Behavior of Elastic Scattering Amplitudes at High Energies

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A Laplace representation of the high-energy elastic scattering amplitude is introduced, which seems convenient for describing the observed sharp decrease of the differential cross sections in the forward direction. It is found that the imaginary part of the determining function of the Laplace transform has a Gaussian peak, with an exponential tail outside. This behavior of the determining function is discussed from a more general point of view, on the basis of another integral representation of the diffraction part of the scattering amplitude. The momentum transfer dependence of the elastic scattering amplitude is also investigated.

§ 1. Introduction

High-energy elastic scattering exhibits a sharp diffraction peak in the forward direction, which is common to all observed processes. According to the Brookhaven group the experimental formula

\[ \frac{d\sigma}{dt} = (d\sigma/dt)_{t=0} \exp(bt + ct^2) \]  

(1.1)
gives good fits to the differential cross sections for \(-1 < t < 1 \) (GeV/c)\(^2\), where \(-t\) is the invariant momentum transfer squared.\(^1\) Both \(b\) and \(c\) are positive and \(c/b^2\) is small compared to one. (Antiproton-proton scattering is exceptional; here \(c\) is consistent with zero.) On the other hand, the Cornell-Brookhaven group has made extensive measurements of \(p-p\) elastic scattering up to the maximum momentum transfer.\(^2\) There seems to be no complicated structure in the observed differential cross sections.

If we bear these points in mind, it is tempting to investigate the expansion of elastic scattering amplitudes at high energies by exponential functions of \(t\) in place of oscillating Bessel functions which have been widely used in various impact parameter formalisms.\(^3\) This is the motivation of the present work. We propose to represent the scattering amplitude \(A(s, t)\) at high energies in the form

\[ A(s, t) = \int da e^{au}G_1(s, a) + \int da e^{au(u-u_0)}G_2(s, a). \]  

(1.2)

\(s, t\) and \(u\) are the conventional Mandelstam variables and \(u_0\) is given by \((m_1^2 - m_2^2)/s\), where \(m_1\) and \(m_2\) are the masses of two colliding particles. Equation (1.2) is to be compared with the impact parameter expansion of Blankenbecler and Goldberger (BG).\(^3\)
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\[ A(s, t) = \int_0^\infty db \, J_0[b(-t)^{1/2}] H_t(s, b^2) \]

\[ + \int_0^\infty db \, J_0[b(-u)^{1/2}] H_u(s, b^2). \]  

(1.3)

For the sake of simplicity we ignore complications due to spin throughout.

In § 2 we introduce \( G_t(s, a) \), which appears in Eq. (1.2), by a Laplace transform of the absorptive part \( A_t(s, t) \) of \( A(s, t) \). Some mathematical properties of \( G_t(s, a) \) as a function of \( a \) are investigated. A necessary condition for the validity of Eq. (1.2) and its domain of convergence are obtained. In § 3 a plausible form for the imaginary part, \( G_{t, r}(s, a) \) of \( G_t(s, a) \) is derived by physical intuition for fixed \( s \). This leads to Eq. (1.1) for small \( |t| \) under the assumption that \( A(s, t) \) is dominated by its imaginary part \( A_r(s, t) \) at small momentum transfers. In § 4 this phenomenological form of \( G_{t, r}(s, a) \) is discussed from a more general point of view. \( A_r(s, t) \) is decomposed into the two parts: the diffraction and the background parts. It is found that the expression for \( G_{t, r}(s, a) \) obtained in § 3 occupies a special position in the new representation. The momentum transfer dependence of the elastic scattering amplitude is investigated on the basis of this representation. A few remarks on the present approach are given in the last section.

§ 2. Representation of scattering amplitudes by a Laplace transform

The sharp exponential decrease of differential cross sections indicated by Eq. (1.1) is common to all the experimentally observed elastic scattering processes. Therefore, it seems worthwhile to investigate the expansion of high-energy elastic scattering amplitudes in terms of exponential functions of \( t \) in place of the oscillating Bessel functions which are used in the impact parameter formalisms.\(^{3-5}\) To do this we start from the dispersion relation of Mandelstam,

\[ A(s, t) = \frac{1}{\pi} \int_{t_0}^\infty \frac{A_t(s, t')}{t' - t} dt' + \frac{1}{\pi} \int_{u_0}^\infty \frac{A_u(s, u')}{u' - u} du', \]  

