Prediction of longitudinal dispersion coefficient using laboratory and field data: relationship comparisons

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ABSTRACT

Knowledge of dispersion of pollutants in streams is necessary for the determination of both the acceptable limits of effluent input and the concentration along the river course. In the far-field, the primary variation of concentration is in one direction and termed longitudinal dispersion; it is independent of the geometrical configuration and type of source. The longitudinal dispersion coefficient represents the dispersive characteristics of a stream and is required to compute the pollutant concentration at downstream locations of the streams. The longitudinal dispersion coefficient can be estimated either from the pollutant concentration profile, stream velocity profile or channel and flow parameters. Many laboratory and field studies have been carried out by several investigators to develop relationships for the longitudinal dispersion coefficient in terms of the known hydraulic characteristics of the stream. This paper evaluates the accuracy of the existing empirical relationships for the prediction of longitudinal dispersion coefficient, using a large volume of data that cover a wide range of flow and channel parameters.

Key words | dispersion coefficient, longitudinal dispersion, mixing, open channel, pollutant concentration

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NOTATION

a	proportionality constant
A	flow area
AS	absolute sensitivity
B	width of channel
b	transverse size of the roughness elements
C	cross-sectional averaged pollutants concentration
C_b	Chezy's coefficient for bed roughness
CC	coefficient of correlation
C_w	Chezy's coefficient for side roughness
D	cross-sectional averaged depth
d	local depth of flow
D_L	longitudinal dispersion coefficient
e	transverse gap between the elements
E^2	efficiency of correlation
h	height of roughness element
h_1	spur length
I	dimensionless integral constant
k	shape factor of the stream
k_S	equivalent sand roughness
l	characteristic length associated with transverse shear
L_B	overall bend length measured along the centerline of the channel
MAE	mean absolute error
MAPE	mean absolute percentage error
MRSE	mean root square error
MSE	mean square error
Q	discharge
q	discharge per unit width
r	pipe radius
R_b	hydraulic radius with respect to bed used for bed roughness
r_c	radius of curvature of bend
RE	relative error
RS	relative sensitivity
R_w	hydraulic radius with respect to sides used for side roughness

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S bed slope
s center to center longitudinal spacing of roughness elements
t time
U cross-sectional averaged velocity
U_* shear velocity
x longitudinal distance
y distance in transverse direction
z distance in the vertical direction
ϕ velocity deviation ratio
η dimensionless shape factor
κ von Karman constant
$\bar{u}$ depth-averaged velocity
$\bar{e}_z$ depth-averaged vertical eddy diffusivity
ε_t local transverse mixing coefficient
E_t cross-sectional average value of ε_t
u'' deviation of local depth-averaged velocity from cross-sectional averaged velocity
D'_m maximum depth in cross-section
$\overline{u''^2}$ mean of square of deviation of depth-averaged velocity from cross-sectional averaged velocity
u' velocity deviation from depth-averaged velocity

INTRODUCTION

The prediction of pollutant concentration downstream of its disposal into the stream is crucial for maintaining a suitable water quality standard. When the pollutant enters streams intentionally or unintentionally, it disperses in every direction due to diffusion and spatial velocity distribution. First the pollutant mixes completely in the vertical direction and then in the transverse direction. After the transverse mixing has taken place, the primary variation of concentration is in one direction; dispersion from that section onwards is referred to as longitudinal dispersion and is independent of the geometrical configuration and type of source. The longitudinal dispersion is primarily due to vertical and transverse velocity shear (Rutherford 1994). Some of the longitudinal spread may be due to a transverse shear resulting from secondary currents. The contribution of longitudinal turbulent diffusion to the longitudinal mixing is generally unimportant because the shear flow dispersion caused by the velocity gradient is much more than the mixing caused by turbulence alone. Moreover, the mixing coefficients due to turbulent mixing and shear flow are additive; the longitudinal turbulent mixing can therefore be neglected.

Invoking the conservation of mass of pollutant, the governing equation for longitudinal dispersion process in a uniform and steady flow for a neutrally buoyant and conservative pollutant is defined (Taylor 1954; Toprak & Cigizoglu 2008):

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = D_L \frac{\partial^2 C}{\partial x^2} \tag{1}$$

where C, t, U, x and D_L are cross-sectional averaged pollutant concentration, time, cross-sectional averaged velocity, longitudinal distance and longitudinal dispersion coefficient, respectively. The longitudinal dispersion coefficient represents the dispersive characteristics of a stream. Several analytical and numerical solutions of Equation (1) for different boundary conditions are available in the literature (Ahmad *et al.* 1999a). Using these solutions, the concentration of a pollutant at downstream locations can be calculated with knowledge of concentration profile at upstream location, flow parameters, geometry of channel and longitudinal dispersion coefficient. D_L can be estimated either from the pollutant concentration profile, stream velocity profile or channel and flow parameters. Error in the estimation of D_L results in inaccurate prediction of downstream concentration profile.

This paper evaluates the accuracy of the existing empirical relationships for D_L using a large volume of data which includes a wide range of flow and channel parameters.

PREVIOUS STUDIES

Determination of D_L using concentration profiles

The longitudinal dispersion coefficient can be obtained by using curves of the temporal variation of concentration (C–t curves) at two or more stations downstream of the injection point. The change of moment method, routing procedure, the diffusive transport method and numerical methods such as that proposed by Ahmad *et al.* (1999a) are available for the determination of D_L using *C–t* curves and other properties of the stream.

The change of moment method is based on the solution of Taylor's one-dimensional dispersion equation and does not make any assumption regarding the initial concentration distribution of the pollutant (Fischer *et al.* 1979). For the one-dimensional dispersion method, the dispersion coefficient is proportional to the rate of change of variance of the *C–t* curves. This method requires a minimum of two *C–t* curves at two stations to find D_L. The method has the drawback that long tails usually observed in *C–t* curves contribute significantly to the second moment, although they correspond to a small mass of flow. The D_L calculated by this method is therefore not very accurate (Fischer 1968).

Fischer (1968) proposed the routing procedure which uses an analytical solution of Equation (1) within a numerical integration. In this method an initial value of D_L is selected and, using a *C–t* curve of the upstream station, the *C–t* curve at the downstream station is obtained by numerical integration. The predicted and observed *C–t* curves at the downstream station are compared. The D_L is obtained by iteration for the best matching of the predicted and observed *C–t* curves. This method also requires a minimum of two *C–t* curves at two stations for the computation of D_L, and provides a better estimate of D_L than the moment method.

