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ELEMENTARY MODELS FOR POPULATION GROWTH AND DISTRIBUTION ANALYSIS

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Abstract—A version of the Lotka-Volterra interaction model is adapted to describe population growth and migration processes in a two-region system. The regions are identified as a metropolis and its non-metropolitan hinterland. Several conditions on growth and migration regimes are imposed. The time behavior of the systems are analyzed, noting especially situations where total depopulation or population explosion eventually occur in one or both populations. Neither growth control nor migration control alone results in a condition of long-run stability in both regions. If at least a momentary condition of zero growth is achieved in both regions, it is possible to maintain finite populations if each population follows a logistic natural growth process and migration flow is proportional to the volume of interaction. It is necessary also that the natural increase limitation is strong relative to migration rates. This result holds even if one population has a net migration advantage over the other.

Analyses of population growth and of population distribution have traditionally been done by separate groups of researchers. A wide variety of population models exists, some focusing on the decision to migrate, some on age structure, some on child spacing, and so on. Frequently, migration and natural increase interact. The crowding of urban environments and depopulation of rural ones is due both to reproductive and migratory processes. Migration is age and sex-selective, affecting gross reproduction rates. Population size, on the other hand, is sometimes used as a measure of migration attractiveness. Little if any theoretical work exists that discusses combined models of population growth and inter-regional distribution.

This paper is an examination of some ideas in population distribution and growth theory with the purpose of determining what kinds of population distributions result when various growth and migration processes are postulated. The shortcomings that will appear in several formulations of the basic model to be presented are valuable in pointing out gaps in conceptual thinking about population processes. An understanding of the various processes and their interactions is necessary in order to offer some foundation for population forecasting.

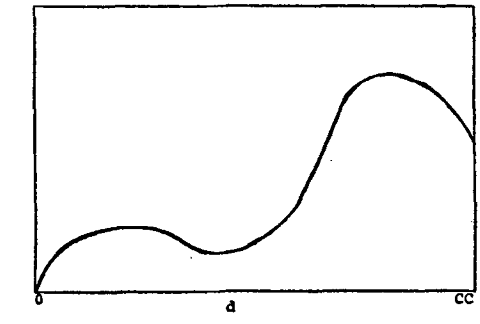
Population Distribution

Any consideration of population distribution is not independent of the numbers of people to be distributed. Unless population growth is controlled, it makes no sense to talk about population distribution. On the other hand, given that population growth is not excessive there are some population spatial distributions that may be preferred over others. A limited population size is a necessary but not a sufficient condition for an acceptable spatial distribution. The term "acceptable" refers to the desirability of the population-settlement structure to the people who must inhabit it. Analyses of space preferences of migrants do show what kinds of surroundings people will accept but they do not show what people would prefer if other alternatives were available. Since so little is known about what is acceptable, this term is defined in the broadest sense in this paper. A population size will be called acceptable for some already inhabited area if it is neither zero nor infinite.

The question of how population is spatially distributed also depends upon scale. Projections of the inter-regional or even inter-state distribution of population may reveal over-all densities and numbers that are not excessive. However, human beings do not exist at that scale. The relevant scale for humans probably includes the area within which a person would circulate in an average week. It would seem desirable to provide just the "right" amount of variance in the frequency distribution of population density in individual circulation areas. In Figure 1, wilderness environments are represented at the extreme left, central cities at the extreme right. The acceptable values of parameters of this distribution are the ones that are unknown.

In the past, models of population behavior over time have not been incorporated into the essentially static models

FIG. 1—Hypothetical Density-Frequency Relationship for an Urbanized Region (0, wilderness environment; CC, central city)



of migration which are found in gravity and potential theory (Olsson, 1965). Although migration itself is a study of changing distribution over time, the time perspective of most models has been short. Much recent work with migration has focused on predicting behavior of individual movers (Moore, 1969). Although this disaggregate focus is preferable to the earlier models on many grounds, it goes even further from the analysis of long-term spatial results of population redistribution and growth.

A Growth and Distribution Model

A classic population model that can be interpreted as a combined migration and growth process is the Lotka-Volterra differential equation model of population interaction. Various forms of this model have been employed in ecology, economics and epidemiology, as well as in demography. Recently, Keyfitz (1968, Chapter 12) has given a summary of this model and its demographic applications.

Consider a closed environment in which all population is classified as belonging either to a metropolitan area (M) or to its non-metropolitan hinterland (H). Each area can grow in two ways: by net migration received from the other or by natural increase. A first approximation to the population growth-migration system follows.

