speed and density, as discussed in Section 2. Figs 4 and 5 show the inferred relative squared sound-speed and density differences, as functions of the fractional radius, between the Sun and three solar models used as reference models in the inversion. These were computed as the GARSON5 model, but characterized by different values of $G'$. The vertical error bars correspond to the standard deviations based on the errors in the mode sets, whereas the horizontal bars give a measure of the localization of the solution.

The inversion results for the sound speed indicate that the differences between the solar models and the Sun are extremely small, except below the base of the convection zone (0.71 $R_\odot$), where mixing beyond the convection zone might reduce the composition gradients established by settling (e.g. Elliott & Gough 1999; Brun, Turck-Chièze & Zahn 1999). The results obtained with the use of the models with the standard $G' = G_0$ and with $G' = 0.999 G_0$ are similar at all radii (we do not show the symmetric case $G' = 1.001 G_0$). This indicates that with the present uncertainty with which oscillation frequencies are measured, helioseismic inversions do not allow us to distinguish between similar models that differ only by 0.1 per cent in the value of $G'$. Moreover, the sound-speed inversion results for the extreme case of a model with $G' = 1.0065 G_0$ visibly deviate...
to the non-homologous part of fixed mass in equation (4). Thus, the inversion is only sensitive is also implied by the fact that the inversion imposes the constraint some subtlety. It follows from the discussion after equation (2) that the adiabatic oscillation frequencies are determined by the scaled density difference contains a term in \( \delta_{G} \). On the other hand, for \( G' = 0.999 G_{0} \) and \( G' = 0.999 G_{0} \) are consistent within the errors at all radii; thus the helioseismic inversion for the density hardly allows us to judge the relative correctness of the two models at this level of change in \( G \). On the other hand, for \( G' = 1.0065 G_{0} \) the relative density differences between the model and the Sun are increased by about 30 per cent relative to the standard case.

5 DISCUSSION

The great precision of the helioseismic inferences of the solar interior structure provides the possibility of investigating physical effects or values of parameters that affect solar structure. The analysis of any given physical effect is usually carried out through the computation of solar models, which are compared with the helioseismic inferences. This is often done by means of ‘seismic solar models’, obtained by applying helioseismically determined corrections to a reference model; the resulting profile of, for example, sound speed is then compared with sound-speed profiles in the models under investigation. Although this is a simple and efficient technique, it may suffer from systematic errors introduced by the finite resolution of the inversion.

Also, there might be a risk that the resulting seismic model may depend on the reference model assumed in the inversion. It was demonstrated by Basu, Pinsonneault & Bahcall (2000) that the inferred structure was relatively insensitive to the reference model, even when they assumed substantially different models, e.g. neglecting diffusion. They concluded that the contribution of the reference model, the inversion method, and that of the oscillation measurement errors overall contributed equal amounts of 0.02–0.04 per cent to the total rms uncertainty of the sound-speed profile. Degl’Innocenti et al. (1997) arrived at a similar conclusion, but emphasized that outside the energy-producing core the first two factors were more important. However, close to the centre the frequency errors were dominant. This is also where the variations in \( G \) have the largest effect (Fig. 1).

In our application of the ‘solar physics laboratory’ we have used the more cumbersome, but more consistent, method of using each model as a reference in the inversion. In this way we obtain a well-defined average of the difference between the Sun and that particular model, thus providing a measure of the physical effects being investigated.

Given the intrinsic uncertainty in laboratory determinations of the gravitational constant \( G \), and recently obtained values that differ substantially from each other, it is natural to attempt to constrain the value of \( G \) through helioseismic analyses. However, both our homology analysis and the numerical results show that the potential helioseismic effects of changing \( G \) are subtle. In particular, the modifications to the sound speed, on which the oscillation frequencies predominantly depend, are at most 20 per cent of the change in \( G \) and concentrated in the core of the model. They result predominantly from indirect effects arising from the change in composition in the core, in part related to the changed initial composition required to obtain the correct solar luminosity at the present age. There are somewhat larger non-homologous changes in the density, of a magnitude similar to the change in \( G \). In the extreme case of Michaelis et al. (1996), who proposed a change in \( G \) of 0.65 per cent relative to the normally accepted value, this has a formally significant effect on the helioseismically inferred density, leading generally to substantially worse agreement between the model and the Sun. Smaller, and perhaps more reasonable, changes in \( G \) have noticeable, but barely significant, effects on solar structure, at the present level of precision of helioseismic inferences. This was also concluded by Ricci & Villante (2002) and Lopes & Silk (2003). A fundamental assumption in these studies is that \( G \) does not vary with time. It is interesting to recall that Guenther et al. (1998) used analysis of solar oscillation frequencies to constrain the temporal variation in \( G \), at a level of precision highly significant compared with other tests.

