On the migration-induced resonances in a system of two planets with masses in the Earth mass range

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ABSTRACT
We investigate orbital resonances expected to arise when a system of two planets, with masses in the range 1–4 M\textsubscript{⊕}, undergoes convergent migration while embedded in a section of gaseous disc where the flow is laminar. We consider surface densities corresponding to 0.5–4 times that expected for a minimum mass solar nebula at 5.2 au. For the above mass range, the planets undergo type I migration. Using hydrodynamic simulations, we find that, when the configuration is such that convergent migration occurs, the planets can become locked in a first-order commensurability for which the period ratio is \((p + 1)/p\) with \(p\) being an integer and migrate together maintaining it for many orbits. Slow convergent migration results in commensurabilities with small \(p\) such as 1 or 2. Instead, when the convergent migration is relatively rapid as tends to occur for disparate masses, higher \(p\) commensurabilities are realized such as 4:3, 5:4, 7:6 and 8:7. However, in these cases the dynamics is found to have a stochastic character with some commensurabilities showing long-term instability with the consequence that several can be visited during the course of a simulation. Furthermore, the successful attainment of commensurabilities is also a sensitive function of initial conditions. When the convergent migration is slower, such as occurs in the equal-mass case, lower \(p\) commensurabilities such as 3:2 are obtained, which show much greater stability.

Resonant capture leads to a rise in eccentricities that can be predicted using a simple analytic model that assumes the resonance is isolated, constructed in this paper. We find that, once the commensurability has been established, the system with an 8:7 commensurability is fully consistent with this prediction.

We find that very similar behaviour is found when the systems are modelled using an \(N\)-body code with simple prescriptions for the disc–planet interaction. Comparisons with the hydrodynamic simulations indicate reasonably good agreement with predictions for these prescriptions obtained using the existing semi-analytic theories of type I migration.

We have run our hydrodynamic simulations for up to \(10^3–10^4\) orbits of the inner planet. Longer times could only be followed in the simpler \(N\)-body approach. Using that, we found that, on the one hand, an 8:7 resonance established in a hydrodynamic simulation could be maintained for more than \(10^5\) orbits. On the other hand, other similar cases show instability leading to another resonance and ultimately a close scattering.

There is already one known example of a system with nearly equal masses in the range of several Earth masses, namely the two pulsar planets in PSR B1257+12, which are intriguingly, in view of the results obtained here, close to a 3:2 commensurability. This will be considered in a future publication.

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Future detection of other systems with masses in the Earth mass range that display orbital commensurabilities will give useful information on the role and nature of orbital migration in planet formation.

Key words: planetary systems: formation.

1 INTRODUCTION

The increasing number of extrasolar multiplanet systems, their diversity and dynamical complexities provide a strong motivation to study the evolution and stability of such systems. One of the important features connected with planetary system evolution is the occurrence of mean motion resonances, which may relate to conditions at the time of or just after the process of formation. There are some well-known examples of systems of celestial bodies exhibiting mean motion resonances both within our Solar system [Neptune and Pluto; Io, Ganymede and Europa (see e.g. Goldreich 1965) and outside such as Gliese 876 (Marcy et al. 2001), HD 82943 (Mayor et al. 2004) and 55 Cancri (McArthur et al. 2004). As the latter examples involve planets with masses in the Jovian mass range, most of the investigations appropriate to extrasolar planets have focused on giant planets. However, it is likely that planetary systems around other stars may harbour planets with masses in the Earth mass range as well. These should be revealed by future space-based missions, such as Darwin, COnvection, ROtation and Transits (COROT), Kepler, Space Interferometry Mission (SIM) and Terrestrial Planet Finder (TPF).

Meanwhile, it is important to establish the main features of the evolution of low-mass planets embedded in a gaseous disc and, in particular, to determine the types of resonant configurations that might arise when a pair of such planets evolves together. The disc–planet interaction naturally produces orbital migration through the action of tidal torques (Goldreich & Tremaine 1980; Lin & Papaloizou 1986), which in turn may lead to an orbital resonance in a multiplanet system, e.g. (Snellgrove, Papaloizou & Nelson 2001; Lee & Peale 2002; and see also Ji et al. 2003). For low-mass planets, the disc undergoes small linear perturbations that induce density waves that propagate away from the planet. The angular momentum these waves transport away results in rapid orbital migration called type I migration (Ward 1997). In this type of migration, when the disc is laminar and inviscid, the planet is embedded and the surface density profile of the disc remains approximately unchanged. The rate of migration is proportional to the mass of the planet and the mass of the central star, and inversely proportional to the disc surface density at the local sound speed and angular velocity. Here, \( \tau_r = \frac{r_p}{\dot{r}_p} = \frac{W_m}{m_{\text{planet}}} \frac{r_p}{\Sigma_0 \Omega_p} \left( \frac{c}{r_p \Omega_p} \right) \Omega_p^{-1} \).