In gravitational models of migration, it is often assumed that a place attracts migrants due to its size. The larger it becomes, the more migrants it attracts, presumably due to the proliferation of jobs produced by the multiplicative effects of size. This probably has some validity, at least over certain ranges of the city size distribution. Suppose it is true in general. The migration components in the two-region case may then be written as

$$\frac{d_1H(t)}{dt} = a_1H(t) + a_{12}M(t) \qquad (1)$$

$$\frac{d_1M(t)}{dt} = a_{21}H(t) + a_2M(t) \qquad (2)$$

with $a_1, a_2 > 0$ and $a_{12}, a_{21} < 0$.

The natural increase component is represented by the balance between births and deaths, or

$$\frac{d_2H(t)}{dt} = [b_H(t) - d_H(t)]H(t) \qquad (3)$$

$$\frac{d_2M(t)}{dt} = [b_M(t) - d_M(t)]M(t) \qquad (4)$$

where $b(t)$ and $d(t)$ are the crude birth and death rates at time t. The simplest natural growth process in a closed population assumes that the balance between births and deaths is a constant in time (Keyfitz, 1968, p. 171), and thus growth is always a constant fraction of population size. The variable growth rates in Equations (3) and (4) are replaced by the respective constant coefficients, a_1' and a_2', both of which are positive, indicating natural increase. Since by form of the demographic equation the migration and natural components are additive, the complete growth equations for the Metropolis-Hinterland systems are:

$$\frac{dH(t)}{dt} = \frac{d_1H(t)}{dt} + \frac{d_2H(t)}{dt}$$

$$= a_{11}H(t) + a_{12}M(t) \qquad (5)$$

$$\frac{dM(t)}{dt} = \frac{d_1M(t)}{dt} + \frac{d_2M(t)}{dt}$$

$$= a_{21}H(t) + a_{22}M(t) \qquad (6)$$

where $a_{11} = a_1 + a_1'$ and $a_{22} = a_2 + a_2'$. The constants a_{11} and a_{22} are positive and as noted above, a_{21} and a_{12} are negative. Each population is increasing exponentially due to inmigration and natural increase and decreasing negative exponentially due to outmigration. These two equations may be taken together to form a second order differential equation in M or H. This has a corresponding characteristic equation

$$\lambda^2 + (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0. \qquad (7)$$

The two roots are

$$(\lambda_1, \lambda_2) = \frac{(a_{11}+a_{22}) \pm \sqrt{(a_{11}-a_{22})^2 + 4a_{12}a_{21}}}{2}. \qquad (8)$$

Since the characteristic determinant is positive, the solutions to the equations are

$$H(t) = c_{11}e^{\lambda_1 t} + c_{12}e^{\lambda_2 t} \qquad (9)$$

$$M(t) = c_{21}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t}. \qquad (10)$$

Either both roots λ_1 and λ_2 are positive, or else one is positive and one is negative. The geographical situation that results, then, is eventual population explosion in both places. The exact course of the process depends on the initial conditions of population balance which are summarized in the constant c_{ij}'s.

The conclusion is that if the population-migration system were to adhere to this model, the result would be unacceptable in terms of both population distribution and size. It would be impossible for either area M or H to have a finite population.

A Logistic Growth and Migration Model

The drastic results of the above model are tempered by its lack of resemblance to actual processes. There are at least two ways that it can be made more realistic.

One modification is suggested by the nature of the migration process. Much of migration apparently relies on an information flow network which spreads messages about work opportunities, housing, etc., between friends or associates in one area and those in another (Hagerstrand, 1957). The amount of migration between two areas at a point in time is some function of the amount of information flowing between them which in turn is more or less proportional to the product of the two population sizes, defining the number of interpersonal contacts between two populations. The more disparate the two populations become, the smaller is the flow. Migration is greatest when they are of equal size. The migration component is expressed as:

$$\frac{d_1H}{dt} = k_{12}H(t)\cdot M(t) \tag{11}$$

$$\frac{d_1M}{dt} = k_{21}H(t)\cdot M(t) \tag{12}$$

where k_{12} and k_{21} may be of any sign.

A second modification is to note that population growth need not conform to the explosive process of equations (5) and (6). If growth can be checked, the natural increase component might be expressed by the second-degree equations:

$$\frac{d_2H}{dt} = k_1H(t) - k_{11}H^2(t) \tag{13}$$

$$\frac{d_2M}{dt} = k_2M(t) - k_{22}M^2(t). \tag{14}$$

The natural growth rate declines proportionately to the square of size, but increases proportionately to size raised to the first power. Such a population would get neither too large nor too small. Equations (13) and (14) are the differential equations of a pair of logistic curves.