Fischer (1968) also proposed the diffusive transport method for the determination of D_L. By definition, D_L is the average mass transport through the entire section divided by the mean concentration gradient. The diffusive transport method provides accurate estimate of D_L but the accuracy of the method is limited by two factors: (1) the channel should be uniform and its geometry must be accurately defined; and (2) measurement of concentration must be made at enough points in the cross-section to adequately define the concentration variation. However, it is difficult to obtain measurements of all the above-mentioned parameters.

The finite difference numerical scheme proposed by Ahmad *et al.* (1999a) can be used to determine D_L using *C–t* curves. This scheme is based on the combined operator approach and is also valid for non-uniform flows. In this scheme, an accurate solution of the advection process is obtained by developing a variable spatial grid so that the root of the trajectory of the concentration characteristics passed through the computation nodes. The solution of the diffusion process is achieved with the help of the Crank–Nicolson scheme by using a weighting coefficient in the temporal axis for the stability of the scheme. The optimum value of D_L is determined for maximum agreement between the predicted and observed *C–t* curves.

Determination of D_L using velocity profiles

Velocity profile in the stream is also used for the determination of D_L. In his classic papers, Taylor (1953, 1954) showed that after balance between advection and diffusion is reached, the longitudinal dispersion coefficient in a vertical plane shear flow is given by Rutherford (1994):

$$D_L = -\frac{1}{d}\int_0^{z=d} u' \int_0^{z} \frac{1}{\bar{e}_z} \int_0^{z} u' \, dz \, dz \, dz \tag{2}$$

where d is local depth of flow; z is distance in the vertical direction; $\bar{e}_z$ is depth-averaged vertical eddy diffusivity; and u' is velocity deviation from depth-averaged velocity $\bar{u}$. Elder (1959) applied Taylor's analysis to two-dimensional shear flow and found that, for a vertical logarithmic velocity profile in an infinitely wide channel, $D_L = 5.93DU*$ where D is cross-sectional averaged depth and U_* is shear velocity. Elder's result does not describe dispersion in natural river channels where transverse shear (due to the variation of depth-averaged velocity) across the cross-section dominates the longitudinal dispersion process.

The vertical shear extends over the depth of flow and transverse shear extends over the width of the channel. In a natural stream (where width >> depth), the deviation of depth-averaged velocity from the cross-sectional averaged velocity is integrated over the width of the stream. This results in a larger value of D_L than if computed from the vertical velocity distribution alone. The effect of vertical velocity variation is therefore neglected in natural streams for the computation of D_L (Fischer *et al.* 1979). Fischer *et al.* (1979) found that the transverse profile of the longitudinal velocity is 100 or more times as important in producing longitudinal dispersion as the vertical profile in the natural streams. Fischer (1967) and Fischer *et al.* (1979) gave the following quantitative estimate of D_L in a real stream by neglecting the vertical velocity profile entirely and applying Taylor's analysis to the transverse velocity profile:

$$D_L = -\frac{1}{A}\int_0^{y=B} du'' \int_0^{y} \frac{1}{\varepsilon_t d}\int_0^{y} du''\,dy\,dy\,dy \qquad (3)$$

where A is flow area; y is distance in transverse direction; B is width of channel; u'' is deviation of local depth-averaged velocity from the cross-sectional averaged velocity U; and ε_t is local transverse mixing coefficient.

Equation (3) is rather difficult to use because detailed transverse profiles of both velocity and cross-sectional geometry are required. As a result, Fischer *et al.* (1979) suggested the following equation:

$$D_L = I\frac{l^2\phi U^2}{E_t} \qquad (4)$$

where I is dimensionless integral constant; l is characteristic length associated with transverse shear; E_t is cross-sectional average of ε_t; and ϕ is velocity deviation ratio which is defined $\phi = \overline{u''^2}/U^2$ where $\overline{u''^2}$ is the mean of square of deviation of depth-averaged velocity from the cross-sectional averaged velocity.

Fischer *et al.* (1979) recommended values of $l = 0.7B$; $I = 0.07$; and $\phi = 0.2$ as reasonable for real streams. Bogle (1997) showed that $I = 0.07$ and $\phi = 0.2$ are high values compared to I and ϕ computed for velocity profiles of Sacramento delta channels.

Ahmad (2003) estimated the longitudinal dispersion coefficient using Taylor and Fischer triple integral methods for the velocity profiles of Sacramento delta channels. It was found that the vertical shear contributes significantly to the longitudinal dispersion process and the contribution of transverse shear depends upon the variation of depth-averaged velocity across the cross-section of the stream rather than the aspect ratio.

Estimation of D_L using flow and channel parameters

Many laboratory and field studies have been carried out by several investigators to develop relationships for D_L in terms of the known hydraulic characteristics of the stream. Empirical relationships for the computation of D_L recommended by the 21 investigators are presented in Table 1. Recently proposed relationships are discussed below and details of others can be found elsewhere (Singh *et al.* 1992; Ahmad 1997; Seo & Cheong 1998; Swamee *et al.* 2000).

Seo & Cheong (1998) collected 59 datasets from 26 streams in the USA from the published reports of Godfrey & Frederick (1970), Yotsukura *et al.* (1970), McQuivey & Keefer (1974) and Nordin & Sabol (1974). They used the routing procedure developed by Fischer (1968) to calculate the observed dispersion coefficient from the field data. Using the one-step Huber method for non-linear multi-regression and the collected data, they proposed an empirical expression as given in Table 1 for the prediction of longitudinal dispersion coefficient D_L, demonstrating its superiority over other reported relationships.

Deng *et al.* (2001) emphasized the importance of the transverse turbulent mixing in addition to other variables of Fischer's triple integral expression and derived expressions for ε_t and D_L as given in Table 1.

Tayfur & Singh (2005) developed an artificial neural network (ANN) model to predict the longitudinal dispersion coefficient in the natural streams and rivers using 71 datasets of hydraulic and geometric parameters and dispersion coefficients measured in 29 streams and rivers in the USA. They claimed that the dispersion coefficient values predicted by the ANN model satisfactorily compared with the observed values and that the model was superior to other predictors for D_L.