(2.1)

with subtractions if necessary. We introduce a function defined by the Laplace transform of \( \pi^{-1} A_t(s, t') \) with respect to \( t' \),

\[ G_t(s, a) = \frac{1}{\pi} \int_{t_0}^\infty e^{-at'} A_t(s, t') dt'. \]  

(2.2)

The integral converges for \( \text{Re} a > 0 \) and \( G_t(s, a) \) is analytic there if \( A_t(s, t') \) is bounded by a polynomial of \( t' \) for large \( t' \), which we assume. Since \( s \) is
fixed at a (large) physical value throughout this work, we shall often omit it as a variable for the sake of simplicity.

It is easy to see that

$$\lim_{a \to \infty} e^{at} G_t(a) = 0 \quad \text{for Re } t < t_0. \quad (2.3)$$

Indeed, for \(\nu > 0\) we have

$$G_t(a) \sim \Gamma(1+\nu) a^{-(1+\nu)} e^{-at_0} \quad \text{as } a \to \infty \quad (2.4)$$

if

$$\pi^{-1} A_t(t') \sim (t' - t_0)^\nu \quad \text{as } t' \to t_0 + 0. \quad (2.5)$$

The behavior of \(G_t(a)\) at small \(a\) is more complicated because \(A_t(t')\) is very likely to oscillate infinitely many times as \(t'\) tends to infinity. We note only that, for \(\text{Re } a > -1\),

$$G_t(a) \sim \Gamma(1+\alpha) a^{-(1+\alpha)} \quad \text{as } a \to 0 \quad (2.6)$$

if

$$\pi^{-1} A_t(t') \sim t'^\alpha \quad \text{as } t' \to \infty. \quad (2.7)$$

If the asymptotic behavior of \(A_t(s, t')\) for large \(t'\) is determined by a top Regge trajectory \(\alpha(s)\) of the \(s\) channel, then \(\alpha\) is given by \(\alpha(s)\). Although there is no proof, it is often conjectured that \(\alpha(s)\) tends to \(-1\) as \(s \to \infty\).

In general Eq. (2.1) needs subtractions, but we assume that no subtraction is necessary in Eq. (2.1) for sufficiently large \(s\). This is true if (2.7) holds with \(\alpha, \text{Re } \alpha < 0\). We then obtain

$$\frac{1}{\pi} \int_{t_0}^{\infty} \frac{A_t(t')}{t' - t} dt' = \int_0^{\infty} e^{at} G_t(a) da. \quad (2.8)$$

Because of (2.4) the integral on the right-hand side of the above equation converges for \(t, \text{Re } t < t_0\), and is analytic there. The same argument holds for \(G_u(a)\) as for \(G_t(a)\). We have only to replace \(t\) by \(u - u_0\). We thus obtain the representation (1.2), in which the first term converges for \(t, \text{Re } t < t_0\), and the second one for \(u, \text{Re } u < u_0\). The physical region, \(t \leq 0\) and \(u \leq u_0\), lies inside the intersection of the analyticity domains of the two terms in Eq. (1.2). The first term is important in the forward direction, while the second one is important in the backward direction.

We normalize \(A(s, t)\) so that it is given by

$$A(s, t) = i^{3/2} f(k, \theta), \quad (2.9)$$

where the partial wave expansion of \(f(k, \theta)\) is given by

$$f(k, \theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos \theta) \quad (2.10)$$
with the partial wave amplitudes

\[
\eta_i(k) = (2i)^{-1}[\gamma_i \exp(2i \theta_i) - 1].
\]  

(2.11)

Then we obtain

\[
(\delta^{3/2}/k) \eta_i = \frac{1}{2} \int_{-1}^{1} A(t) P_i(z) \, dz
\]

\[
= \int_0^{\infty} da \left\{ G_i(a) \frac{1}{2} \int_{-1}^{1} e^{as} P_i(z) \, dz + (t-u+u_0) \right\}.
\]

