The essential difference between the two methods in question now becomes apparent. By using \( U(x, y) \) instead of \( u(x, y) \) to evaluate \( \frac{2P}{2x} \), Schultz-Grunow and Breuer approximate the term in \( (42) \) which contains the integral by

\[
(\alpha + \gamma) \frac{2(1 + \Omega \eta)^2}{2(1 + \Omega \eta)^2}
\]

Thus they deal with a third-order differential equation which in general is not equivalent to the Massey and Clayton fourth-order differential equation.

Consider the special case of constant edge velocity, \( U_0 = \text{const} \), or \( \alpha = -1 \) and \( \gamma = -1 \). The term in question now vanishes.

The slope of the curves \( C_f \) versus \( \Omega \) as plotted in Fig. 9 of [6] agrees with the slope cited in Table 2 of [14]. (Note that the foregoing cases are the only ones for which a comparison can be made.) From this we deduce that the method of reference [6] and the method of matched asymptotic expansions, as presented in reference [14], are apparently equivalent for all values of the parameter \( \gamma \). We saw earlier that the method of reference [7] agrees for only one special case and otherwise disagrees with [6].

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References


B. S. Massey and B. R. Clayton

The authors have added significantly to our knowledge of laminar boundary layers on curved surfaces by their results for the special case \( \alpha = 0 \). We are justified in their investigations confirm the validity of our own work on the effect of surface curvature. It does appear, however, that in discussing their results the authors make some unwarranted generalizations.

We must dissent from the statements following equation (5). The sequence of differentiating and ordering does matter. It is true that in the particular example examined here the same result is obtained whatever the sequence, but this fact does not justify the conclusion that ordering before differentiation is, in general, valid.

Consistency of procedure, which the authors emphasize, seems to be so soon forgotten. Equations (1)-(3) are said to involve the assumptions that \( kL \) is of \( O(1) \) and \( \delta = \delta/L \) is small (presumably in comparison with unity). Thus \( kL \) is also considered small. Yet results are presented (Fig. 3) in which \( \Omega \eta = kn \) goes as high as 0.5. The authors are not alone in their inconsistency:

\[ \Delta \text{MASSEY AND CLAYTON [6]} \]

\[ \circ \text{SCHULTZ-GRUNOW AND BREUER [7]} \]

\[ \square \text{NARASIMHA AND OJHA [14]} \]
Taulbee and Patel, using similar initial premises, give results in which $\varphi_2$ is as high as 3.6.

By good fortune, as we have shown, the authors' equation (12) is valid for all values of $k$b provided only that the Reynolds number of the flow is sufficiently large. Their results can therefore be accepted as accurate for very large Reynolds numbers. Nevertheless, the methods by which both Taulbee and Patel and the present authors arrive at their equations must be regarded as untrustworthy.

One would have welcomed some comment on the accuracy of the equation. The case $\alpha = \varphi_2$ here examined, is peculiar in that the complete single equation of motion (that is, the equation obtained when no approximations whatever are made) yields similar solutions for all values of $x$. In the more general case, on the other hand, for all values of $\gamma/\alpha$ less than 3 and other than unity, similar solutions are approached asymptotically with increasing distance downstream. The authors' equation (12) in fact is strictly true only for infinite Reynolds number.

The authors' comment, in the paragraph following equation (40), that “the only restriction is that $(u_i/v)^2$ be large enough for a boundary-layer approximation to be valid” seems a little too ingenious, for not only is the approximation unspecified but the question “How large is large enough?” is left unanswered. We have yet to be persuaded that the value $u_i/v = 1344$ used in Fig. 7 is large enough for reasonable accuracy of the results, and we should like to know the authors’ justification of this choice of $u_i/v$.

Nor do we find convincing the authors’ conclusion that the methods of Narasimha and Ojha are equivalent. The method of matched asymptotic expansions as used both by Van Dyke [8] and by Narasimha and Ojha [14] involves a linearizing of the equation whereas our method does not. Consequently, the methods are not equivalent for large values of curvature. A result of the linearizing technique of Narasimha and Ojha is the linear relation between $f''(0)$ and $\varphi_2$ illustrated in the authors' Fig. 8. As Narasimha and Ojha explicitly state, however, no significance can be attached to the slope of the line except at $\Omega = 0$. In fact, as we have shown [6], for curvature of large magnitude, the relation between $f''(0)$ and $\Omega$ becomes appreciably nonlinear.

The authors have supposed that, because the results of Narasimha and Ojha correspond closely to our own for values of $\gamma$, the methods are equivalent for all values of $\gamma$. Here again is a dangerous argument from the particular to the general.

As expected, the authors' results indicate that for $\alpha = 0$ the effects of curvature alone on displacement and momentum thickness and on the skin-friction coefficient are qualitatively similar to those for $\alpha > 0$. We see no reason why the additional effects of the displacement of the main stream should not also be qualitatively similar. Even so, since these effects may be quite easily taken into account when the solutions are exactly known, we regret the omission from the paper of any consideration of these displacement effects.

Although we have been somewhat critical of arguments used by the authors we do welcome the publication of their results. They add valuable detail to those already presented by Taulbee and Patel, and appear to be entirely reliable for large values of Reynolds number.