Toprak & Savci (2007) and Toprak & Cigizoglu (2008) used fuzzy logic and ANNs, respectively, for the prediction of D_L.

Based on the dimensional and regression analyses, Kashefipour & Falconer (2002) developed another empirical relationship for D_L using published data from 29 rivers in the USA. They showed that the new expression has the least root mean square error (MSE), the highest coefficient of correlation (CC) and the best discrepancy ratio (DR) compared to the relationships proposed by Seo & Cheong (1998), McQuivey & Keefer (1974), Fischer (1975) and Koussis & Rodriguez-Mirasol (1998). The relationship is found to be especially suited to wide rivers, where predictions are very

Table 1 | Relationships for longitudinal dispersion coefficient

No.	Investigator	Relationships	Remarks
1	Taylor (1954)	$D_L = 10.1U{*}r$	Pipe flow, dispersion mainly due to shear flow (r is pipe radius)
2	Elder (1959)	$D_L = 6.3U{*}D$	For wide channel
3	Sumer (1969)	$D_L = 6.23U{*}D$	Considering velocity profile and vertical turbulent diffusion
4	Yotsukura & Fiering (1964)	$D_L = 9.0\,U{*}D$ to $13.0\,U{*}D$	For hydraulically rough and smooth boundary
5	Thackston & Krenkel (1967)	$D_L = 7.25U{*}D\left(\frac{U}{U{*}}\right)^{0.25}$	Using both the laboratory and the field data
6	Fischer (1966)	$D_L = 0.011\frac{U^2B^2}{DU{*}}$	Using data for smooth and rough laboratory flume and field data
7	McQuivey & Keefer (1974)	$D_L = 0.058\frac{Q}{SB}$	Based on data from 18 rivers (S is bed slope and Q is discharge)
8	Jain (1976)	$D_L = \frac{U^2T^3}{kAU{*}}$	Considering shape of the stream, $k = 0.1$–0.2 and increases with B/D
9	Liu (1977)	$D_L = \frac{Q^2}{2U{*}R^3}\left(\frac{U{*}}{U}\right)^2$	Considering transverse mixing coefficient, $\varepsilon_t = 0.23DU{*}$
10	Marivoet & Craenenbroeck (1986)	$D_L = 0.0021\frac{U^2B^2}{DU{*}}$	Based on data from canals
11	Sooky (1969)	$D_L = K'_1 + K' + K''$ $K'_1 = 0.2222\frac{U{*}}{\kappa^3}\frac{D'_m}{a}\eta$ $K'' = \frac{1}{9}\kappa U{*}D'_m$ $K' = aK''$	Considering shape of the stream and velocity distribution (D'_m is max depth in cross-section; a is proportionality constant; η is dimensionless shape factor; κ is von Karman constant)
12	Fukuoka & Sayre (1973)	$\frac{D_L}{DU{*}} = 1.0\left(\frac{Br_c^3}{L_B^2D^2}\right)^{0.86}$ and $\frac{D_L}{RU{*}} = 0.8\left(\frac{r_c^2}{L_BD}\right)^{1.4}$	Considering the meandering effect (r_c is radius of curvature of bend; L_B is overall bend length measured along the centerline of the channel)
13	Hou & Christensen (1976)	$\frac{D_L}{RU{*}} = 1.5239 - 0.1395\log\frac{R}{k_S} + 0.0081\left(\log\frac{R}{k_S}\right)^2$	Considering roughness of bed (k_S is equivalent sand roughness)
14	Magazine *et al.* (1988)	$\frac{D_L}{R_bU} = \frac{D_L}{R_wU} = 75.86(P_r)^{-1.632}$ $P_r = \frac{C_b}{\sqrt{g}}\left(\frac{s}{h}\right)^{0.3}\left(\frac{s}{b}\right)^{0.3}\left(1.5 + \frac{e}{h}\right)$ for rough-side channels $P_r = 1.5\frac{C_w}{\sqrt{g}}\left(\frac{s}{h_1}\right)^{0.3}\left(\frac{s}{D}\right)^{0.3}$ for rough-bed channels	Studied large-scale bed and side roughness (C_w is Chezy's coefficient for sides roughness; C_b is Chezy's coefficient for bed roughness; s is center-to-center longitudinal spacing of roughness elements; h is height of roughness element; h_1 is spur length; b is transverse size of the roughness elements; e is transverse gap between the elements; R_b is hydraulic radius with respect to bed used for bed roughness; R_w is hydraulic radius with respect to sides used for side roughness)
15	Beltaos (1978)	$\frac{D_L}{RU{*}} \propto \left(\frac{B}{R}\right)^2$	Used Sooky's findings

(*continued*)

Table 1 | Continued

No.	Investigator	Relationships	Remarks
16	Asai *et al.* (1991)	$\frac{D_L}{DU_*} = 2.0\left(\frac{B}{D}\right)^{1.5}$	By analyzing field and laboratory data
17	Seo & Cheong (1998)	$\frac{D_L}{BU_*} = 5.915\left(\frac{B}{D}\right)^{0.62}\left(\frac{U}{U_*}\right)^{1.428}$	Using data from 26 streams in the USA
18	Ahmad *et al.* (1999b)	$\frac{D_L}{q} = 0.4\left(\frac{B}{R}\right)^{2.12}\left(\frac{U}{U_*}\right)^{-0.72} S^{0.19}$	Using large volume of laboratory and field data
19	Deng *et al.* (2001)	$\frac{D_L}{BU_*} = \frac{0.15}{8\varepsilon_t}\left(\frac{B}{D}\right)^{5/3}\left(\frac{U}{U_*}\right)^2$ where $\varepsilon_t = 0.145 + \left(\frac{1}{3,520}\right)\left(\frac{B}{D}\right)^{1.38}\left(\frac{U}{U_*}\right)$	By giving importance of the transverse turbulent mixing in addition to other variables of Fischer's triple integral expression
20	Kashefipour & Falconer (2002)	$\frac{D_L}{BU_*} = 10.612\left(\frac{U}{U_*}\right)^2$	Using published data from 29 rivers in the USA
21	Sahay & Dutta (2009)	$\frac{D_L}{BU_*} = 2.0\left(\frac{B}{D}\right)^{0.96}\left(\frac{U}{U_*}\right)^{1.25}$	Using published data from 29 rivers in the USA

close to the measured dispersion coefficients. They also pointed out that the ratio of the cross-sectional mean velocity to the shear velocity is the most influencing parameter for the accurate prediction of the longitudinal dispersion coefficient.