(1)

Here, \( M_* \) is the mass of the central star, \( m_{\text{planet}} \) is the mass of the planet orbiting at distance \( r = r_p \), \( \Sigma_0 \) is the disc surface density at \( r = r_p \), and \( \Omega_p \) are the the local sound speed and angular velocity, \( \Omega , \) at \( r = r_p \) respectively. The numerical coefficient \( W_m = 0.3704 \).

It is important to note that type I migration appropriate to a laminar disc may lead to short migration times in standard model discs that may threaten the survival of protoplanetary cores (Ward 1997). However, in a disc with turbulence driven by the magnetorotational instability, the migration may be stochastic and accordingly less effective (Nelson & Papaloizou 2004). None the less, there is considerable uncertainty as to the extent of turbulent regions in the disc resulting from uncertainties in the degree of ionization (e.g. Fromang, Terquem & Balbus 2002) so that type I migration appropriate to a laminar disc may operate in some regions. We also note that, because it is inversely proportional to the disc surface density, the migration time becomes long in low surface density regions. Accordingly, for this first study of resonant interactions of planets in the mass range 1–30 \( M_\oplus \) embedded in a gaseous disc, we shall consider only migration induced by a laminar disc.

Figure 1. The initial configuration: two planets with masses \( m_1 \) and \( m_2 \), respectively, in circular orbit around a central star with mass \( M_* \) at distances \( r_1 \) and \( r_2 \) are embedded in a gaseous disc.

Figure 2. The evolution of the ratio of semimajor axes for the two planets with masses \( m_1 = 4 \, M_\oplus \) and \( m_2 = 1 \, M_\oplus \). Starting from the lower curve and going upwards, the curves correspond to the initial surface density scalings \( \Sigma_0 = \Sigma_4, \Sigma_0 = \Sigma_2, \Sigma_0 = \Sigma_1 \) and \( \Sigma_0 = \Sigma_{0.5} \), respectively.
The density waves excited by a low-mass planet with small eccentricity also lead to orbital circularization (e.g. Artyomowicz 1993; Papaloizou & Larwood 2000) at a rate that can be estimated to be given by (Tanaka & Ward 2004)

\[ \frac{\dot{e}}{W_c} \left( \frac{\epsilon}{r_\text{p}^3 \Omega_p} \right)^2. \]

Here, the numerical coefficient \( W_c = 0.289 \).

It is expected from equation (1) that two planets with different masses will migrate at different rates. This has the consequence that their period ratio will evolve with time and may accordingly attain an orbital circularization. In the situation where the migration is such that the orbits converge, subsequently become locked in a mean motion resonance (e.g. Nelson & Papaloizou 2002; Kley, Peitz & Bryden 2004).

In the simplest case of nearly circular and coplanar orbits, the strongest resonances are the first-order resonances, which occur at

\[ \frac{p_1}{p_2} = \frac{m_2}{m_1} \approx \frac{1}{m_2/m_1}. \]
Figure 4. As for Fig. 3 but when the planets are embedded in a disc with $\Sigma_0 = 2\Sigma_2$. Note the passage through the 5:4 and 6:5 resonances at $t \sim 4000$ yr and $t \sim 8000$ yr, respectively.

Resonance overlap occurs when the difference of the semimajor axes of the two planets is below a limit with half-width given, in the case of two equal-mass planets, by Gladman (1993) as

$$\frac{\Delta a}{a} \approx \frac{2}{3p} \approx 2 \left( \frac{m_{\text{planet}}}{M_*} \right)^{2/7},$$

with $a$ and $m_{\text{planet}}$ being the mass and semimajor axis of either one of them, respectively. Thus, for a system consisting of two equal

locations where the ratio of the two orbital periods can be expressed as the ratio of two consecutive integers, $(p + 1)/p$, with $p$ being an integer. As $p$ increases, the two orbits approach each other and the strength of the resonance increases. In addition, the distance between successive resonances decreases as $p$ increases. The combination of these effects ultimately causes successive resonances to overlap and so, in the absence of gas, leads to the onset of chaotic motion.
4 $M_\oplus$ planets orbiting a central solar mass, we expect resonance overlap for $p \gtrsim 8$. Conversely, we might expect isolated resonances in which systems of planets can be locked and migrate together if $p \lesssim 8$.

However, note that the above discussion does not incorporate the torques producing convergent migration or eccentricity damping and thus may not give a complete account of the forms of chaos that might be expected. Kary, Lissauer & Greenzweig (1993) have discussed the case of small particles migrating towards a much more massive planet and indeed conclude that chaotic behaviour is more extensive in the non-conservative case. This occurs because the higher $p$ resonances can be unstable thus preventing long-term trapping. The instability arises because, if locked in resonance, the orbit of a planet has some eccentricity. Making a slight perturbation to the orientation of the orbit produces an impulsive change to the semimajor axis at the next conjunction.
This in turn affects the phase of the next conjunction and the next impulsive change to the semimajor axis. For resonances of high enough $p$, this sequence results in instability and chaotic behaviour. Kary et al. (1993) give an estimate of when this form of instability occurs as

$$p > 0.667 \left( \frac{8 \pi e m_{\text{planet}}}{3 M_\star} \right)^{-1/5}. \quad (4)$$

Here, $m_{\text{planet}}/M_\star$ refers to the mass ratio for the most massive planet and $e$ is the orbital eccentricity for the planet with the smallest mass. Taking values appropriate to our simulations of $e \sim 0.04$ and $m_{\text{planet}}/M_\star \sim 10^{-5}$ gives $p \gtrsim 8$. This is similar to the non-dissipative estimate.

