Research Note

The Condition at the Surface of a Thin Conductor in which Currents are Induced

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Consider a thin conducting sheet of integrated surface conductivity $\sigma$ (not necessarily constant) subjected to a time varying magnetic field of vector potential $\mathbf{A}^e$ which induces currents in the sheet of vector potential $\mathbf{A}^i$. It was found by the author (Ashour 1971a) that at the surface of the sheet

$$[\partial \mathbf{A}^i / \partial n] = 4\pi \sigma (\mathbf{A}^e_s + \mathbf{A}^i_s) / \partial t$$  \hspace{1cm} (1)

where $\partial / \partial n$ denotes differentiation along the normal to the sheet, the subscript $s$ designates the tangential component and $[F]$ denotes the change in the function $F$ as the sheet is crossed normally. If the sheet forms a part of a closed surface, then the condition at the non-conducting part is obtained by equating the L.H.S. of (1) to zero. The boundary equation (1) was also obtained earlier by Smythe (1968) for the special case of the plane sheet.

It is the purpose of this note to indicate that the boundary condition (1) needs modification in certain cases including the case when axial symmetry does not exist. We shall limit our discussion to plane and spherical sheets only.

The relation (1) was obtained by finding that each of its sides equals $-4\pi i_s$, where $i$ is the surface current density in the sheet. This is in fact true for the L.H.S. provided that the normal component $A_s^i$ of $\mathbf{A}^i$ is zero at the sheet. In the case of plane or spherical sheets this condition ($A_n^i = 0$) is satisfied not only on the sheet but also outside it. As for the R.H.S. of (1), noting that

$$\mathbf{E} = -\left[ (\partial \mathbf{A} / \partial t + \text{grad} \ P) \right]$$

$$\mathbf{J} = k \mathbf{E}$$

where $\mathbf{E}$ is the electric field, $P$ the electric potential, $\mathbf{J}$ the volume current density and $k$ the volume conductivity, we obtain by integration through the thickness of the sheet and by noting that at the surface of the sheet $\text{grad} \ P = P_0$ where $P_0$ is the value of $P$ at the sheet:

$$-4\pi i_s = -4\pi \sigma \mathbf{E}_s = 4\pi \sigma \left[ \partial (\mathbf{A}^e_s + \mathbf{A}^i_s) / \partial t + \text{grad} \ P_0 \right].$$

Hence the R.H.S. of (4) should replace that of (1).

Clearly $P_0$ can be eliminated by taking the curl of both sides of the modified surface condition after dividing by $\sigma$. It can easily be verified that what we obtain in this way is the well-known Price relation (Price 1949) which is expressed in terms of the scalar magnetic potentials of the fields.

To obtain the equation satisfied by $P_0$ in the sheet, we note that since $A_n^i$ is assumed zero everywhere, $\text{div} \mathbf{A}^i = 0$ everywhere (because $\text{div} \mathbf{A}^i = 0$). Also $i_s$ and

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\(A^\varepsilon\) (but not necessarily \(A_s^\varepsilon\)) are non-divergent. Hence the divergence of \(i_s/\sigma\) gives at the sheet

\[
V_s^2 P_0 = -i_s \cdot \text{grad} \sigma^{-1} + \partial^2 (h_2 h_3 A_n^\varepsilon)/h_2 h_3 \partial t \partial n
\]  

(5)

with the usual notation of orthogonal curvilinear co-ordinates.

Thus \(P_0\) is a surface function which satisfies (5) and is finite and single valued at every point of the sheet. On the other hand, \(P\) is a function of the three co-ordinates satisfying Laplace's equation, is finite and single valued everywhere in space and is equal to \(P_0\) at the sheet. If the sheet is closed these are sufficient conditions to determine \(P\). If the sheet is confined to a part of a closed surface, the condition \(P = P_0\) is replaced on the non-conducting part of the surface by the continuity of \(\partial P/\partial n\).

We shall now consider the solution of (5) in certain special cases, and also in the general case.

(i) Axi-symmetric problems

In this case each of \(i_s\) and \(A_s\) has one component only in the direction of increase of the azimuthal angle and their magnitudes, and that of \(\sigma\), are independent of that angle. Hence it is easily seen from (4) that grad \(P_0 = 0\).