Sahay & Dutta (2009) derived an expression for prediction of the longitudinal dispersion coefficient in natural rivers using a genetic algorithm and published data from 29 rivers in the USA. They observed that the performance of their relationship is better than the relationships proposed by Seo & Cheong (1998), Deng *et al.* (2001), Fischer (1975) and Kashefipour & Falconer (2002).

Riahi-Madvar *et al.* (2009) developed a new flexible tool to predict the longitudinal dispersion coefficient using adaptive neuro-fuzzy inference system (ANFIS). They found that dispersion coefficient values predicted by the ANFIS model satisfactorily compared with the observed values and also provides better prediction than the relationships proposed by Elder (1959), Fischer (1967), Liu (1977), Seo & Cheong (1998), Koussis & Rodriguez-Mirasol (1998), Deng *et al.* (2001) and Kashefipour & Falconer (2002).

Ahmad *et al.* (1999b) proposed a relationship for D_L using a large volume of data collected by several investigators in the field as well as from the laboratories. Similar to relationships proposed by other investigators, Ahmad *et al.*'s (1999b) relationship can also be expressed in terms of non-dimensional parameters D_L/BU_*, B/D and U/U^*. Note that most of the investigators used limited datasets to propose a relationship for D_L.

DESCRIPTION OF COLLECTED DATA

To evaluate the accuracy of the existing relationships for the prediction of longitudinal dispersion coefficient using flow and channel parameters, a wide range of longitudinal dispersion laboratory and field data has been collected from the different sources (Table 2). Depending on the availability of flow and channel properties, concentration profile of tracer, time of travel of tracer, etc., these datasets have been grouped into the following three categories:

1. Table 2, 1–8: complete observed concentration profiles of tracer (*C*–*t* curves) are available along with flow and channel properties.
2. Table 2, 9–11: complete observed *C*–*t* curves are *not* available, but salient points of *C*–*t* curves are known and the flow properties and channel properties are also known.
3. Table 2, 12–16: *C*–*t* curves are not known but values for D_L, flow properties and channel properties are known.

Singh (1987) and Ahmad (1997) performed experiments in rectangular flumes of bed width 0.40 and 0.20 m, respectively, in the Hydraulics Laboratory of the

Table 2 | Range of data used in this study

No.	Investigators	Type of study	No. of data	D_L (m^2 s^{-1}) Min. Max.	*B* (m) Min. Max.	*D* (m) Min. Max.	*Q* (m^3 s^{-1}) Min. Max.	*S* (–) Min. Max
1	Singh (1987)	Laboratory	28	0.00147 0.0805	0.4 0.4	0.034 0.155	0.008 0.041	0.0013 0.00603
2	Ahmad (1997)	Laboratory	13	0.02273 0.05678	0.20	0.058 0.1388	0.00589 0.02188	0.001488 0.004546
3	Fischer (1966)	Laboratory	10	0.01 0.5306	0.317 1.100	0.021 0.184	0.00603 0.05101	0.000267 0.00104
4	Beltaos & Day (1976)	Field	6	11.0 20.0	9.96 54.6	2.49 3.74	70.51 80.94	0.0001 0.0001
5	Nordin & Sabol (1974)	Field	13	11.53 1,417.81	12.34 739.14	0.25 17.51	1.26 7,126.0	0.000007 0.0006
6	Berkas (1982)	Field	5	14.14 81.465	29.87 80.60	1.25 2.393	35.2335 39.62	0.000273 0.00352
7	James & Helinsky (1984)	Field	3	10.35 12.75	9.36 12.95	0.30 0.61	0.62 2.22	0.00341 0.00341
8	McQuivey & Keefer (1976)	Field	4	139.68 295.07	867.00 867.00	16.76 16.76	18,454.27 21,636.54	0.000002 0.000002
9	Graf (1984)	Field	22	8.85 2,880.0	8.83 118.77	0.17 42.25	0.4896 45.1385	0.00235 0.000289
10	Calandro (1978)	Field	23	0.52 6,800.0	7.92 268.22	0.24 3.66	1.11 251.45	0.000007 0.000365
11	Taylor (1970)	Field	9	4.29 350.0	34.9 48.46	0.29 0.88	2.10 21.24	0.000556 0.000956
12	Hou & Christensen (1976)	Field and laboratory	45	0.01914 46.445	0.90 67.96	0.15 7.46	0.2269 118.657	0.0000002 0.001334
13	Dobran (1982)	Field	4	1.24 3.36	8.6 12.0	0.285 0.30	0.95 1.18	0.00087 0.001583
14	McQuivey & Keefer (1974)	Field	36	4.64 891.84	12.5 201.17	0.25 3.84	0.99 933.0	0.00015 0.00355
15	Miller & Richardson (1974)	Laboratory	8	0.0522 6.12	0.5974 0.5974	0.1247 0.132	0.0231 0.0606	0.001 0.0296
16	Liu (1977)	Field	13	6.5 1,490.0	12.4 183.95	0.38 3.05	0.99 957.0	0.000121 0.00292

Civil Engineering Department of the University of Roorkee. They used water tracing (WT) rhodamine as a tracer and fluorometer for the measurement of tracer concentration. The experiments were carried out in clear-water flows, as well as in sediment-laden flows.

Fischer (1966) conducted the laboratory experiments in re-circulating rectangular and trapezoidal flumes with smooth as well as rough bed and sides at the WM Keck Laboratory of Hydraulics and Water Resources, California Institute of Technology. He used salt solution as a tracer for the study and measured the salt concentration by conductivity probe.

Beltaos & Day (1976) conducted the longitudinal dispersion study using the WT rhodamine fluorescent dye in Lesser Slave River, Alberta, which is relatively prismatic in the study reach with a constant discharge for long period. Nordin & Sabol (1974) compiled the longitudinal dispersion data in terms of channel and flow parameters along with *C*–*t* curves from the published and unpublished sources for the Antietam Creek, Monocacy River, Red River, Sabine River and Mississippi River. W. R. Berkas (personal communication, 1982) reported longitudinal dispersion data for the Chariton river from Rathbun reservoir to Prairie Hill, Missouri. James & Helinsky (1984) reported the three dye studies on the Jones Fall between Lake Roland Dam and a point 7.2 km downstream at Baltimore, Maryland. McQuivey & Keefer (1976) reported the dispersion study data of the Clinch River, Missouri River and Mississippi River.