When this form of instability operates, long-term stable trapping in commensurabilities is not possible. However, a system can remain in one for a long time before moving into another higher $p$. 

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**Figure 6.** As for Fig. 3 but for the case when the planets are embedded in a disc with $\Sigma_0 = \Sigma_{0.5}$. 

commensurability. Furthermore, detailed outcomes are very sensitive to input parameters, which is characteristic of chaotic motion. Slight changes can alter the sequence of commensurabilities a system resides in making the issue of their attainment acquire a probabilistic character. Both our hydrodynamic simulations and $N$-body calculations show evidence of this behaviour. Although there may be islands of apparent stability, a system in this regime may ultimately undergo a scattering and exchange of the orbits of the two planets so that we should focus on stable lower $p$ commensurabilities as being physically possible planetary configurations. The arguments given above suggest that these must have $p \lesssim 8$. Our calculations indicate that the limit is even smaller.

Hydrodynamic simulations of disc–planet interactions, in which the discs are modelled as flat two-dimensional objects with laminar flow governed by the Navier–Stokes equations and that incorporate a migrating two giant planet system that evolves into a 2:1 commensurability have been performed by Kley (2000), Snellgrove et al. (2001) and successfully applied to the GJ876 system. In addition to performing supplementary simulations, Papaloizou (2003) has developed an analytic model describing two planets migrating in resonance with arbitrary eccentricity. In this model, the eccentricities are determined as a result of the balance between migration and orbital circularization. However, disc tides were considered to act on the outer planet only. Capture of giant planets into resonances has also been recently investigated numerically by Kley et al. (2004).

In this paper, we extend the types of two-planet systems considered to include those with orbital resonances formed when the masses of the planets are 1–30 $M_\oplus$. Then for discs with aspect ratio $H/r = c/(r\Omega) \sim 0.05$ as in standard protostellar disc models, the orbital migration will be induced by type I migration.

In Section 2, we describe initial set-up for our calculations giving the properties of the system in which two planets interact with each other and a gaseous disc.

In Section 3, we present the results of two-dimensional numerical calculations showing how convergent differential migration of two planets can lead to resonance trapping followed by migration in which the resonance is maintained for the duration of the simulation, which is characteristically around $2 \times 10^3$ orbits. By considering simulations with a range of planet masses and initial conditions, we are able to explore a variety of commensurabilities and show the sensitivity of the attainment of higher $p$ commensurabilities to the input parameters, this being indicative of stochastic behaviour.

In Section 4, we compare our results with simplified $N$-body integrations that model the effects of the disc through the addition of terms to cause migration and eccentricity damping (see e.g. Snellgrove et al. 2001; Lee & Peale 2002; Adams & Laughlin 2003). The comparison enables us to calibrate these terms and then apply the much faster $N$-body calculations to consider a wider range of planet masses and disc surface densities for a longer evolution time than is possible for the hydrodynamic simulations.

We summarize and discuss our results in the context of potentially observable commensurabilities in Section 6.

In addition, we generalize the analytic model of Papaloizou (2003) to incorporate migration and circularization effects for both planets and also to consider general first-order commensurabilities in the Appendix. Thus this model can be applied to systems such as those considered here for which the planets have comparable mass and disc interactions may be important for either of them.

2 NUMERICAL SIMULATIONS OF MIGRATING PLANETS IN RESONANCE

We have performed simulations of two interacting planets together with an accretion disc with which they also interact. The simulations performed here are of the same general type as that performed by Snellgrove et al. (2001) of the resonant coupling in the GJ876 system induced by orbital migration caused by interaction with the disc. For details of the numerical scheme and code adopted, see Nelson et al. (2000). For the simulations performed here, we adopt $n_s = 384$ and $n_g = 512$ equally spaced grid points in the radial and azimuthal directions, respectively.

We use a system of units in which the unit of mass is the central mass $M_*$, the unit of distance is the initial semimajor axis of the inner planet, $r_\text{in}$, and the unit of time is $2\pi(GM_*/r_\text{in}^2)^{-1/2}$, this being the orbital period on the initial orbit of the inner planet. When $r_\text{in}$ corresponds to 1 au, this unit of time is 1 yr. In dimensionless units, the inner boundary of the computational domain was at $r = r_\text{in} = 0.33$ and the outer boundary was at $r = r_\text{max} = 4$. In all simulations, the disc model is locally isothermal with aspect ratio $H/r = 0.05$ and the kinematic viscosity is set to zero.