Since \( t = -2k^2(1-z), \ u = -2k^2(1+z) + u_0 \) and

\[
\frac{1}{2} \int_{-1}^{1} e^{as} P_i(z) \, dz = \frac{1}{2} \frac{1}{2l+1} \int_{-1}^{1} e^{as} \frac{d^l}{dz^l} (z^2 - 1)^l \, dz
\]

\[
= \frac{x^l}{2^{l+1}l!} \int_{-1}^{1} e^{as} (1-z^2)^l \, dz
\]

\[
= (\pi/2x)^{l/2} I_{l+1/2}(x) = i_l(x),
\]

(2.12)

where \( i_l(x) \) is the modified spherical Bessel function of order \( l \), we obtain

\[
(\delta^{3/2}/k) \eta_i = \int_0^{\infty} da \, e^{-2ak^2} i_l(2ak^2) \{ G_i(a) + (-1)^l G_n(a) \}.
\]

(2.13)

Next we discuss the connection of our \( G_i(a) \) with the impact parameter amplitude \( H_i(b^2) \) of BG. By means of Weber's formula

\[
e^{at} = \frac{1}{2a} \int_0^{\infty} bdb \, e^{-b^2/4a} J_1[b(-t)^{1/2}],
\]

(2.14)

which is valid for \( a > 0 \) and for \( t \leq 0 \), we can rewrite Eq. (2.8), after changing the order of integrations, as

\[
A_i(t) = \int_0^{\infty} bdb \, J_1[b(-t)^{1/2}] \int_0^{\infty} da \, e^{-b^2/4a} G_i(a).
\]

For simplicity we denoted the left-hand side of Eq. (2.8) by \( A_i(t) \). Therefore, if we put

\[
H_i(b^2) = \int_0^{\infty} da \, e^{-b^2/4a} G_i(a),
\]

(2.15)

we obtain the impact parameter expansion
\[ A_i(t) = \int_0^\infty bdb J_0[b(-t)^{1/2}]H_i(b^2). \]  

(2·16)

In order to confirm that \( H_i(b^2) \) defined by Eq. (2·15) is the impact parameter amplitude of BG, we substitute Eq. (2·2) into Eq. (2·15). After changing the order of integrations we find

\[
H_i(b^2) = \frac{1}{\pi} \int_{t_0}^{\infty} dt' A_i(t') \int_{0}^\infty \frac{da}{2a} \exp[-at' - b^2/4a]
\]

\[
= \frac{1}{\pi} \int_{t_0}^{\infty} dt' K_0(bt'^{1/2}) A_i(t'),
\]

(2·17)

where \( K_0(x) \) is the modified Bessel function of the second kind of order zero. Equation (2·17) is identical with Eq. (7·7) of BG. It may be added that

\[ H_i(b^2) \sim \{\Gamma(1 + \alpha)/2\} (4/b^2)^{1+a} \quad \text{as} \quad b \to 0 \]  

(2·18)

if \( G_i(a) \sim a^{-(1+a)} \) as \( a \to 0 \). Under the same assumption, \( \Re \alpha < 0 \), as before, the integral (2·16) converges at the lower limit of the integration. It is interesting to note that \( H_i(b^2) \) could become infinite as \( b \to 0 \). This is in contrast to the fact that the impact parameter amplitudes at the origin of references 4 and 5 are equal to the S-wave amplitude and hence should always be finite.\(^*\)

§ 3. **Diffraction scattering and the imaginary part of \( G(a) \)**

The forward peak of elastic differential cross sections is usually supposed to be due to pure imaginary shadow scattering. In order to see the form suggested by experiment for the imaginary part \( G_i(a) \) of \( G(a) \), let us assume that the imaginary part \( A_i(t) \) of the scattering amplitude predominates over the real part \( A_R(t) \) for small momentum transfers \( 0 \leq -t \leq 0.6 \text{(BeV/c)}^2 \), say.

With this assumption the exponential decrease of \( d\sigma/dt \) can be obtained when

\[ A_i(t) \approx A_i(0) e^{\gamma t} \]

(3·1)

for small \( |t| \). It may be noted that we need not consider the second term in Eq. (1·2), which is negligible compared to the first one in the region of low momentum transfers. Hereafter we shall often omit the suffix \( t \) to \( G_i(a) \). We hope this will not give rise to any confusion.