Graf (1984) reported the dye travel time and longitudinal dispersion data in 10 Illionois streams from 1975 to 1982 in the form of travel time of the leading edge, peak concentration, trailing edge and recovery ratio of the tracer. Calandro (1978) reported the travel time and dispersion of solutes in 18 streams in Louisiana that were measured by injecting a fluorescent water tracer. Taylor (1970) reported the three time-of-travel studies using a fluorescent dye in Monocacy River in Maryland during 1967–1968 at 95, 70 and 35% dependable flow.

Hou & Christensen (1976) carried out dispersion studies in natural and man-made streams and in a flume of bed width 0.30 m. They conducted field studies in 12 different streams and canals at 10 locations in Florida. They used WT rhodamine as a tracer and fluorometer with strip recorder for the measurement of tracer concentration. Dobran (1982) conducted the longitudinal dispersion study in Miljacka River (Bosnia and Herzegovina), a mountainous stream of length of 1,463 m, and used sodium bichromate as a tracer. McQuivey & Keefer (1974) studied the time of travel of tracer in 18 natural streams in 14 states and used the Fischer routing method to estimate the longitudinal dispersion coefficient. Miller & Richardson (1974) conducted laboratory experiments in a rectangular channel of bed width 0.60 m at the Engineering Research Center at Colorado State University. They used WT rhodamine as a tracer and fluorometer with strip recorder for the measurement of tracer concentration. Liu (1977) used the dispersion study data from Clinch River, Copper Creek and Powel River from Fischer (1968), Godfrey & Frederick (1970) and Yotsukura *et al.* (1970) to propose a relationship for the longitudinal dispersion coefficient.

The *C*–*t* curves for datasets (1) were developed with the use of travel time of the leading edge, peak concentration, trailing edge, peak concentration and area of *C*–*t* curves or recovery ratio of the tracer. D_L is computed using the available *C*–*t* curves for dataset (1) and generated *C*–*t* curves for dataset (2) by the Ahmad *et al.* (1999a) scheme. In this scheme, the initial approximate value of D_L is calculated by the change of moment method. A grid for D_L is then generated taking the initial grid size to be equal to one-tenth the initial D_L value. The observed *C*–*t* curve at the first station is used as input and *C*–*t* curves at the other downstream stations are predicted by the Ahmad *et al.* (1999a) scheme. The agreement between predicted and observed *C*–*t* curves is judged in terms of the error (ERS), which is defined:

$$\mathrm{ERS} = \left(1 - \frac{\text{Common area between observed and predicted } C-t \text{ curves}}{\text{Area of observed } C-t \text{ curve}}\right) \times 100\% \tag{5}$$

The common area between observed and predicted *C*–*t* curves is shown in Figure 1. Using the value of D_L at each of the grids, the error ERS is calculated. The value of D_L producing the minimum value of ERS is considered the optimum value for the chosen grid. This derived value of D_L is refined by further bisecting the search grid size until the desired accuracy is attained, i.e. the percentage error in ERS for two consecutive D_L is less than 0.5 (Ahmad *et al.* 1999a).

COMPARATIVE ANALYSIS

Of the various relationships proposed by previous investigators for the prediction of longitudinal dispersion coefficient (Table 1), six recently proposed relationships were evaluated for their accuracy using 226 laboratory and

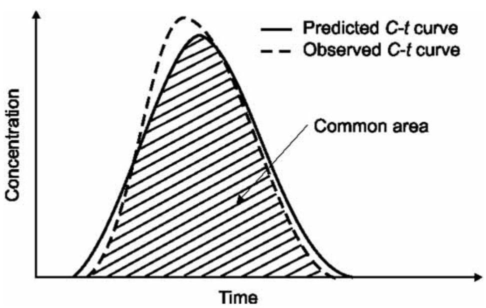

Figure 1 | Common area between observed and predicted *C*–*t* curves.

field datasets collected in the present study (summarized in Table 2). These include relationships proposed by Sahay & Dutta (2009), Seo & Cheong (1998), Asai *et al.* (1991), Ahmad *et al.* (1999b), Kashefipour & Falconer (2002) and Deng *et al.* (2001). For a quantitative evaluation of the difference between observed and predicted D_L from the selected relationships, the discrepancy ratio (as defined by White *et al.* (1973)) was used as an error measure, defined:

$$DR = \log\left(\frac{D_L(\text{predicted})}{D_L(\text{observed})}\right) \quad (6)$$

For $DR = 0$, the predicted D_L is identical to the observed value. For positive (negative) values of DR, the predicted value of the dispersion coefficient is greater (smaller) than the observed value. Accuracy is defined as the frequency of cases for which the discrepancy ratio is within a suitable range for the total number of data. Discrepancy ratios for each relationship for the 226 laboratories and field datasets are shown in Figure 2. It can be noted from Figure 2 that the frequency of data within $DR = \pm 0.5$ are 87, 82, 136, 150, 76 and 88 and within $DR = \pm 1.0$ are 214, 197, 221, 222, 153 and 209 for Sahay & Dutta (2009), Seo & Cheong (1998), Asai *et al.* (1991), Ahmad *et al.* (1999b),

