Partially Connected Faddeev-Weinberg-Rosenberg Equation

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In order to give a foundation to the cluster approximations, the Faddeev-Weinberg-Rosenberg formalism is applied. The general partially connected integral equation is obtained. This equation is not fully connected, so it is in general not solvable. Nevertheless, it may be useful for examining and/or introducing approximations.

§ 1. Introduction

In the past few years, many authors have focussed their interest upon the treatment of integral equations for the N-body scattering amplitudes. These discussions have revealed that only fully connected kernels can construct solvable equations, so efforts have been devoted to the construction of an equation of this type. Weinberg and Rosenberg have finally succeeded in constructing an N-body integral equation having the desired property.

The present paper applies the Faddeev-Weinberg-Rosenberg formalism to the examination of the cluster approximations. The cluster equation is obtained. In this equation, we stop the process of connecting the kernels at an intermediate stage. We call this type of equation a partially connected F-W-R equation. Since this type of equation is only partially connected, it is in general not solvable. We can introduce approximations into this type of equation, and continue the connecting process until we get the fully connected F-W-R equations.

In § 2, the partially connected F-W-R equation is obtained. In § 3, we discuss the relation between the equation obtained, Eq. (2·3), and various existing approximations. In § 4, some concluding remarks are given. Typical example of several-particle systems interacting through the two-body force is found in nuclear system. In particular cluster treatment proves its utility in Be8. To take such an example makes the discussion visual.

§ 2. Partially connected F-W-R equation

Let the total system be divided into $M(S)$ subsystems, which may be called clusters. These clusters are denoted by $S_\alpha (\alpha = 1, 2, \cdots, M(S))$. When we treat Be8, one method of dividing the total system is $Be^8 \rightarrow He^4 + He^4$, where $S_1 = S_2 =$
He$^4$ and $M(S) = 2$. Of course, many other such divisions are possible. In what follows, the division of the total system into $S_1, \ldots, S_{M(S)}$ is denoted by $A(M)$, the corresponding number $M(S)$ classifying such a division.

By a procedure parallel to the first paper by Faddeev, the $N$-body Lippmann-Schwinger equation for the division $A(M)$ can be rewritten as

$$ T(A(M)) = \frac{\sum \varphi}{\tau_{\varphi}(M)} T(i; A(M)), \quad (2.1) $$

where $i$ denotes a pair of particles. For a division $A(M)$, the amplitude is defined so that particles within a subsystem $S_\alpha$ interact with each other, but not with particles in other subsystems, and so the summation runs only over the pairs of particles which are found in any one of the $S_\alpha$'s (this condition is expressed by the notation $i < A(M)$).

Suppose no subsystem of $A(m)$ is further divided by the division $A(M)$. To express this condition we use the notation $A(m) < A(M)$. Taking the example of Be$^8$, if $A(M)$ stands for He$^4$+He$^4$, one such $A(m)$ is $d+d$+He$^4$.

Then the amplitude $T(i; A(M))$ satisfies the integral equation

$$ T(i; A(M)) = T(i; A(m)) (1 + G_0 \sum_{A(m) < A(M)} T(j; A(M))), \quad (2.2) $$

for $A(m) < A(M)$. Here $G_0$ is the Green's function in free space. By the notation $A(m) < j$ we mean that we cannot find the pair $j$ in any subsystem of $A(m)$.

Further, we introduce the vector $m(M)$ with $M(S)$ components to classify the division $A(m)$ which follows $A(M)$. The magnitude of the $k$-th component of $m(M)$ denotes the number of parts into which the $k$-th subsystem of $A(M)$ is divided by $A(m)$. Taking again the above example, the division He$^4$+He$^4$ $\rightarrow (d+d) + (n+n+p+p)$ is expressed as $m(M) = (2, 4)$.

Now we shall prove that, if a general form of the partially connected F-W-R equation exists, it should take the form

$$ T(i; A(m)) = \sum_{\mu(m) \geq \mu_0(m)} R(i; A(\mu)) + \sum_{\mu(m) < \mu_0(m)} \mu_0 G_0 \sum_{A(m) < A(M)} T(j; A(m)), \quad (2.3) $$

where $\mu_0(m)$ is a fixed $m(S)$-component vector. Only when $\mu(S)$ exceeds $m(S)$ by unity is this equation the fully connected F-W-R equation. The notation $\mu(m) \geq \mu_0(m)$ means that each component of $\mu(m)$ is not smaller than the corresponding component of $\mu_0(m)$. ($\mu(m) = \mu_0(m)$ denotes that each component of one is equal to that of the other.) We need no definition of the quantity $R(\cdots)$, because the equation (2.3) itself defines $R(\cdots)$; only when Eq. (2.3) holds are the $R$'s meaningful.

We prove this equation by mathematical induction.

PROOF: We insert Eq. (2.3) into Eq. (2.2) and sum it over all possible redivisions (those $A(m)$'s which satisfy $A(m) < A(M)$) with fixed $m(M)$ and $\mu_0(M)$ (the latter condition allows some freedom to $\mu_0(m)$; for example, if $m(M) = (2, 2)$ and $\mu_0(M) = (3, 2)$, two cases are allowed, namely (1, 2, 1, 1)
and (2, 1, 1, 1)).