Since \( A_i(t) \) is given by

\[ A_i(t) = \int_0^\infty da e^{at} G_i(a), \]

(3·2)

the simplest way to get (3·1) would be to take \( G_i(a) \) such that

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\[ G_\Gamma(a) \propto \delta(a-\gamma/2) \]  \hspace{1cm} (3.3)

This choice would lead to an impact parameter amplitude of a Gaussian form

\[ H_\Gamma(b) \propto \gamma^{-1} e^{-b^2/2\delta^2} \]  \hspace{1cm} (3.4)

The form (3.3) is not admissible, because we know that \( G(a) \) should be analytic in the right half-plane. It seems plausible, however, that the most important contribution to \( A_\Gamma(t) \) near \( t=0 \) comes from \( G_\Gamma(a) \) in the neighborhood of \( a=\gamma/2 \). We may take \( G_\Gamma(a) \) of a resonance form

\[ G_\Gamma(a) \propto \frac{1}{(a-\gamma/2)^2+i\delta} \]  \hspace{1cm} (3.5)

near \( a=\gamma/2 \). This is also inadmissible because \( G(a) \) would have two complex poles near \( a=\gamma/2 \). Since we are considering approximate forms of \( G_\Gamma(a) \), we might have to accept complex poles if they are far away from the real axis. Such is not the case for (3.5) unless \( \delta \) is very large.

On the basis of these considerations, let us now assume that \( G_\Gamma(a) \) has the following Gaussian form:

\[ G_\Gamma(a) \propto \exp[-(a-\gamma/2)^2/(2\delta^2)] . \]  \hspace{1cm} (3.6)

Since (3.6) is supposed to be valid only in the neighborhood of \( a=\gamma/2 \), it is not legitimate to use the Gaussian form in the whole range of \( a \). For further improvements of (3.6), however, it is instructive to know the \( A_\Gamma(t) \) which is obtained when (3.6) is used for all \( a \). Substituting (3.6) to Eq. (3.2), we get

\[ A_\Gamma(t) \propto \exp[(\gamma t + \delta t^2)/2] \{1 + \text{Erf}[(2\delta)^{-1/2}(\gamma/2 + \delta t)] \}, \]  \hspace{1cm} (3.7)

where \( \text{Erf}(x) \) is the error function defined by

\[ \text{Erf}(x) = 2\pi^{-1/2} \int_0^x e^{-y^2} dy . \]  \hspace{1cm} (3.8)

For small \( |t| \) the factor inside the curly bracket in (3.7) may be regarded as constant in comparison with the exponential factor in front of it if \( 4\delta/\gamma^2 \ll 1 \). Therefore we can write \( A_\Gamma(t) \) for small \( |t| \) as

\[ A_\Gamma(t) \approx A_\Gamma(0) \exp[(\gamma t + \delta t^2)/2] . \]  \hspace{1cm} (3.9)

Under the assumption that \( A_\Gamma(t) \ll |A_R(t)| \) for small \( |t| \), the experimental formula (1.1) is obtained if we equate \( \gamma \) and \( \delta \) with \( b \) and \( c \), respectively. Experimentally we have \( 4c/b^2 \sim 0.1 \), which is consistent with our previous assumption, \( 4\delta/\gamma^2 \ll 1 \). It is important that \( \delta \) must be positive from its meaning as the width of a Gaussian peak in \( G_\Gamma(a) \). This agrees with the experimental finding, \( c>0 \). It is to be mentioned that \( \vec{p}-p \) scattering is exceptional. In this case the differential cross sections measured up to 0.6 (BeV/c)\(^2\) are consistent with
$c=0$. This fact is somewhat puzzling from the standpoint of the present work. It would be desirable to extend the measurements to larger $|t|$, up to 1.0 (BeV/$c^2$), say.

The Gaussian form (3.6) for $G_t(a)$ has a fatal defect from a theoretical point of view, if it is adopted in the whole range of $a$. The function $A_t(t)$ obtained from it does not have any branch point in the complex $t$ plane and is indeed an entire function of $t$. This means that there exists no $A_{t,r}(t')$ which gives the form (3.6) to $G_t(a)$. When we attempt to improve (3.6), we have to take into account the following two points. First, $G_t(a)$ should decrease exponentially for large $a$. Second, (3.6) gives $A_t(t)$ too large values for large $|t|$. (In fact, since $G_{t,r}(a)$ tends to a constant as $a\to0$, we should have $A_{t,r}(t)\sim1/t$ as $t\to-\infty$.) On the other hand, it can be seen from Eq. (3.2) that the behavior of $A_t(t)$ for large $|t|$ is essentially determined by $G_t(a)$ with small $a$. Thus we conclude that $G_t(a)$ should be much smaller for small $a$ than the one indicated by (3.6).