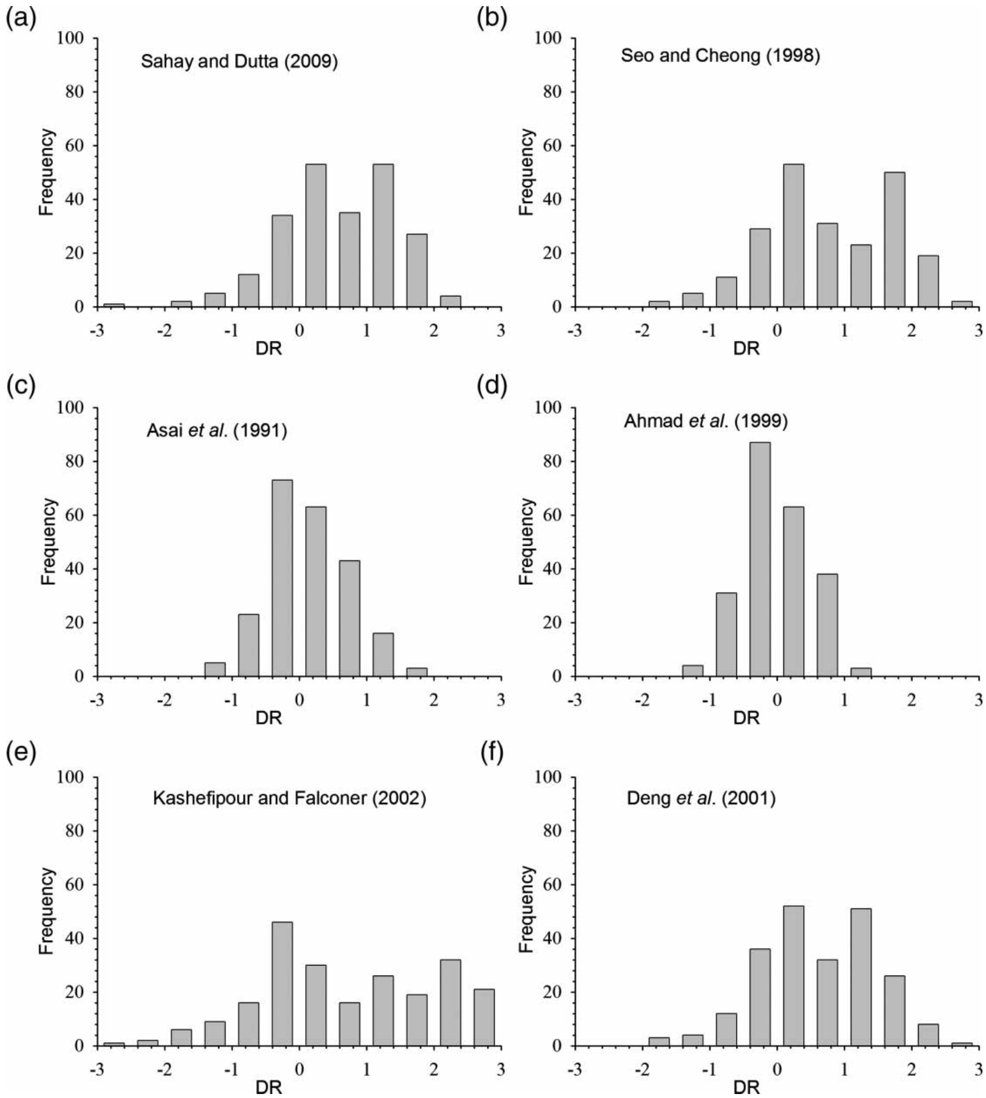

Figure 2 | Distribution of DR for selected relationships.

Kashefipour & Falconer (2002) and Deng *et al.* (2001), respectively. This clearly indicates that the relationship of Ahmad *et al.* (1999b) has maximum accuracy, followed by Asai *et al.* (1991).

A close examination of Table 1 reveals that the most promising dimensionless parameters affecting the dimensionless longitudinal dispersion coefficient are B/D or B/R and U/U_*. The variation of DR with B/D and U/U_* for each relationship are therefore shown in Figures 3 and 4. The relationships proposed by Sahay & Dutta (2009), Seo & Cheong (1998), Kashefipour & Falconer (2002) and Deng *et al.* (2001) overestimate the D_L for $B/D < 50$ and $U/U_* > 6.5$ and underestimate for $B/D > 50$ and $U/U_* < 6.5$. Data points are evenly distributed about $DR = 0$ for the entire range of B/D and U/U_* studied in this study for Asai *et al.* (1991) and Ahmad *et al.* (1999b). In the case of Asai *et al.* (1991) compared to Ahmad *et al.* (1999b), as depicted by Figures 3(c), 3(d), 4(c) and 4(d), a scattering of points is more evident. It is therefore concluded that the relationship of Ahmad *et al.* (1999b) is better than the other relationships for a wide range of B/D and U/U_*, followed by Asai *et al.* (1991).

The accuracy of selected relationships is also checked using the statistical performance indices, which are used to measure the extent of the agreement between the