2.1 Initial configuration and computational set-up

Our initial set-up shown in Fig. 1 includes the central star with a mass $M_*$ and two orbiting planets with masses $m_1$ and $m_2$, respectively. The two planets are embedded in the disc, which is a source of planet orbit migration. They are initialized on circular orbits with the central mass taken to have a fixed value of 1 $M_\odot$. The gravitational potential was softened with softening parameter $b = 0.8H$. This results in the formation of an equilibrium atmosphere around the embedded planet, which then does not accrete. This softening also allows an adequate representation of type I migration in two-dimensional discs (see e.g. Nelson & Papaloizou 2004). The disc in which planets are initially embedded has the initial surface density, $\Sigma(r)$, profile specified by

$$\Sigma(r) = \begin{cases} 0.1\Sigma_0[15(r - r_\text{min})/r_\text{min} + 1] & \text{if } r_\text{min} \leq r \leq 8r_\text{min}/5, \\ \Sigma_0 & \text{if } 8r_\text{min}/5 < r < 4.5r_\text{min}, \\ \Sigma_0(4.5r_\text{min}/r)^{1.5} & \text{if } r \geq 4.5r_\text{min}. \end{cases}$$

Figure 7. The semimajor axis ratio when $m_1 = 4 M_\oplus$, $m_2 = 1 M_\oplus$ with $\Sigma_0 = \Sigma_{0.5}$ (upper curve) and $\Sigma_0 = \Sigma_4$ (lower curve).
where $r_{min}$ is the inner edge of the computational domain. The planets are located in the flat part of this distribution. We use four different values for the maximum value of the surface density, $\Sigma_0$, namely $2 \times 10^3 \ (5.2 \text{ au} / r_2)^2 \ \text{kg m}^{-2}$: the standard value attributed to the minimum mass solar nebula at 5.2 au when $r_2 = 5.2$ au, which we denote $\Sigma_1$; then $\Sigma_{0.5} = 0.5 \Sigma_1$; $\Sigma_2 = 2 \Sigma_1$; and finally $\Sigma_4 = 4 \Sigma_1$. The radial boundaries were taken to be open.

3 TWO-DIMENSIONAL HYDRODYNAMIC SIMULATIONS: A SURVEY

We have performed simulations of two interacting planets embedded in a disc with which they also interact. This interaction leads to spiral wave excitation, energy and angular momentum exchange between the planetary orbits and the gaseous disc, which in turn results
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Figure 9. The semimajor axis ratio for $m_1 = 4 \, M_\oplus$ and $m_2 = 1 \, M_\oplus$ with surface density scaling parameter $\Sigma_0 = \Sigma_{0.5}$ (upper curve) and $\Sigma_0 = \Sigma_1$ (lower curve).

in orbital migration and eccentricity damping. We have carried out simulations for a variety of planet masses, initial orbital separations and initial surface density scaling in order to explore the possible outcomes. Because these simulations are very computationally demanding, the extent of the survey is limited. None the less, some characteristic features emerge. We describe some of these results below.

3.1 One pair of planets in three different resonances

In order to investigate how the orbital evolution depends on the surface density of the disc in which the planets are embedded, we have considered two planets with masses $m_1 = 4 \, M_\oplus$ and $m_2 = 1 \, M_\oplus$. We assume that previous evolution brought the two planets to a configuration with circular orbits with $r_1/r_2 = 1.2$. This separation is slightly smaller than that required for a strict 4:3 commensurability. The faster migration expected for the outer planet (see equation 1) will make the distance between the planets smaller and, as a consequence of such convergent migration, it is possible that they may become locked in a first-order mean motion resonance (e.g. Goldreich 1965) such as 5:4, 6:5, 7:6, 8:7,..., $(p + 1):p$... and subsequently migrate together maintaining the commensurability for a considerable period of time or perhaps indefinitely, depending on how close the system is to a stochastic regime.

3.2 Attainment of commensurability

Papaloizou (2003) has given a simple approximate analytic solution for two migrating planets locked in a 2:1 commensurability. Only the outer planet was presumed to interact with the disc. As a result of the resonant interaction, the eccentricities of both planets grew with time until the effects of circularization due to disc tides balanced this. For the situation studied here, both planets are embedded in the disc and may have significant interaction with it. Furthermore, higher $p$ commensurabilities than 2:1 occur for planet masses in the Earth mass range. Thus, in the Appendix, we generalize the solution so that it applies to general first-order commensurabilities and allows for disc tides to act on both planets. In particular, we show that when the eccentricities stop growing, provided they are not too large, they must satisfy

$$e_1^2 + e_2^2 = \frac{m_2}{m_1} \frac{a_1^2}{a_2^2} \left( \frac{f}{1 - f/t_{mig}} \right) - \left( \frac{f}{1 - f/t_{circ}} \right)$$

where $f = m_2 a_1 / [(p + 1)(m_2 a_1 + m_1 a_2)]$.

Here, the semimajor axes and eccentricities of the two planets ($i = 1, 2$) are $a_i$ and $e_i$. The migration rates (assumed directed inwards) and circularization times induced by the disc tides are

$$t_{mig} = |n_i/n_1| = |2a_i/(3a_1)| = 2\pi/3$$

and

$$t_{circ} = |e_i/e_1|$$

respectively. The mean motions are $n_i$.