Taking into account these considerations, we replace $\delta$ in (3.6) by $2\delta a/\gamma$. This leads to

$$G_t(a) \propto \exp[-\gamma(a-\gamma^2 t)/(4\delta a)] = e^{2\gamma a} \exp[-t_1 a - \gamma^2 t_1/(4a)],$$

where $t_1$ is defined by

$$t_1 = \gamma/(4\delta).$$

Mathematically it is convenient to consider $G_t(a)$ of the form

$$G_t(a) \propto \frac{(\gamma/2)^s}{2a^{s+v}} e^{2\gamma a} \exp[-t_1 a - \gamma^2 t_1/(4a)],$$

which is slightly more general than (3.10). Substituting (3.12) into Eq. (3.2), we get

$$A_t(t) \propto (\gamma/2)^s e^{2\gamma a} \int_0^\infty \frac{da}{2a^{s+v}} \exp[-(t_1-t)a - \gamma^2 t_1/(4a)]$$

$$= (1-t/t_1)^{s/2} e^{2\gamma a} K_s[\gamma t_1(1-t/t_1)^{1/2}],$$

which has a cut on the real positive axis in the complex $t$ plane extending from $t_1$ to $+\infty$. It follows from (3.13) that

$$\frac{A_t(t)}{A_t(0)} = (1-t/t_1)^{s/2} K_s[\gamma t_1(1-t/t_1)^{1/2}]/K_s(\gamma t_1).$$

Since $\gamma t_1 = \gamma^2/(4\delta) \gg 1$, we can simplify Eq. (3.14) by means of the asymptotic expansion of $K_s(x)$,

$$K_s(x) \sim (\pi/2x)^{1/2} e^{-x} [1 + O(1/x)],$$

where
for large $x$. We obtain
\begin{equation}
\frac{A_{f}(t)}{A_{f}(0)} = (1 - t/t_{1})^{(\nu-1)/\lambda} \exp[-\gamma t_{1}\{(1 - t/t_{1})^{1/2} - 1\}] .
\end{equation}
(3·15)

$t_{1}(=b/4c)$ is of the order of 1 (BeV/c)$^{2}$ except for $\bar{p}-p$ scattering, for which it will be larger. It is easy to see that (3·15) reduces to (3·9) for small $|t|$, unless $\nu$ is exceedingly large. By taking the logarithm of the both sides of (3·15), we get
\begin{equation}
\ln[A_{f}(t)/A_{f}(0)] = -\gamma t_{1}\{(1 - t/t_{1})^{1/2} - 1\} + \{(2\nu - 1)/4\} \ln(1 - t/t_{1}) .
\end{equation}
(3·16)

§ 4. Diffraction part of the scattering amplitude

In the previous section, guided mostly by physical intuition, we arrived at the expression for $G_{f}(a)$,
\begin{equation}
G_{f}(a) = \frac{A_{f}(0)}{K_{v}(\gamma t_{1})} \frac{1}{2a} \Gamma_{v}^{-1}(\gamma/2a)^{1+v} \exp[-t_{1}a - \gamma^{2}t_{1}/(4a)] ,
\end{equation}
which is easily obtained from (3·12) by writing the proportionality constant explicitly. Owing to the optical theorem $A_{f}(0)$ is related to the total cross section by
\begin{equation}
A_{f}(0) = (ks^{1/2}/4\pi) \sigma_{T} .
\end{equation}
(4·2)

In this section we want to discuss the meaning of Eq. (4·1) from a more general standpoint than in § 3. We also investigate what form $A_{f}(t')$ will take when $G_{f}(a)$ is given by Eq. (4·1) or by its superposition with respect to $\gamma$.