Combining the natural and migration components gives:

$$\frac{dH}{dt} = k_1H(t) - k_{11}H^2(t) \pm k_{12}H(t)\cdot M(t) \tag{15}$$

$$\frac{dM}{dt} = k_2M(t) - k_{22}M^2(t) \pm k_{21}H(t)\cdot M(t). \tag{16}$$

These equations may be evaluated by setting the growth rate equations (15) and (16) equal to zero, indicating an equilibrium situation. There are three equilibrium results which would be termed unacceptable:

$$H(t) = 0; M(t) = k_2/k_{22} \tag{17}$$

$$M(t) = 0; H(t) = k_1/k_{11} \tag{18}$$

$$H(t) = 0 = M(t). \tag{19}$$

They correspond to complete depopulation of either the hinterland (17) or the metropolis (18) or both (19). Equations (17) and (18) represent situations where total population is controlled and nonzero but the spatial distribution is unacceptable, whereas (19) is unacceptable for both reasons.

Keyfitz explains a fourth solution to this equation. Setting equations (15) and (16) equal to zero and by dividing respectively by H and M the result is:

$$0 = k_1 - k_{11}H \pm k_{12}M \tag{20}$$

$$0 = k_2 - k_{22}M \pm k_{21}H. \tag{21}$$

Let the sign of k_{12} be negative and that of k_{21} be positive, or in other words, postulate a net migration from the hinterland to the metropolis. This requires k_1 to be very large compared to k_{11} and hence the tendency toward a population ceiling in the hinterland would be very gradual if the metropolitan population were large. It appears that the metropolis could have a stable equilibrium but only if $k_2 + k_{21}H = k_{22}M$, that is if hinterland growth kept pace with metropolitan growth.

The actual solution is obtained by supposing a solution of the form $H = H_1$ and $M = M_1$, where both H_1 and M_1 are positive, and substituting these into a two-dimensional Taylor series expansion of the solution. Let $h = H - H_1$ and $m = M - M_1$. The ordinary differential equations which are obtained from taking the first three terms of the Taylor series are:

$$\frac{dh}{dt} = (-k_{11}H)h(t) + (-k_{12}H)m(t) \tag{22}$$

$$\frac{dm}{dt} = (k_{21}M)h(t) + (-k_{22}M)m(t). \quad (23)$$

Using the same technique for deriving a second-order equation as before and substituting into equation (7) the result is:

$$\lambda^2 + (k_{11}H + k_{22}M)\lambda + (k_{12}Hk_{21}M + k_{22}Mk_{11}H) = 0. \quad (24)$$

The two roots are

$$(\lambda_1, \lambda_2) = \frac{-(k_{11}H + k_{22}M) \pm \sqrt{(k_{11}H - k_{22}M)^2 - 4k_{12}Hk_{21}M}}{2}. \quad (25)$$

The solutions are:

λ_1 and λ_2 real and unequal;
$\lambda_1 = \lambda_2$ and both are real;
λ_1 and λ_2 are complex conjugates.

In the case that the roots are real and equal, either λ_1 or λ_2 must be negative. The equation of $h(t)$ is of the same form as equations (9) and (10). One root will be positive when $k_{12}k_{21} > k_{11}k_{22}$ and the equilibrium will be unstable. This indicates a condition of migration rates whose geometric mean is greater than that of the rates of natural increase limitation.

In the case of identical real roots the root is negative and the solution, say for h, is of the form

$$h(t) = (c_1 + c_2t)e^{-\lambda t} \quad (26)$$

where the c's are determined from the initial conditions.

If $(k_{11}H - k_{22}M)^2 < 4k_{12}Hk_{21}M$, the roots are complex of the form

$$(\lambda_1, \lambda_2) = p \pm qi$$

and the solution is

$$h(t) = e^{-(k_{11}H+k_{22}M)t}[A \cos (qt) + B \sin (qt). \quad (27)$$

where q is the term under the radical of equation (25) and A and B are constants which depend on initial conditions. This equation describes damped oscillations around the equilibrium. Cycles occur when there are small differences between the hinterland and the metropolis in the relative amount of natural increase that has been avoided and the volume of migration is not small. The metropolitan and hinterland populations will fluctuate, eventually approaching a stable level. The rate of approach toward equilibrium increases as the coefficients k_{11} and k_{22} increase.