(ii) Two-dimensional problem

This case is similar to the axi-symmetric case because each of \(i_s\) and \(A_s\) has one component only in the constant direction of the \(x\)-axis say, and these components, together with \(\sigma\), are independent of \(x\). Hence grad \(P_0 = 0\) as in (i).

(iii) Uniform conductivity (including infinite conductivity)

When \(\sigma\) is constant (5) gives

\[
V_s^2 P_0 = \partial^2 (h_2 h_3 A_n^\varepsilon)/h_2 h_3 \partial t \partial n
\]  

(6)

The solution of (6) clearly depends on \(A_n^\varepsilon\). We shall consider two special cases as examples.

First, a spherical sheet \(r = a\) (\(r, \theta, \phi\) are spherical polar co-ordinates) and a uniform inducing field normal to the axis \(\theta = 0\). In this case \(A_n^\varepsilon = 0\) and the dependence on \(\phi\) throughout is a factor \(\exp (i\phi)\). Hence

\[
V_s^2 P_0(\theta, \phi) = 0
\]  

(7)

which leads to

\[
P_0(\theta, \phi) = (C \tan \theta/2 + D \cot \theta/2) \exp (i\phi)
\]  

(8)

where \(C, D\) are constants. The expression (8) is infinite for \(\theta = 0\) and \(\theta = \pi\) and thus must be excluded if the sheet is the complete surface \(r = a\). If however the sheet is confined to the cap \(0 \leq \theta \leq \alpha\), then, since \(P_0(\theta, \phi)\) must be finite at the sheet only,

\[
P_0 = C \tan (\theta/2) \exp (i\phi)
\]  

(9)

is a possible solution.

Next we consider induction in a sheet coinciding with the plane \(z = 0\) (\(z, \rho, \phi\) are cylindrical co-ordinates) by a dipole \(M\) situated on the \(z\)-axis at distance \(c\) from the sheet and with axis normal to the \(z\)-axis. In this case \(A_s^\varepsilon \neq 0\) and (6) reduces to

\[
V_s^2 P_0(\rho, \phi) = 3M c \rho \exp (i\phi)/(\rho^2 + c^2)^{3/2}.
\]  

(10)

The complete solution of (10) is given by

\[
P_0(\rho, \phi) = (\alpha \rho + \beta \rho^{-1} + M c/\rho \sqrt{\rho^2 + c^2}) \exp (i\phi).
\]  

(11)

where \(\alpha\) and \(\beta\) are constants. For \(P_0\) to be finite at \(\rho = 0\), we must have

\[
\beta = -\dot{M}.
\]  

(12)
Hence if the sheet is confined to the disk \( z = 0, 0 \leq \rho \leq a \), a possible solution is

\[
P_0(\rho, \phi) = \{x\rho - \tilde{M}\rho/\sqrt{(\rho^2 + c^2) \left(\sqrt{(\rho^2 + c^2)} + c\right)}\} \exp(i\phi).
\]  

(13)

If the whole plane \( z = 0 \) is the conducting sheet then \( \alpha \) must be taken zero in (13) since otherwise \( P_0 \) will not be finite at \( \rho = \infty \).

(iv) General case

In the general case the solution of equation (5) will consist of two parts: the first is the solution of \( \nabla_s^2 P_0 = 0 \) and the second a 'particular integral' depending on \( A^s, \sigma \text{ and } i_4 \). Hence there will be 'coupling' between \( A^s \) and \( P_0 \).

The modified boundary condition has been applied to two problems whose solutions are known otherwise, namely induction by a non-symmetric axial dipole in a perfectly conducting circular disk and a perfectly conducting spherical cap. The results, which could not have been obtained if \( P_0 \) was taken zero, agreed with the earlier solutions (Ashour 1965a, b). The details are not given here.

The boundary condition in the form (1) has been applied by the author (Ashour 1971a) to certain axi-symmetric and two-dimensional problems of induction in thin sheets. It follows from (i) and (ii) above that all results of that paper are correct and need no modification. It has also been applied by the author (Ashour 1971b) to induction in a non-uniform hemispherical shell whose conductivity has axial symmetry in the two cases when the inducing field is uniform and is either parallel or normal to the axis of symmetry. Again it follows that the results for the axially-symmetric case are correct. However, the modified boundary condition must be used in the non-symmetrical case. Hence, the analysis and results based on it given in the paper for that case need re-examination. Work is proceeding on this now.

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References