In addition, near a $(p + 1):p$ resonance, the resonant angles $\phi = (p + 1)\lambda_1 - p\lambda_2 - \pi/3$, $\psi = (p + 1)\lambda_1 - p\lambda_2 - \pi/3$ and $\pi - (\lambda_2 - \lambda_1)$ librate about equilibrium values which could be near 0 or $\pi$ mod 2$\pi$. Dissipative effects may be responsible for some shift. Here, the mean longitudes and the longitudes of the pericentre of the two planets ($i = 1, 2$) are $\lambda_i$ and $\pi_i$, respectively. For giant planets like GJ876, these all librate about values close to zero. However, for lower mass planets, the libration of $\pi_1 - \pi_2$ in particular may be about $\pi$ mod 2$\pi$ (e.g. Snellgrove et al. 2001; Lee & Peale 2002).

All the simulations carried out here indicate libration about a value closer to $\pi$ than 0.

Which resonance is established depends on the rate of relative migration. Roughly speaking, one expects that locking occurs only if the relative migration is slow enough that the time to migrate through the resonance is longer than a characteristic libration period. The resonances become stronger as $p$ increases, so that increasing the relative migration rate tends to cause the attainment of higher $p$ commensurabilities. However, if the $p$ becomes high enough ($p \sim 8$ for the planet masses considered), the planets enter a chaotic region (see equations 3–4) of phase space, making commensurabilities unstable in the long term (although a system may remain in the vicinity of one for a considerable time) and introducing sensitivity of detailed outcomes to initial conditions. In such cases, a scattering may ultimately occur that interchanges the positions of the planets. Thus, very high surface densities with very rapid migration rates do not favour attainment of very stable commensurabilities.

In order to study these effects, we have considered the pair of planets evolving in discs with $\Sigma_0 = \Sigma_{0.5}$, $\Sigma_0 = \Sigma_1$, $\Sigma_0 = \Sigma_2$ and $\Sigma_0 = \Sigma_4$. The evolution was followed until a resonance was established. The results are summarized in Fig. 2, where the evolution of the ratio of the semimajor axes, which starts from the value 1.2 for all four cases, is shown. The fastest migration (steepest slope) corresponds to the case where the two planets with $m_1 = 4 \, M_\oplus$ and $m_2 = 1 \, M_\oplus$ are embedded in a disc with $\Sigma_0 = \Sigma_4$ and the slowest corresponds to a disc with $\Sigma_0 = \Sigma_{0.5}$. The planets in the disc with $\Sigma = \Sigma_4$ become trapped in 8:7 resonance. If the disc surface density is 2 times smaller, then a 7:6 resonance is attained; if it is 4 or 8 times smaller, then the resonance attained is 5:4. These results are fully consistent with the idea that higher $p$ resonances are associated with faster relative migration rates.

However, this situation corresponds to the evolution during the first 18 000 yr and it is not obvious that the planets will remain in the same resonances in their subsequent evolution. This is because, in addition to possibly being in a chaotic regime, the ratio of migration time to local orbital period varies with disc parameters and accordingly disc location. Consequently, the relative migration might become relatively faster as the evolution proceeds, resulting in a shift to a higher $p$ resonance. We comment that a slowing of relative migration, as long as it does not lead to a reversal, will not result in a transition to a lower $p$ resonance. Therefore, changes in effective local migration rates will tend to lead to higher $p$ commensurabilities and ultimately chaos.

The whole evolution for the fastest migration rate is illustrated in Fig. 3. We show there the evolution of the individual semimajor axes, eccentricities, angle between apsidal lines and one of the resonant angles. A surface density contour plot of the disc near the end of the simulation is given in Fig. 3 together with a comparison between the initial and final surface density profiles.

The inner planet did not migrate significantly until 8:7 resonance was attained at about 6000 yr. Subsequently, the two planets migrated inwards together maintaining the commensurability. The system appears to be close to the chaotic regime as estimated by use of equations (3)–(4). During 18 000 yr of evolution, the outer planet changed its location from a dimensionless radius of 1.2 to 0.9. The eccentricity of the outer planet increased slightly and that of the inner planet substantially, reaching at around 10 000 yr the balanced value of 0.04 and at later times oscillating around this value. At the end of the simulation shown here, the ratio of eccentricities, \(e_1/e_2\), is equal to 0.125. The local peaks in the values of inner planet eccentricity occurring at 2000, 4000 and around 5400 yr correspond

Figure 10. As for Fig. 6 but here the planets initially have \(a_1/a_2 = 1.23\).
Figure 11. As for Fig. 5 but when the planets are initiated with $a_1/a_2 = 1.23$. Note that the system fails to become trapped in the 4:3 and 5:4 commensurabilities but none the less shows associated strong resonant interaction for long periods of time around $t = 4000$ yr and $t = 14000$ yr, respectively.

to the planet passing through the 5:4, 6:5 and 7:6 resonances, respectively. After about 8000 yr, the angle between the apsidal lines is oscillating around 212° and the resonant angle $\phi$ around 264°. The amplitude of the oscillations for both angles is decreasing with time. Similar behaviour can be seen in the evolution of these planets when embedded in a disc with lower surface density (see Fig. 4).