Thus the only unstable equilibrium of population size or distribution in this process is when the natural increase control coefficients are small compared with migration rates. The principal difference of this model over equations (5) and (6) lies in the supposition that there exists a mechanism which effectively limits natural increase. If this mechanism exists, and even if the metropolitan population does have a migration-attractiveness advantage over the hinterland, the long-run result will be a finite-size population in both regions. It is not necessary that the migration attractiveness of the metropolitan center be subject to control, since the size of the migration stream dwindles as the hinterland population becomes a relatively smaller proportion of the total two region system. This built-in regulator will not work unless natural increase control is of a greater magnitude than migration.

Migration Regulator

It can be shown that a mechanism regulating net migration flow is not sufficient to control population distribution if it fails to limit size. Suppose some mechanism were put into effect which could alter the spatial distribution of migration opportunities or in some more direct way could control population movement. The goal would be some type of population balance between the metropolis and hinterland to avoid excessive crowding of the former and large scale depopulation of the latter. Whenever the difference between M and H was greater than some amount, this "regulator" would reverse the direction of net migration. If natural increase were uncontrolled in both populations, then the equations would be:

$$\frac{dH(t)}{dt} = a_{11}H(t) + a_{12}(M(t) - bH(t)) \tag{28}$$

$$\frac{dM(t)}{dt} = a_{21}\left[H(t) - \frac{1}{b}M(t)\right] + a_{22}M(t). \tag{29}$$

a_{12}, and a_{21}, are the regulator coefficients determining the size of the migration stream. b is the value setting the bound on the amount by which the M and H populations may differ, a_{11} and a_{22} are the constant rates of natural increase. Each coefficient is taken to be positive.

The most sensitive regulator, corresponding to equality of population distribution between M and H would set $b = 1$. Rearranging terms in equations (28) and (29) yields

$$\frac{dH(t)}{dt} = (a_{11} - a_{12})H(t) + a_{12}M(t) \tag{30}$$

$$\frac{dM(t)}{dt} = a_{21}H(t) + (a_{22} - a_{21})M(t). \tag{31}$$

The only possibility for a stable solution is if $a_{11} < a_{12}$ and $a_{22} < a_{21}$. The characteristic equation

$$\lambda^2 + (a_{12} + a_{21} - a_{11} - a_{22})\lambda + (a_{11}a_{22} + a_{11}a_{21} - a_{12}a_{22}) = 0, \tag{32}$$

has two distinct real roots,

$$\lambda_1, \lambda_2 = \frac{(a_{11} - a_{12} + a_{22} - a_{21}) \pm \sqrt{(a_{11} - a_{22} + a_{21} - a_{12})^2 + 4a_{21}a_{12}}}{2}. \tag{33}$$

The resulting equations for M and H are of the same form as equations (5) and (6). In order that both λ_1 and λ_2 be negative the solution requires that $a_{11}(a_{22} - a_{21}) > a_{12}a_{21}$. Since it was assumed that $a_{21} > a_{22}$, the term on the left is negative, which cannot be true if all the a_{ij}'s are individually greater than zero. No stable solutions are possible since the roots are real and they cannot both be negative. The population balance system of equations (30) and (31) is incapable of alone producing an acceptable spatial distribution because it leads to population explosion. The previous model, symbolized in equations (15) and (16) which postulated a natural increase control but imposed no migration regulator proves to be a superior model, since it has several possible stable solutions. By controlling size it also indirectly controlled distribution. If growth control is achieved and if migration follows according to the information flow model, and provided the proper constants are in the model, then the spatial distributions will be acceptable.

Zero Natural Growth Processes

Suppose the population of both M and H could so effectively limit natural increase as to result in zero natural population growth. If migration followed the attraction to size process of equations (1) and (2) then the growth and migration equations for the Metropolitan and Hinterland populations would be

$$\frac{dH_1}{dt} = a_{11}H(t) - a_{12}M(t); \qquad \frac{dM_1}{dt} = -a_{21}H(t) + a_{22}M(t); \tag{34}$$

$$\frac{dH_2}{dt} = 0; \qquad \frac{dM_2}{dt} = 0. \tag{35}$$

Since a condition of zero growth exists,

$$H(t) + M(t) \equiv c \tag{36}$$

for all t.

Substituting equation (36) into (34), the complete growth equations will be

$$\frac{dH(t)}{dt} = a_{11}c - (a_{11} + a_{12})M(t) \tag{37}$$

$$\frac{dM(t)}{dt} = -a_{21}c + (a_{21} + a_{22})M(t). \tag{38}$$

Integrating equation (38) yields

$$M(t) = \frac{a_{21}c + ke^{(a_{21}+a_{22})t}}{a_{21} + a_{22}} \tag{39}$$

where k is the net migration at $t = 0$. This equation can clearly reach its maximum value, c, in a finite time. When $M(t) = c$, then $H(t) = 0$, which would be an unacceptable spatial distribution. The same conclusion could be obtained by letting $H(t) = c - M(t)$ and integrating $dH(t)/dt$.