**Authors' Closure**

The authors wish to thank the discussers for their comments, some of which have pointed to areas in need of clarification.

The first point has to do with the sequence in which ordering and cross-differentiation are applied when deriving our equation (5). Massey and Clayton [6] chose to cross-differentiate first and then to neglect certain terms according to an order-of-magnitude procedure. We have neglected terms first, differentiating subsequently. Since our equation (5) is the same as their equation (26) we concluded that the sequence of the two processes does not matter. Note that we have never claimed that this is true “in general.” It should be emphasized however that our equation (5) is the basic boundary layer momentum equation of [6] and is thus general for steady, incompressible, laminar boundary layers on curved surfaces.

Concerning the order of magnitude of our variables we retained terms of $0(1)$ and $0(\delta)$ assuming that $kL \sim 0(1)$. In addition, we retained the last term of equation (2), which is of $0(\delta^3)$ because then in the limit the outer flow satisfies the boundary layer equations. From this it does not follow, as the discussers have concluded, that the equations are invalid for larger curvatures. In fact, it may be easily shown that the same equation (5) is obtained when $kL$ is of $0(1/\delta)$. We have refrained from claiming the validity of our analysis for $kL$ of $0(1/\delta)$ because we had some difficulty envisioning the physical significance of a self-similar boundary layer analysis for radii of curvature smaller than $\delta$. Also, for large curvature difficulties arise with the matching conditions at the outer edge (see $\S 6$ of [14]). Thus although the results for $\Omega = 0.1$ presented by us in Fig. 5 are valid solutions, they ought to be viewed with caution as they probably represent outer limits for the validity of the theory. We observe that some of the results of reference [6] are open to the same criticism.

In discussing the accuracy of equations and results the discussers rely substantially on their analysis presented in [15]. In this paper they show that the terms which we omitted by them in deriving the similarity equation (49) of [6] can be put into two groups: one proportional to $(Ue/v)^{-1}$ (i.e., of $0(\delta^2)$) and the other proportional to $(Ue/v)^{-2}$ (i.e., of $0(\delta^4)$). The magnitudes of the proportionality factors, which are complex functions of $\Omega, \gamma, \alpha, f_1$, and its derivatives, is never shown although it is said that they have been evaluated for one particular point within the boundary layer. Likewise, they do not discuss higher order effects on the external flow or the higher order matching conditions at the outer edge. Thus the significant conclusion of [15] is that the order of magnitude analysis of [6] is valid. It seems to us that in order to establish the accuracy of boundary layer solutions generated from equation (49) of [6] one must compare such solutions with those generated from full Navier-Stokes equations or with carefully obtained experimental data.

Concerning the inferences which may be drawn about our results on the basis of those presented in [15] we note that since the terms which we neglected are approximately at most as large as $(u_i/v)^{-1}$, our choice of 1344 for the Reynolds number (see Fig. 7) is not refuted by [15]. However, upon reviewing the known solution of Navier-Stokes equations by Millsaps and Polhausen [16], we believe at present that a more judicious choice would have been say 5000, which would still yield the same qualitative picture. A pictorial representation of a sample flow was our only intention in presenting the Fig. 7.

Narasimha and Ojha [14] have studied the same family of solutions as did Massey and Clayton [6]. However, reference [14] employed the method of matched asymptotic expansions which uses a perturbation method on both the inner and the outer flow. The first approximation to the inner flow is described by a nonlinear equation and the second approximation by a linear equation in velocities which turns out to yield a linear relation between $f''(0)$ and $\Omega$. Narasimha and Ojha [14] contend that for skin friction their method, and all other second order methods (including [6]), cannot be expected to yield significant results except for the slope of the $f''(0)$ versus $\Omega$ curve at $\Omega = 0$. In the comments above Massey and Clayton state that their results given in

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*Numbers 15 and 16 in brackets designate Additional References at end of paper.*

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[6] show appreciable nonlinearity for large curvature. Since they do not define what is large curvature we assume that perhaps $\Omega = 0.1$ is large since that is the largest value used in their Fig. 9. The relation between $f'(0)$ and $\Omega$ in this figure is linear within the reading accuracy for the entire range of $\Omega$ and $\gamma$. It is conceivable that the commentators have since obtained some additional data which corrects their findings presented in Fig. 9 of [6]. It is also pertinent to note that the work of Taulbee and Patel, which starts with the same governing equation as [6] except that it is specialized to a particular value of $\alpha$, indicates a linear variation of $f'(0)$ with $\Omega$ up to $\Omega = 0.4$.

We have shown that the results of references [14] and [6] agree very well for $f'(0)$ versus $\Omega$ for values of the parameter $\gamma = 1, -0.6, -1, -1.36$. This range of values of $\gamma$ includes the largest and the smallest values presented in reference [6] as well as two intermediate values. This correlation of results obviously does not constitute a rigorous proof that the two methods are equivalent. However, we believe that it is sufficient evidence to conclude that the two methods give basically the same results for the variation of the skin friction coefficient with curvature over the range of parameters which have been considered by the authors.

Additional References