When $\Sigma_0 = \Sigma_1$, leading to a slower rate of migration, as before, the inner planet did not migrate significantly until a commensurability was achieved and maintained. A 7:6 resonance was reached after 10 000 yr and subsequently maintained. The passage through the 5:4 and 6:5 commensurabilities are clearly marked by a local increase of inner planet eccentricity and also in the behaviour of the angle between apsidal lines. In fact, after the passage through 5:4 resonance, the eccentricities do not completely decay, potentially making the outcome of subsequent resonance passages probabilistic (Kary et al. 1993). In this case, the planets relative migration
is slower and the planets stay longer in the vicinity of these resonances. At the end of the evolution, the eccentricities ratio is $e_1/e_2 = 0.14$. The eccentricity of the inner planet is slightly higher than it was in the case with $\Sigma_0 = \Sigma_4$. However, the eccentricities continue to grow and the equilibrium is not quite established at the end of the simulation. The amplitude of oscillations in both the angle between the apsidal lines and the resonant angle $\phi$, is smaller than in the previous case. For the two simulations that started with even lower disc surface densities such that $\Sigma_0 = \Sigma_1$ and $\Sigma_0 = \Sigma_{0.5}$, the evolution is shown in Figs 5–6.

When $\Sigma_0 = \Sigma_1$, the evolution of two planets leads to locking in 5:4 resonance. This happens at around 8000 yr. Before that, the inner planet migrated very slowly. After becoming trapped in the commensurability, the inner planet semimajor axis changes in a characteristic oscillatory manner. These oscillations are clearly visible in all other related plots (e.g. Fig. 5). The eccentricity ratio at the end of the simulations is $e_1/e_2 = 0.18$. The eccentricities are still growing. Similar behaviour is seen in the angle between the apsidal lines, which starts to librate at time 8000 yr around 180° and at time 18 000 yr oscillates around 212°. The resonant angle changes have very large amplitude and seem to grow with time, which might indicate that the planets will not remain in this resonance. In this context, note that equation (1) indicates that $t_{\text{mig}}n_{\text{j}}$ increases inwards for a uniform surface density. This should favour resonance locking at smaller radii.

Also, the planets in a disc with $\Sigma_0 = \Sigma_{0.5}$ end in a 5:4 commensurability. It occurs after approximately 14 000 yr. Because of slow migration rates in low surface density discs, the subsequent evolution could not be continued for long enough to be able to make a definitive statement about the final outcome. However, the oscillations in the resonance angles are much smaller and more closely centred around $\pi$ than is the case when $\Sigma_0 = \Sigma_1$. This could indicate greater stability of the 5:4 resonance in this case.

3.3 Comparison with the analytic model

It is of interest to compare the eccentricities obtained above when the planets are trapped in a commensurability with what is expected when resonant effects and disc tides are in balance. In the simple analytic model given in the Appendix, when the eccentricities are small these satisfy equation (5). In principle, orbital circularization acting on both planets should be included. However, for the simulations presented here, $e_2 \gg e_1$ and, even though the circularization time may be smaller for the possibly more massive outer planet, the effect can be neglected in comparison with that associated with the inner planet. Accordingly, we set $e_1 = 0$. Then equation (5) gives the simple relation

$$e_2^2 \left( \frac{m_2 a_1}{m_1 a_2} + \frac{f}{t_{\text{mig}}} \right) = \left( \frac{1}{t_{\text{mig}}} - \frac{1}{t_{\text{mig}}^2} \right) \frac{f}{3},$$

(6)

where we recall that $f = m_2 a_1/[(p + 1)(m_2 a_1 + m_2 a_2)]$. We further simplify matters by assuming that $p$ is large. Then we may set $a_1 = a_2 = n_1 = n_2$ and neglect $f$ on the left-hand side of equation (6); in the same spirit, we replace by $m_2/[(p + 1)(m_2 + m_1)]$ on the right-hand side. Then we obtain

$$e_2^2 = \left( \frac{1}{t_{\text{mig}}} - \frac{1}{t_{\text{mig}}^2} \right) \frac{m_1}{3(p + 1)(m_2 + m_1)}.$$  

(7)

Interestingly, equation (7) shows that it is the rate of convergent migration of the two planets that determines the eccentricities, which are small when this is small in agreement with our results. Further, other things being equal, the eccentricities decrease with increasing $p$. We also note that use of equations (1) and (2) gives $t_{\text{c}} = (\tau_f/W_f)(H/r)^2$. Thus, after recalling that in general the e-folding rate for mean motion is a factor of 1.5 greater than that for radius, or $t_{\text{mig}} = 2\tau_f/3$, we obtain

$$e_2^2 = \left[ \frac{m_1}{m_2} \left( \frac{a_1}{a_2} \right)^{1/2} - 1 \right] \frac{m_1}{0.578(p + 1)(m_2 + m_1)}(H/r)^2.$$  

(8)

We apply this to the case illustrated in Fig. 3 for the two planets with masses $m_1 = 4 M_\oplus$ and $m_2 = 1 M_\oplus$ embedded in a disc with initial surface density scaling $\Sigma_0 = \Sigma_1$ and obtain $e_2 = 0.037$ in reasonable agreement with the simulations.