$A_{f}(t)$ is expressed in terms of the imaginary part $A_{f,R}(t')$ of $A_{f}(t')$ as
\begin{equation}
A_{f}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A_{f,R}(t')}{t'-t} dt' + (t\rightarrow u) .
\end{equation}
(4·3)

As before we shall not consider the second term in the above equation. It seems convenient to write $A_{f}(t$) as a sum of two terms,
\begin{equation}
A_{f}(t) = A_{f,P}(t) + A_{f,B}(t) .
\end{equation}
(4·4)

Although we cannot at present give a well defined meaning to the above decomposition, $A_{f,P}(t)$ is supposed to become dominant in the diffraction region and $A_{f,B}(t)$ represents the background. In the light of our result obtained in § 3, we assume that $A_{f,P}(t)$ has a cut on the real positive axis from $t_{1}$ to $\infty$;
\begin{equation}
A_{f,P}(t) = \frac{1}{\pi} \int_{t_{1}}^{\infty} \frac{A_{f,R}(t')}{{t'}-t} dt' .
\end{equation}
(4·5)

The important point of this assumption is that the smallest branch point $t_{1}$ of $A_{f,P}(t)$ is considerably larger than $t_{0}$, $t_{0} = 4\mu^{2}$ ($\mu$ is the piont mass). $t_{1}$ is to
be regarded as a phenomenological parameter.

By means of the formula\textsuperscript{b}

\[
\frac{1}{x^2+y^2} = \left( \frac{y}{x} \right) \int_0^\infty bdb \, K_v(by) J_v(bx),
\]

which is valid for any \( v, \Re v > -1 \), we can write

\[
\frac{1}{t'-t} = \left( \frac{t_1-t}{t'-t_1} \right)^{\nu/2} \int_0^\infty bdb \, K_v[b(t_1-t)^{1/2}] J_v[b(t'-t)^{1/2}],
\]

where we keep \( t \) and \( t' \) so that \( t'>t_1>t \). Substituting Eq. (4.7) into Eq. (4.5) and changing the order of integrations, we get

\[
A_r^P(t) = (t_1-t)^{-\nu/2} \int_0^\infty bdb \, K_v[b(t_1-t)^{1/2}] F(b; v),
\]

where \( F(b; v) \) is defined by

\[
F(b; v) = \frac{1}{\pi} \int_{t_1}^\infty dt' (t'-t_1)^{-\nu/2} J_v[b(t'-t)^{1/2}] A_r^P(t').
\]

Since Eq. (4.9) can be rewritten as

\[
F(b; v) = \frac{2}{\pi} \int_0^\infty x dx J_v(bx) x^{-\nu} A_r^P(t_1+x^2),
\]

we obtain

\[
\frac{2}{\pi} A_r^P(t_1+x^2) = x \int_0^\infty bdb \, J_v(bx) F(b; v).
\]

Rigorously speaking, it is possible that the integrals which appear in Eqs. (4.8) to (4.10) do not exist. Since we do not have enough knowledge at present about \( A_r^P(t') \) for large \( t' \) and \( F(b; v) \) for small \( b \), we assume the existence of these integrals for \( v, 1 \geq \nu \geq 0 \), say.

Corresponding to the decomposition (4.4), \( G_r(a) \) is written as a sum of two terms

\[
G_r(a) = G_r^P(a) + G_r^H(a),
\]

where

\[
G_r^P(a) = \frac{1}{\pi} \int_{t_1}^\infty e^{-at'} A_r^P(t') \, dt'
\]

\[
= e^{-at_1} \int_0^\infty bdb \, F(b; v) \int_0^\infty dx \, x^{1+\nu} e^{-a_2x} J_v(bx)
\]
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\[ = (2a)^{-1} e^{-\alpha} \int_0^\infty db \, b^{1+v} e^{-\alpha/4} F(b; \nu). \]  

(4.12)

It will be seen that Eq. (4.1) is obtained if we take \( F(b; \nu) \) such that

\[ F(b; \nu) \propto b^{-1} \delta (b - \gamma_t/t_1). \]  

(4.13)

It follows from Eqs. (4.10) and (4.13) that

\[ \pi^{-1} A_{t'1}(t') \propto (x'/2) J_v(bx), \]  

(4.14)

where \( x = (t' - t_1)^{1/2}. \)

In a similar way to Eq. (4.11), we write \( H_t(b^2) \) as

\[ H_t(b^2) = H_t^p(b^2) + H_t^p(b^2), \]  