Then one is left with a constant times
\[ X(m(M); i(M)) T(i; \Delta(M)) = \sum_{\mu(M) \in \mu_k(M)} \{ X(m(M); \mu(M)) R(i; \Delta(\mu)) + Y(m(M); \mu(M)) R(i; \Delta(\mu)) G_0 \sum_{\Delta(\mu) \not\subset \Delta(M)} T(j; \Delta(M)) \} \]  
(2.4)

Here \( X(m(M); \mu(M)) \) denotes the number of possible \( \Delta(\mu) \)'s for a fixed \( m(M) \) and \( \mu(M) \), and is given by
\[ X(m(M); \mu(M)) = \Pi_{k=1}^{M(M)} \{ x(m_k(M); \mu_k(M)) / (m_k(M)!) \} \]  
(2.5)

where, \( x(a, b) \) is defined by the recurrence formula
\[ x(1; b) = 1 \],
\[ x(a, b) = a^b - \binom{a}{a-1} x(a-1, b) - \binom{a}{a-2} x(a-2, b) - \cdots - \binom{a}{1} x(1, b) \]  
(2.5')

and
\[ Y(m(M); \mu(M)) = X(m(M); \mu(M)) - \sum_{k=1}^{M(M)} X(m(M); \mu(M) - k) \]  
(2.6)

where \( k \) denotes the unit vector along the \( k \)-th axis, and \( \Delta(i) \) denotes the division which separates all the particles from each other except for the pair \( i \).

Knowing the relation (2.6), and assuming that the equation (2.3) holds for the given class of divisions denoted by \( \mu_0(M) \), we see that Eq. (2.4) reduces to Eq. (2.3) with \( \Delta(m) \) replaced by \( \Delta(M) \) (note that \( \Delta(m) \subset \Delta(M) \)).

On the other hand, one can in principle construct more highly connected kernels by solving the fully connected F-W-R equations step by step.

Thus the proof is completed. Now we realize that our \( R \)'s are identified to the \( T^{(\alpha)} \)'s of the Rosenberg equation.

\section{3. Approximations}

The partially connected F-W-R equation can be applied to discuss existing

\[^{a)} \text{If a quantity } z \text{ satisfies the relation} \]
\[ a_i z = \sum_{i=1}^{n} \{ a_i x_i + (a_i - a_{i+1}) y_i \} \quad \text{(this stands for (2.4))} \quad \text{(a}_{n+1}=0) \]

\[ z = (\sum_{j=1}^{n} x_j) + y_i \quad \text{(corresponding to Eq. (2.3))} \]

for any \( i \) less than \( n \), it is easily seen that
\[ z = (\sum_{j=1}^{n} x_j) + y_n \]
holds. Here \( i \) is replaced by \( n \).
approximations.
i) The structureless (or hard) cluster models$^{11}$ begin by approximating the total system by a few particular cluster decompositions, which prepare partially connected kernels for the total system. The selection of these cluster modes is mainly dependent on physical insight. Then they are approximated by their lowest poles; these poles originate from the most highly connected part of $T(\Delta(M)), R(\Delta(M))$. In most cases, the subsequent calculations are carried out by the variational method. The equation (2·3) suggests the possibility of decomposing clusters into smaller clusters, and so on.

ii) Cluster models with inner parameters may include further corrections to the approximated kernels.$^{12}$ In general it may be said that variational calculations give the more accurate expectation values the more parameters are included.

iii) The Brueckner approximation$^{13}$ is somewhat different from the above approximations. To obtain the $K$-matrix equation, one should set

$$K_{ij} = V_{ij} + V_{ij}G_0(K_{ij} + K'_{il} + K''_{ij}), \tag{3·1}$$

where $K'_{il} = \sum_{k} K_{ikel}$. Here $i$ or $j$ denotes a particle. To visualise this approximation we rewrite Eq. (3·1) as

$$K(i; S) = T(i; \Delta(i)) (1 + G_0\sum_{j \neq i} K(j; S)), \tag{3·2}$$

where the notation is replaced by that of § 2, $\Delta(i; S-i)$ denotes the division that separates the pair $i$ from the residual system, and $S$ denotes the total system without any division.

The approximations involved are:

a) Only $R(i; \Delta(i; S-i))$ is taken into account in the fully connected F-W-R equation, with respect to each pair.

b) $R(i; \Delta(i; S-i))$ is approximated by $R(i; \Delta(i))$.

Strictly speaking, the latter approximation causes undesirable delta functions to appear. However, this difficulty is removed by the use of the expectation values in the non-degenerate vacuum and self-consistent calculations. Bethe's correction using the Faddeev equation$^{14}$ is parallel to the above discussion, except for the fact that the three-body effect is also included in the calculation.

§ 4. Discussion

In any given case, exact computation for an interacting several-particle system is in practice impossible. The present work gives a classification of divisions by the vector $\mu_r(M)$, which leads us to an (at least formally) impartial method of introducing approximations.

Approximating the $R$'s by their poles together with renormalization is
equivalent to projection upon the cluster states. A more precise treatment
requires further corrections. The next step is to study the behaviour of the
R's in the neighbourhood of their poles.

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