3.4 Dependence on the initial separation of the planets

Here, we investigate the dependence of attained commensurabilities on the initial radial separation of the planets. To do this, we have performed simulations with planets of the same mass as above, but starting with orbital separations a little larger than that required for strict 3:2 and 4:3 commensurabilities. In the former case, the two planets were initiated with $r_1 = 1.32$ and $r_2 = 1.00$. Two initial surface density scalings were considered, namely $\Sigma_0 = \Sigma_{0.5}$ and $\Sigma_0 = \Sigma_4$. The evolution of the ratio of the semimajor axes for both cases is plotted in Fig. 7. For $\Sigma_0 = \Sigma_4$, the end-state has the planets in a 9:8 resonance. The evolution for $\Sigma_0 = \Sigma_{0.5}$ is shown for comparison. The result in the $\Sigma_0 = \Sigma_4$ case is a clear indication that the system gets into the chaotic regime as a value of $p$ of the attained resonance is increased when the initial semimajor axis ratio was larger. Stability would have implied that the same resonance should have been attained. Unfortunately, the migration rate in the $\Sigma_0 = \Sigma_{0.5}$ case was too slow for us to be able to attain a commensurability with the available computational resources and confirm or otherwise the stability of the 4:3 commensurability.

When the ratio of the initial semimajor axes is changed from 1.2 to 1.32, the two planets embedded in a disc with $\Sigma_0 = \Sigma_4$ become trapped in a different resonance (compare Figs 3 and 8). This is again consistent with the presence of stochastic behaviour as a different migration history leads to different end states and it suggests that...
Figure 13. The evolution of semimajor axes, eccentricities, angle between apsidal lines and resonant angle for two planets with equal mass, $m_1 = m_2 = 4M_\oplus$, migrating towards a central star and embedded in a disc with $\Sigma_0 = \Sigma_1$ (four upper panels). The mean surface density profile of the disc near the end of the simulations (solid line), together with the initial surface density profile (dashed line) and a surface density contour plot near the end of the simulation are given in the lowest left- and right-hand panels, respectively.

having crossed different resonances before attaining a current one may influence the final outcome.

A similar experiment has been performed but now placing the planets close to 4:3 resonance. In this case, we have chosen discs with lower surface density, such that $\Sigma_0 = \Sigma_{0.5}$ and $\Sigma_0 = \Sigma_1$ (see Fig. 9). In the former case, the system enters into 4:3 resonance; while in the latter, the system passes through the 4:3 resonance and becomes trapped in the 6:5 resonance.

When $\Sigma_0 = \Sigma_{0.5}$, the planets enter 4:3 commensurability at a time around 6000 yr as illustrated in Fig. 10. The eccentricities of
both planets grow. The resonant angle oscillates around 260° with large amplitude, higher than it was in the case when the initial planet separation was smaller, namely 1.2 (Fig. 6).

For the disc with the surface density $\Sigma_0 = \Sigma_1$, we can follow in detail the passage of the planets through the 4:3 resonance and the temporary trapping in the 5:4 resonance. In Fig. 11, at time 4000 yr there is a characteristic rapid increase in the eccentricities of both planets followed by a slower decrease. The angle between the apsidal lines also shows the expected behaviour during resonance crossing.

This evolution should be compared with that shown in Fig. 5, for the same pair of planets embedded in the same disc but starting with a smaller initial planet separation. The effect of a prior resonance passage before approaching and becoming relatively stably trapped in the 5:4 resonance is not present in this case. So it is likely that the mutual interaction of planets during resonance crossing influences their subsequent evolution and whether higher $p$ resonances are attained or not and that these planets are either close to or in the chaotic regime. This behaviour may be related to the fact that

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**Figure 14.** The same as for Fig. 13 but the planets initially had $a_1 = 1.23$ and $a_2 = 1$ and they are embedded in a disc with $\Sigma_0 = \Sigma_4$. 

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eccentricities excited during resonance passages do not completely decay between them.

3.5 Slower relative migration: planets with equal masses

As described above, the resonant interaction must balance the tendency towards relative migration of the two planets. This is smaller when the planets have the same mass. Thus, lower \( p \) commensurabilities and less tendency to be driven into a chaotic state might be expected in this case when compared with the situation where the inner planet has significantly smaller mass. In order to investigate this, two planets of mass \( 4 M_\oplus \) were initiated close to 3:2 resonance with \( a_1 = 1.32 \) and \( a_2 = 1 \) in a disc with \( \Sigma_0 = \Sigma_1 \). The evolution of the semimajor axis ratio for the planet orbits is shown in Fig. 12. We also performed simulations for the same masses starting with \( a_1 = 1.23 \) and \( a_2 = 1.00 \) in a disc with \( \Sigma_0 = \Sigma_4 \) and \( a_1 = 1.2 \) and \( a_2 = 1 \) in a disc with \( \Sigma_0 = \Sigma_4 \). The planets become trapped in the nearest available resonance, which is a good indication of...
stability. The evolution of the equal-mass pair of planets is shown in Figs 13–15. The planets attaining 3:2 resonance after around 4000 yr show the following behaviour. Their eccentricities increased until at 8000 yr $e_1 = 0.013$ and $e_2 = 0.007$. They then start to oscillate. At around time 13 000 yr, the inner planet eccentricity dropped to zero for short period of time. The resonant angle oscillates around 0.008. The angle between apsidal lines and the resonant angle oscillate around 180°.