(4.15)

where

\[ H_t^p(b^2) = \int_0^\infty db \, b^{1+v} e^{-\alpha/4} G_t^p(a) \]

\[ = \int_0^\infty db' b'^{1+v} F(b'; \nu) \int_0^\infty \frac{da}{(2a)^{1+v}} \exp\left[ -t_1 a - (b'^2 + b^2) / (4a) \right] \]

\[ = \int_0^\infty db' b'^{1+v} \left\{ t_1 / (b'^2 + b^2) \right\} (3 + \nu/2) K_{1+v} \left[ x' \right] F(b'; \nu). \]  

(4.16)

Therefore, when \( F(b; \nu) \) has the form (4.13), we get

\[ H_t^p(b^2) \propto \gamma^{-1} t_1^{1/2} \left\{ 1 + b^2 / (\gamma^2 t_1) \right\} ^{-(1+v)/2} K_{1+v} \left[ \gamma t_1 \left( 1 + b^2 / (\gamma^2 t_1) \right) ^{1/2} \right]. \]  

(4.17)

Although the choice of \( \nu \) is arbitrary in so far as the relevant integrals exist, the special case \( \nu = 1/2 \) is particularly interesting because the Bessel functions with \( \nu = 1/2 \) are very simple. The formula (4.7) is now expressed as

\[ \frac{1}{t' - t} = \left( t' - t_1 \right)^{-1/2} \int_0^\infty db \exp \left\{ -b(t_1 - t) / (t' - t) \right\} \sin \left\{ b(t' - t_1) / (t' - t) \right\}. \]  

(4.18)

By means of this formula \( A_t^p(t) \) is represented in the form

\[ A_t^p(t) = \int_0^\infty db \exp \left\{ -b(t_1 - t) / (t' - t) \right\} F(b) \]  

(4.19)

for \( t < t_1 \), where \( F(b) \) is defined by

\[ F(b) = (2/\pi)^{1/2} \int_0^\infty dx \sin bx \, A_{t'1}(t_1 + x^2). \]  

(4.20)

for \( b > 0 \) and it is related to \( F(b; 1/2) \) by
Since Eq. (4·20) is a sine transform, we obtain

\[
A_{\gamma,t}(t_1 + x^2) = \int_0^\infty db \sin bx F(b)
\]

for \(x > 0\). We can also show that

\[
G_{\gamma}^P(a) = (2/\pi)^{1/3} (2a)^{-1/3} \int_0^\infty b db e^{-b^3/4a} F(b)
\]

for \(a > 0\) and that

\[
H_{\gamma}^P(b') = t_1^{1/3} \left[ b'/b \right]^{1/3} \exp \left[ - \{t_1(b'/b) \}^{1/3} \right] F(b').
\]

The simplest choice for \(F(b)\) will be to take

\[
F(b) \propto \delta \left( b - \gamma t_1^{1/3} \right).
\]

This leads us to a scattering amplitude of the form

\[
A_{\gamma}^P(t) \propto \exp \left[ - \gamma t_1 (1-t/t_1)^{1/3} \right],
\]

which can be rewritten as

\[
A_{\gamma}^P(t) = A_{\gamma}^P(0) \exp \left[ - \gamma t_1 \{ (1-t/t_1)^{1/3} - 1 \} \right].
\]

In a recent paper by the present author, Eq. (4·26) was used for a phenomenological analysis of the forward diffraction peak.\(^9\) It is evident that the delta function in (4·25) should be replaced in a more refined treatment by a smooth function of \(b\) with a peak at \(b \approx \gamma t_1^{1/3}\). In order to see the effect of a finite width of the peak in \(F(b)\), let us assume that

\[
F(b) = 1/(2\varepsilon), \quad b_0 - \varepsilon \leq b \leq b_0 + \varepsilon,
\]

\[
= 0 \quad \text{otherwise},
\]

except for an irrelevant multiplying constant. We immediately find

\[
A_{\gamma}^P(t) \propto e^{-b_0^3} (\sin \varepsilon y/\varepsilon y),
\]

where \(y = (t_1-t)^{1/3}\), and

\[
A_{\gamma}^P(t') \propto \sin x \varepsilon x/\varepsilon x,
\]

where \(x = (t'-t_1)^{1/3}\). Since \(\varepsilon\) is supposed to be appreciably smaller than \(b_0\), the effect of a finite \(\varepsilon\) is not very remarkable for \(A_{\gamma}^P(t)\) and for \(G_{\gamma}^P(a)\). By contrast, the oscillation amplitude of \(A_{\gamma}^P(t')\) is considerably suppressed for \(t'\) above \(t_1\), say, where \(t_1 \approx t_1 + (\pi/\varepsilon)^3\). The most important contribution to the
diffraction peak appears to come from $A_t(t')$ in the interval, $t_1 \leq t \leq t_2$. We expect $t_1$ to be about $1 \text{(BeV/c)}^2$, while $t_2$ is supposed to be of the order of several $\text{(BeV/c)}^2$.