The most severe population growth control obviously does not provide for an acceptable spatial distribution. The same, unacceptable result would be obtained if a zero natural growth process were combined with the information flow model of migration, with one population having a net migration advantage over the other. It would result in eventual total depletion of one population.

Of the three migration processes discussed in this paper, only the migration regulator of equations (28) and (29) is capable of producing an acceptable spatial distribution under conditions of zero natural growth. This can be shown by integrating the equations

$$\frac{dH(t)}{dt} = a_1(M(t) - bH(t)) \tag{40}$$

$$\frac{dM(t)}{dt} = a_2\left(H(t) - \frac{1}{b}M(t)\right) \tag{41}$$

with $H(t) + M(t) = c$. Like equations (37) and (38) the complete equations have only a migration component. The signs on the constant coefficients indicate a process where one population increases the larger the other becomes and tends to decline, the smaller the other becomes, balancing the distribution.

The second-order equation in H, obtained from equation (40) is

$$\frac{d^2H(t)}{d^2t} + (a_1b + a_2/b)\frac{dH(t)}{dt} = 0, \tag{42}$$

which has the stable solution

$$H(t) = c_1 + c_2e^{-(a_1b+a_2/b)t} \tag{43}$$

and the corresponding result for the metropolitan population is

$$M(t) = c_3 + c_4e^{-(a_1b+a_2/b)t}. \tag{44}$$

This outcome states that under conditions of zero natural growth and regulated migration the expected result does occur. Both populations are finite.

Conclusions

The basic Lotka-Volterra model of population interaction has been examined in the context of migration and natural growth processes. Several combinations of three kinds of natural increase components (constant rate, logistic, zero growth) and three kinds of migration components (attraction to size, information flow, migration regulator) were examined. The models offer an approach to more comprehensive analyses of growth and distribution but they still have definite limitations.

Although the dependence between migration, age structure and sex composition was mentioned in the first paragraph, these effects were not included in the models. Also, all coefficients of natural increase and migration were assumed constant. Another limitation is that only the two-region system was considered. The reason these more general formulations were not discussed is that they become much more difficult to solve analytically, although analytical solutions are possible for some processes that are more complicated than those discussed here. The principle is illustrated well enough with the two-region case. Actual applications of age-specific, n-region models with migration inputs from outside, all with variable coefficients, would be simulated on a digital computer.

The models give no indication as to how these various processes might be effected (e.g. zero natural growth), but rather show what temporal course the population distribution system would follow when certain processes do operate. There are several points worth noting. The strongest conclusion is the intuitive one that there is no possibility of achieving an acceptable spatial distribution when natural increase remains unchecked. On the other hand, a condition of zero natural growth leads to an unacceptable spatial distribution if migration is based either upon the attraction to size or the information flow process. Neither zero growth nor migration control by themselves will produce an acceptable population distribution. Zero growth leads to an acceptable result when it is combined with a migration regulator.

The information-flow/logistic-growth model expressed in equations (15) and (16) may produce acceptable results provided the limit on natural increase is strong enough. However, the solution of the equations for this model depended first on achieving at least a momentary condition of zero growth in both populations. The attraction-to-size/constant-rate model described in equations (1) and (2) has no acceptable solutions. No non-zero values of the constant coefficients can avoid eventual population explosion. On the other hand, under complete control (migration-regulator/zero-growth) no values of the constant coefficients can produce an unacceptable result. This model is expressed in equations (40) and (41). In all other processes either values of the constant coefficients or else the process itself is capable of producing population explosion or extinction.

The ordinary differential equations which have been used to produce these results have yet to be proven very useful in modelling long-term population processes, nor indeed have any models. The kinds of systems that these very simple equations can accurately describe are not the type that involve complex, decision-making processes (Buckley, 1967, Chapter 3). It might be assumed that populations are in some sense free to regulate the constants governing their rates of growth and migration, but may not be as free to change the form of the processes that define the system. These models then indicate which processes are capable of being controlled, which ones cannot be controlled by the coefficients, and which ones will never produce unacceptable solutions for any set of constants that could reasonably be imposed. As expected, when the processes are those which themselves are based on conditions that are likely to exist only when imposed, such as migration regulation, then the constants are much less crucial to the system.

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