The helioseismic effects being subtle, one has to be concerned about the numerical precision of the solar models. While the difference between standard solar and seismic models is of the \( 10^{-5} \) level (Fig. 4), that between models with different \( G \) values is one order of magnitude smaller (Fig. 1). During our calculations it turned out that the precision of the GARSOM models did not suffice to separate numerical effects completely from those arising from variations in \( G \) of the magnitude considered. We therefore had to improve our models in that respect, most importantly by ensuring identical spatial and temporal resolution. The models have been tested by comparing them with those calculated by the solar model code of Christensen-Dalsgaard (Christensen-Dalsgaard et al. 1996), which intrinsically is more accurate than the Garching code, which still is a general stellar evolution code.

A probably more important concern is the question of how the small effect of changing \( G \) compares with other uncertainties in the constituting physics of solar models. We do not consider strong effects like the absence of diffusion, ad hoc mixing or strongly modified electron screening. Such models were already investigated by Basu et al. (2000), who called them ‘deficient models’ because they clearly disagreed with the seismic constraints. We rather concentrate on subtle effects. Basu et al. (2000) also investigated such cases (‘variant models’) and found that, for example, the inclusion of the pre-main-sequence phase increases the rms deviation from
the seismic sound-speed profile from $6.9 \times 10^{-4}$ to $8.5 \times 10^{-4}$, where the largest contribution comes from the inner 30 per cent of the radius. This is already an effect comparable to or even larger than the one shown in Fig. 1. The inclusion of rotation to such an extent as to solve the solar lithium problem (Richard et al. 1996) leads to similar variations. If the Sun initially had a mass larger by 1 to 7 per cent, Sackmann & Boothroyd (2003) found that, depending on the time dependence of mass loss, changes in sound speed of the order of the deviation of the standard solar model might occur, improving in fact the agreement with the seismic sound-speed profile. Relativistic effects in the equation of state change the density profile in the centre by 0.1 per cent, which is an order of magnitude larger than the change visible in Fig. 1, lower panel. This list could be continued (see Basu et al. 2000), but it is already evident that effects of various reasonable uncertainties in the solar model are larger than the small effect due the allowed range of $G$. Finally, even the uncertainty in the empirical solar parameters $R_{\odot}$ and $L_{\odot}$ is still large enough to result in rms variations of the sound-speed profile of the order of a few times $10^{-4}$ (Boothroyd & Sackmann 2003); also, the recent potential revision of the solar surface abundances (e.g. Asplund et al. 2004) has a substantial effect on the comparison between helioseismic inferences and solar models (Basu & Antia 2004). In fact, a thorough investigation of the total error budget in the solar model, as was done by Bahcall & Ulrich (1988), is indicated, but is beyond the scope of this paper.

6 CONCLUSIONS

The extremely accurate and extensive data from observations of solar oscillations provide unique constraints on solar internal structure. From the dependence of the oscillation frequencies on the structure, it follows that the immediately accessible properties are the ‘mechanical’ structure of the Sun, as characterized by, for example, the sound speed and density as a function of distance to the centre. However, of greater interest is obviously to constrain the underlying physical properties and processes that lead to that structure, including also the physical constants on which these processes depend. In such analyses all possible uncertainties affecting the results must in principle be taken into account. Thus, without further assumptions or constraints, we should not expect to be able to obtain as strict a constraint on any given quantity as apparently warranted by the precision of the data. Also, it is obviously important to understand the relation between the effect under consideration and the frequencies.

Newton’s gravitational constant $G$, which enters into the calculation of stellar models, is amongst the most uncertain of the fundamental constants; thus it is of interest to investigate whether significant constraints on its value can be obtained from helioseismic analysis. To contribute to this investigation, motivated also by recently claimed rather discrepant experimental values of $G$, we have considered the effects of changes to $G$ on solar models and carried out helioseismic tests of these models. As an obvious constraint on these calculations, we have kept fixed the product $GM$, known with extreme accuracy from planetary motion, and furthermore fixed the solar radius and luminosity. Homology arguments, as well as analysis of the response of the convection zone to the changes, show that to a first approximation the sound speed is unaffected by the change to $G$. Also, to the extent that the changes to the models follow homology scaling, the adiabatic oscillation frequencies are not affected.

As discussed above, physical inferences based on helioseismology must take into account other potential uncertainties in the modelling. In the case of the temporal variation in $G$, this was noted by Guenther et al. (1998) who, however, argued that any such effect is unlikely to have a significant influence on the constraint that they obtained. In the present case of testing the value of $G$, the effects of changes to $G$ are small compared with, and cannot in any obvious way be separated from, effects of other uncertainties in the assumed physics of the solar interior, such as nuclear reaction parameters or opacities. In Section 5 we have listed a number of such effects and their individual influence on the accuracy of the solar model. We concluded that many of them by far exceed the influence of $G$ as discussed in this paper. Therefore, the solar calibration and the present solar model are clearly not accurate enough to allow a determination of $G$ better than laboratory experiments can do. On the other hand, it is also true that the standard solar model is in good agreement with the seismic Sun (Basu et al. 2000), although with potentially larger discrepancies induced by the revision of the solar surface abundance (Asplund et al. 2004; Basu & Antia 2004). Thus, we still found it interesting to consider the effect of $G$ only, keeping all other model parameters constant.