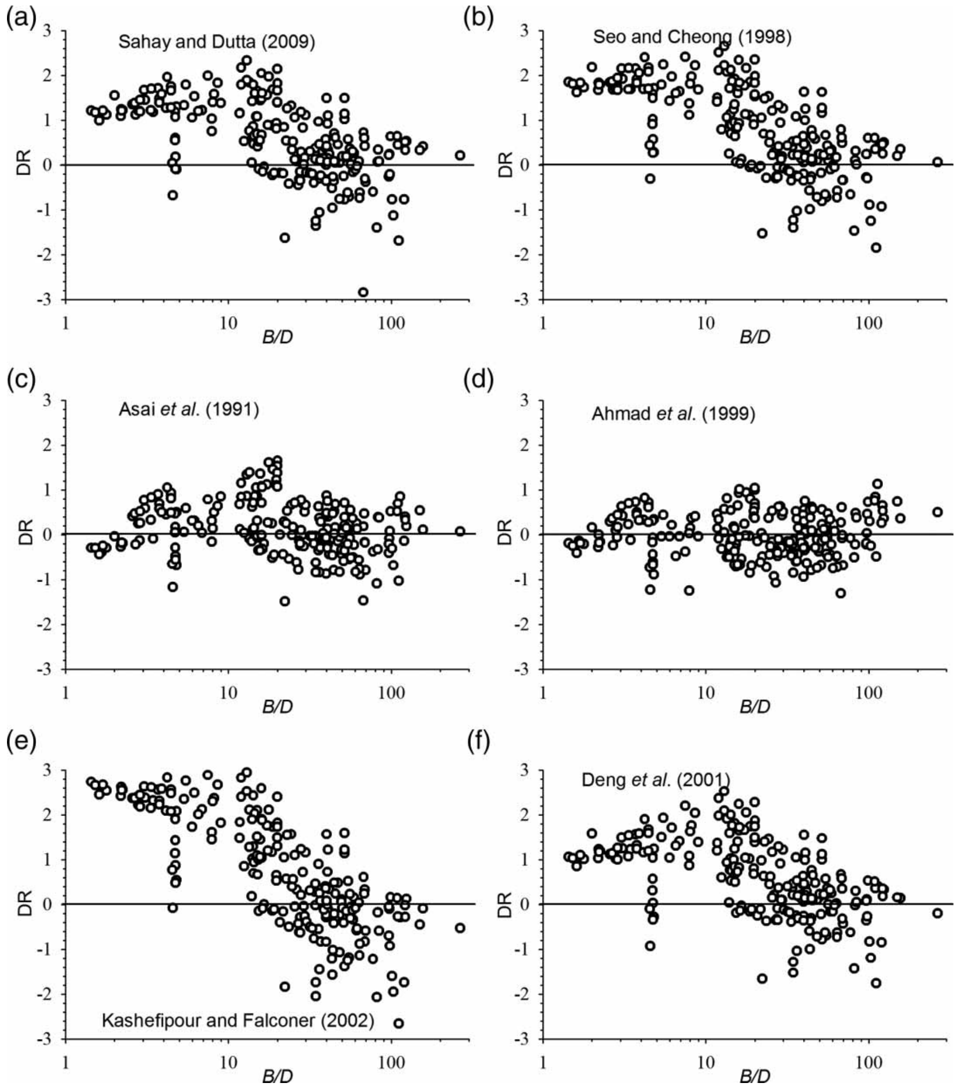

Figure 3 | Variation of computed DR with B/D for selected relationships.

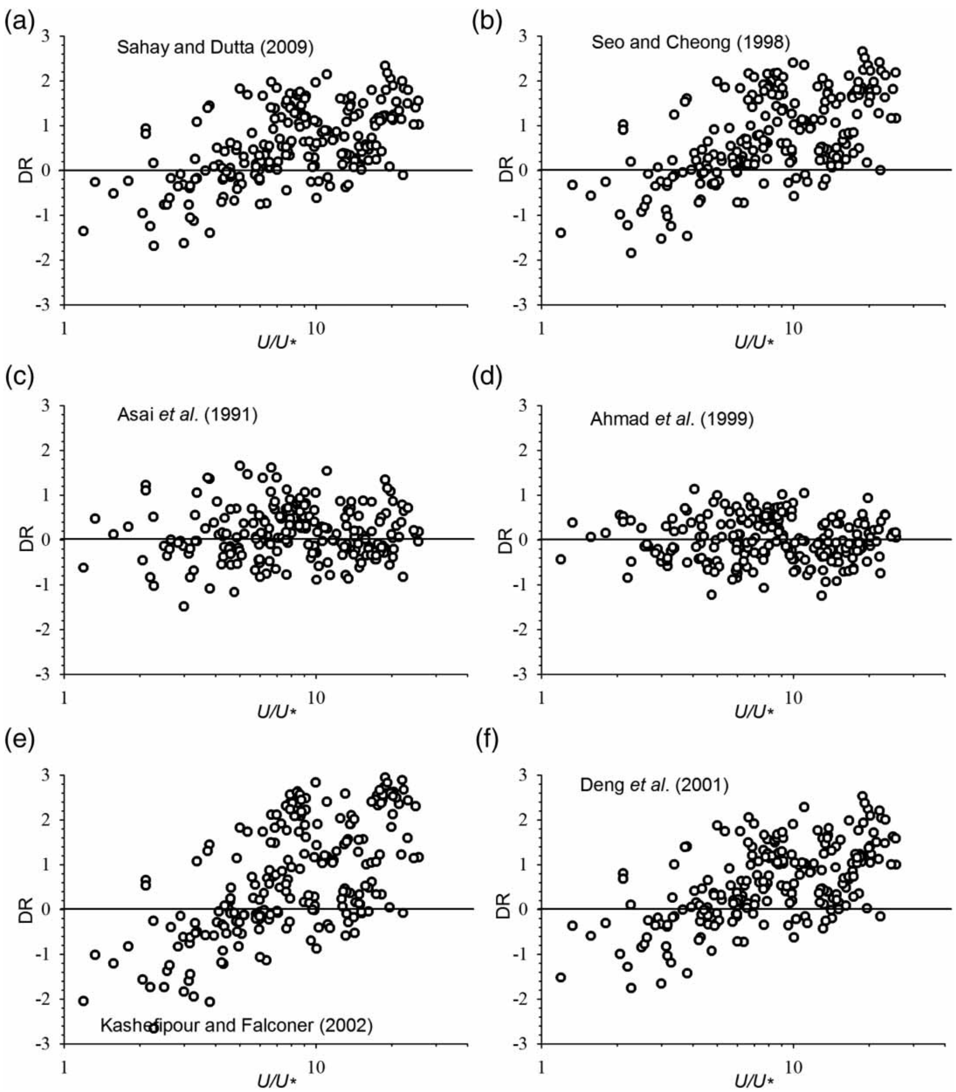

Figure 4 | Variation of computed DR with $U/U*$ for selected relationships.

observed and predicted dispersion coefficients. If Y is the observed value and $\hat{Y}$ is the corresponding predicted value, the different performance indices may be defined (Maier & Dandy 1996; Rajurkar *et al.* 2004; Riahi-Madvar *et al.* 2009):

Coefficient of correlation,

$$\mathrm{CC} = \frac{N\sum Y\hat{Y} - \sum Y \sum \hat{Y}}{\sqrt{N\sum Y^2 - (\sum Y)^2}\sqrt{N\sum \hat{Y}^2 - \left(\sum \hat{Y}\right)^2}} \tag{7}$$

Mean absolute error,

$$\mathrm{MAE} = \frac{1}{N}\sum_{i=1}^{N}\left|Y_i - \hat{Y}_i\right| \tag{8}$$

Mean square error,

$$\mathrm{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(Y_i - \hat{Y}_i\right)^2 \tag{9}$$

Mean root square error,

$$\mathrm{MRSE} = \sqrt{\frac{\sum_{i=1}^{N}\left(Y_i - \hat{Y}_i\right)^2}{N}} \quad (10)$$

Mean absolute percentage error,

$$\mathrm{MAPE} = \frac{100}{N}\sum_{i=1}^{N}\frac{\left|Y_i - \hat{Y}_i\right|}{|Y_i|} \quad (11)$$

Efficiency of correlation,

$$E^2 = \frac{\sum_{i=1}^{N}(Y_i - \bar{Y}_i)^2 - \sum_{i=1}^{N}\left(Y_i - \hat{Y}_i\right)^2}{\sum_{i=1}^{N}(Y_i - \bar{Y}_i)^2} \quad (12)$$