§ 5. Discussion

In the present work the parameter $a$, which seems to have no classical analogue, was introduced in place of the impact parameter. In order to see the physical meaning of $a$ Eq. (2·15) may be helpful. It seems worthwhile to note that the impact parameter amplitude has the Gaussian form (3·4) if $G_t(a)$ behaves like (3·3). Anyhow the present author does not think that it is essential to our approach for the parameter $a$ to be understandable by some classical concept, although it is desirable if possible.

\[
G_{t_1}(a)
\]

Fig. 1. The behavior of $G_{t_1}(a)$ and the three regions.

The behavior of $G_{t_1}(a)$ will be approximately given by that of $G_{t_1}^D(a)$ because $G_{t_1}^D(a)$ is supposed to add only a small background contribution. We saw that $G_{t_1}^D(a)$ has a Gaussian peak at $a = \gamma/2$, which explains the sharp diffraction peak in high-energy elastic scattering. We show the behavior of $G_{t_1}(a)$ in Fig. 1. Since the diffraction scattering is caused by strong absorption due to multiple production, the interval in which the Gaussian peak of $G_{t_1}(a)$ dominates may be called the absorption region. We note here that $\gamma/2$ is almost equal to $(R/2)^2$, where $R$ is the effective radius defined in the previous paper.\(^9\) ($R$ is about $0.85 \times 10^{-13}$ cm for $p$-$p$ scattering and $0.80 \times 10^{-13}$ cm for $\pi$-$p$ scattering.)

It is convenient to divide the whole range of $a$ into three parts: the inner region, the absorption region and the outer region. $G_t(a)$ in the outer region is essentially determined by $A_t(t')$ with small $t'$, as is understood from Eq. (2·2), which gives $G_t(a)$ an exponential tail. As is seen from Eq. (2·8), $A_1(t)$, which is defined by the first term of Eq. (2·1), is determined for large $|t|$ by $G_t(a)$ with small $a$. Experimentally we can never know the asymptotic behavior of $A_1(t)$ because the physical region extends only up to $|t| = 4k^2$ ($2k^2$ for identical
particles) and because $A_{1}(t)$ is very likely to be dominated by $A_{2}(t)$, the second term of Eq. (2.1), in the backward direction. Investigation of large angle scattering ($30^\circ \leq \theta \leq 90^\circ$, say) may, however, give us some information about $G_{t}(a)$ for small but finite $a$.

It was suggested at the end of § 4 that the most important contribution to the diffraction peak will come from $A_{i,t}(t')$ in the interval between $t_{1}$ and $t_{2}$, where $t_{1}$ is about $(\text{BeV}/c)^2$ and $t_{2}$ is supposed to be several $(\text{BeV}/c)^2$. This statement should be regarded as a conjecture rather than a conclusion. A further investigation seems desirable to confirm or reject this conjecture. It is interesting to note that the above interval seems to be roughly the same as the one in which the top Regge trajectory $\alpha_{t}(t)$ of the t channel (the Pomeranchuk trajectory) takes larger values than elsewhere. In this connection it is to be mentioned that the lower end $t_{1}$ of the interval is considerably larger than the branch point $t_{2}(=4\mu^2)$ of $\alpha_{t}(t)$. If we assume that $A_{t}^{p}(t')$ is essentially given by the contribution from the Pomeranchuk trajectory, the point mentioned above might be an indication that $\text{Im} \alpha_{t}(t)$ is small for $t_{1} \leq t \leq t_{2}$. The situation may not be so simple, however, because we have also to take into account the residue function $\beta_{t}(t)$, which becomes complex for $t$ above $t_{2}$. Detailed investigations are undoubtedly required before we can state anything definite.

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References

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