As noted by, for example, Lopes & Silk (2003), further constraints on the structure of the solar core are provided by the measurements of neutrino fluxes, but only to the extent that the details of neutrino oscillations are well understood. Needless to say, we may expect improvements in the observational oscillation-frequency and neutrino constraints, as well as likely further independent improvements in our knowledge of aspects of solar internal physics. In this way, the parameters controlling solar modelling, including the gravitational constant, will undoubtedly become better known; however, it is not obvious that the dominant improvement in the determination of $G$ will arise from the solar data rather than from improved direct experimental measurements.

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**APPENDIX A: CHANGES IN THE CONVECTIVE ENVELOPE**

The homology scalings applied in Section 3 assume that the solar interior is described by a uniform set of physics. This is manifestly not the case in the outer convection zone, whose structure may therefore be expected to change in a non-homologous manner. In this appendix we discuss some aspects of these changes; the analysis is based on a similar analysis by Christensen-Dalsgaard (1997).

The bulk of the convective envelope can be characterized by an adiabatic relation between pressure and density, which we write as $P = K \rho^\gamma$, assuming that $K$ and $\gamma$ are constants. This is approximately satisfied below the ionization zones of hydrogen and helium, for $r \lesssim 0.97 R$. From this relation, and the equation of hydrostatic support, it follows approximately that

$$GM \left( \frac{1}{r} - \frac{1}{R} \right) \simeq \frac{\gamma P}{\gamma - 1} = \frac{\gamma}{\gamma - 1} K^{1/\gamma} P^{1-1/\gamma},$$

where furthermore we have neglected the mass in the convective envelope and assumed $m \simeq M$. From the first part of this equation, it follows that $c^2 = \gamma P/\rho$ at fixed $u$ is unaffected by changes to the model that leave $GM$ and $R$ unchanged, as is the case for the changes to $G$ considered here. It also follows that

$$\frac{\delta P}{P} = \frac{\delta \rho}{\rho} \simeq - \frac{1}{\gamma - 1} \frac{\delta K}{K};$$

in particular, $\delta P/P$ and $\delta \rho/\rho$ are approximately constant in the convection zone (see also Fig. 1).

Further relations between the model changes can be derived by noting that, at the base of the convection zone ($r = r_{cz}$), $\nabla_{rad} = \nabla_{ad}$, which can be taken to be fixed. It follows from equation (15) that, given that $L$ and $GM \simeq GM$ are fixed, the value of $K P/T^4$ at the base of the convection zone is fixed. Assuming again that $\kappa \simeq \kappa_0 P^{2/5} T^{-4/5}$, and using the ideal gas law, this yields the following relation satisfied by the change $\delta P_{cz}$ in the value of the pressure at the base of the convection zone:

$$0 = -\frac{\gamma - 1}{\gamma} \left( 4 + v - \frac{\gamma + \delta}{\gamma - 1} \right) \frac{\delta P_{cz}}{P_{cz}} - \frac{1}{\gamma} \left( 4 + v + \delta \right) \frac{\delta K}{K}$$

From equation (A2) it furthermore follows that

$$\frac{1}{K} \frac{\gamma - 1}{\gamma} \frac{\delta P_{cz}}{P_{cz}} \simeq - \frac{R}{d_{cz}} \frac{\delta r_{cz}}{r_{cz}} \simeq - \frac{\mu}{\gamma - 1} \frac{\delta r_{cz}}{H_{P,cz}},$$

where $H_{P,cz}$ is the pressure scaleheight at the base of the convection zone. In the last equality we again used equation (A2). Using this to eliminate $\delta P_{cz}/P_{cz}$ in equation (A4) we obtain

$$0 = \frac{\delta + 1}{\gamma - 1} K \left( 4 + v - \xi \right) \frac{\delta \mu}{\mu}$$

$$+ \left( 4 + v - \frac{\gamma + \delta}{\gamma - 1} \right) \frac{\gamma}{\gamma - 1} \frac{\delta r_{cz}}{H_{P,cz}}.$$ (A6)

Also, $(4 + v - \xi) \delta \mu/\mu \simeq -(1 + \delta) \delta G/G$ from the homology argument for constant luminosity based on equation (17). Using equation (A3) we finally obtain that in the convection zone

$$\frac{\delta P}{P} = \frac{\delta \rho}{\rho} \simeq - \frac{\delta G}{G} \left( 4 + v - \frac{\gamma + \delta}{\gamma - 1} \right) \frac{\delta r_{cz}}{H_{P,cz}}.$$ (A7)

Here the first term is identical to the result obtained from homology. However, in general the change in the model will result in a non-zero change in the radius at the base of the convection zone and hence in a non-homologous contribution to $\delta \rho/\rho$.

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