In Equations (7)–(12), N is the number of datasets ($N = 226$ in the present study). The values of CC, mean absolute error (MAE), MSE, mean root square error (MRSE), mean absolute percentage error (MAPE) and efficiency coefficient (E^2) of each relationship are listed in Table 3. The CC and E^2 of relationships proposed by Asai *et al.* (1991) and Ahmad *et al.* (1999b) are higher than those of Sahay & Dutta (2009), Seo & Cheong (1998), Kashefipour & Falconer (2002) and Deng *et al.* (2001), and MAE, MSE, MRSE and MAPE are lower. This indicates that the relationships proposed by Asai *et al.* (1991) and Ahmad *et al.* (1999b) perform better than the other relationships in terms of various performance indices. However, Ahmad *et al.*'s (1999b) relationship has an edge over Asai *et al.*'s (1991) relationship. The better performance of Ahmad *et al.*'s (1999b) relationship is due to inclusion of dimensionless terms U/U_* and S in the relationship compared to Asai *et al.*'s (1991) relationship.

SENSITIVITY ANALYSIS

A sensitivity and error analysis has been carried out to identify the most critical dimensionless input parameter of Ahmad *et al.*'s (1999b) relationship of D_L/q affecting the longitudinal dispersion coefficient. This is carried out with the use of average values of dimensionless input and output parameters from 226 datasets. The analysis is based on the assumption that the errors in each input variable are independent. The average values of dimensionless input parameters B/R, U/U_* and S for the data used in the present study are 35.5, 9.93 and 0.00139, respectively.

If an error ΔY in the output is defined as the difference between values of output predicted for inputs X and $X + \Delta X$, then the error can be estimated as the absolute sensitivity $\mathrm{AS} = \Delta Y/\Delta X$. Here the output is $Y = D_L/q$ and input $X = B/R$, U/U_* and S. The error could also be expressed in a relative form $\mathrm{RE} = \Delta Y/Y$. The error ΔY in output is essentially the deviation sensitivity with ΔX being the error. The relative sensitivity can be expressed as $\mathrm{RS} = (X\,\Delta Y)/(Y\,\Delta X)$ (ASCE 1996; Sahay & Dutta 2009).

The sensitivity and error analysis is carried out by changing each input parameter by ±10%. Tables 4 and 5 provide the results of these analyses, which indicates that B/R is the most sensitive parameter followed by U/U_* and S. The relative sensitivity of B/R is about 3.4 times that of U/U_* and about 12.4 times that of S for a 10% increment in X. However, for a 10% reduction in X, the relative sensitivity of B/R is about 2.53 times that of U/U_* and about 10.15 times that of S. The prediction accuracy of the relationship

Table 3 | Performance indices of relationships for the longitudinal dispersion coefficient

Investigators	CC	MAE	MSE	MRSE	MAPE	E^2
Sahay & Dutta (2009)	0.48	131.63	337,423.94	38.64	1,421.25	−7.66
Seo & Cheong (1998)	0.44	173.30	651,348.13	53.68	3,247.41	−15.71
Asai *et al.* (1991)	0.62	53.83	24,780.56	10.47	291.37	0.36
Ahmad *et al.* (1999b)	0.63	53.33	24,096.82	10.33	131.07	0.38
Kashefipour & Falconer (2002)	0.37	158.52	571,186.91	50.27	8,571.19	−13.66
Deng *et al.* (2001)	0.46	126.53	309,397.98	37.00	1,726.11	−6.94

Table 4 | Sensitivity and error analysis of the Ahmad *et al.* (1999b) relationship (X is input parameter; ΔX is 10% increment in X; ΔY is % change in output (D_L/q))

X	ΔX	ΔY	AS	RE	RS
B/R	3.55	9.52	2.68	0.224	2.24
U/U^*	0.993	−2.82	−2.84	− 0.066	−0.66
S	0.0001393	0.78	5,600	0.018	0.18

Table 5 | Sensitivity and error analysis of the Ahmad *et al.* (1999b) relationship (X is input parameter; ΔX is 10% reduction in X; ΔY is % change in output (D_L/q))

X	ΔX	ΔY	AS	RE	RS
B/R	3.55	−8.51	−2.40	−0.20	−2.0
U/U^*	0.993	3.35	3.37	0.079	0.79
S	0.0001393	−0.84	−6,030	−0.0197	−0.197

proposed by Ahmad *et al.* (1999b) therefore depends heavily on B/R, followed by U/U_*.

CONCLUSIONS

The existing empirical relationships for the prediction of longitudinal dispersion coefficient have been evaluated for their accuracy using a large volume of data comprising a wide range of flow and channel parameters. Various performance indices have been computed for the quantitative evaluation of differences between observed and predicted longitudinal dispersion coefficient from the selected relationships. The frequency of data within discrepancy ratio DR = ±0.5 are 87, 82, 136, 150, 76 and 88 and within DR = ±1.0 are 214, 197, 221, 222, 153 and 209 for relationships proposed by Sahay & Dutta (2009), Seo & Cheong (1998), Asai *et al.* (1991), Ahmad *et al.* (1999b), Kashefipour & Falconer (2002) and Deng *et al.* (2001), respectively. The maximum accuracy of Ahmad *et al.*'s (1999b) relationship followed by Asai *et al.* (1991) was observed. The most promising dimensionless parameters affecting the dimensionless longitudinal dispersion coefficient are B/D or B/R and U/U_*. The relationships proposed by Sahay & Dutta (2009), Seo & Cheong (1998), Kashefipour & Falconer (2002) and Deng *et al.* (2001) overestimate the value of D_L for $B/D < 50$ and $U/U_* > 6.5$ and underestimate for $B/D > 50$ and $U/U_* < 6.5$. The relationships of Asai *et al.* (1991) and Ahmad *et al.* (1999b) perform better for the entire range of B/D and U/U_*; however, a scattering of points is evident in the case of Asai *et al.* (1991) compared to Ahmad *et al.* (1999b). Sensitivity analysis indicates that the prediction accuracy of the Ahmad *et al.* (1999b) relationship depends heavily on B/R and U/U_*. The relative sensitivity of B/R is about 3.4 times that of U/U_* and about 12.4 times that of S for a 10% increment in independent parameter X. For a 10% reduction in X the relative sensitivity of B/R is about 2.53 times that of U/U_* and about 10.15 times that of S.

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