The planets placed close to 4:3 resonance become locked in this resonance at around 2000 yr. The eccentricities of both planets change in a similar way. They do not grow much; towards the end of the simulation they oscillate around 0.008. The angle between apsidal lines and the resonant angle oscillate around 180° and 230°, respectively.

The last experiment of this series was as follows. The two planets are initially located just interior to 4:3 resonance. After 12 000 yr, they arrive through convergent migration at the 5:4 resonance and become trapped in it. The eccentricities of the two planets evolve in an oscillatory way. At a time around 16 000 yr, they are very close to zero. Particular features at that time are seen in the evolution of the angle between apsidal lines as well as the resonant angle, which are not properly defined for zero eccentricity.

We also performed calculations with two planets of 30 $M_\oplus$ starting with $a_1 = 1.3$ and $a_2 = 1$. The planets were embedded in a disc with $\Sigma_0 = \Sigma_1$. They attained a stable 3:2 resonance. The behaviour of the ratio of semimajor axes close to the resonance is shown in Fig. 16 and other aspects of the evolution are shown in Fig. 17. The excursions around resonance are larger and noisier in this case. Note too that, in this case, the disc–planet interaction becomes non-linear with significant perturbation to the underlying disc surface density. There is a tendency towards gap formation and the excavation of disc material into two mounds at the gap edges. The inner mound occupies only a small radial region and so may not be well represented with the available numerical resolution in this case.

4 N-BODY INVESTIGATIONS: A SURVEY

In order to study migrating planets with a wider range of masses and disc surface densities, we have performed a resonance survey using an N-body code. The approach is the same as that used by Snellgrove et al. (2001) and Nelson & Papaloizou (2002) and it has also been used by Lee & Peale (2002) and Kley et al. (2004). The reader is referred to those papers for details. The procedure is to model the effects of a disc by incorporating orbital migration and eccentricity damping as described by equations (1)–(2) through the addition of appropriate acceleration terms to the equations of motion. We summarize the parameters and outcomes of some of the N-body simulations in Table 1.

We begin by discussing a calibration procedure that enables a matching of the N-body simulations with the hydrodynamic simulations. The advantage of the N-body simulations is that the time evolution of a system can be followed for much longer periods of time. After discussing a few examples, we move on to discuss the results, shown in Table 1, for the possible commensurabilities to be expected for low-mass planets for particular choices of their masses, disc parameters and migration and eccentricity damping rates.

4.1 Comparison between hydrodynamic simulations and N-body calculations

The hydrodynamical calculations discussed in the previous section allow us to follow the migrations of planets for almost $2 \times 10^6$ yr. As a result of this, we have been able to simulate planets becoming trapped in resonances. The behaviour and stability of the resonance trapping varies with the planet masses and the surface density of the disc in which they are embedded. The outcome of these simulations could be well matched to those of N-body integrations where we incorporate the simple prescriptions for the migration and eccentricity damping given through equations (1)–(2).

Using these expressions in the N-body code, we have extended the hydrodynamic calculations for a longer period of time and studied the long-term stability of the resonances we have found in that context. As a first step, we have adjusted the numerical coefficients in equations (1)–(2) in such a way that the hydrodynamic and N-body approach give the same qualitative evolution.

As an example in Fig. 18 we show the results of the comparison for the case of two planets with masses 1 and 4 $M_\oplus$, respectively, embedded in a disc with initial surface density scaling parameter $\Sigma_0 = \Sigma_1$. The numerical coefficients adopted were $W_p = 0.3647$ and $W_e = 0.225$. These results show good agreement with the hydrodynamical results (see Fig. 3) and display the same 8:7 commensurability. The fitted coefficients were also reasonably close to those expected from the analytic disc–planet interaction theory, confirming both the migration and eccentricity damping times. Though the latter were typically 40 per cent longer than expected from the analytic theory. The evolution was followed using the N-body approach for an additional $1.2 \times 10^6$ yr. The two planets remain in the 8:7 resonance during this time and there is no indication of any significant changes in the monitored quantities for the last $9 \times 10^5$ yr. The long-term evolution of this system is shown in Fig. 19. It is interesting to note that the equilibrium value of eccentricity of the inner planet is around $e_2 = 0.03$, in accordance with the value predicted by the simple analytic model discussed in Section 2 and derived in the Appendix. The angle between apsidal lines is close to 180° the other resonant angles occupy a wide band between 80° and 287°. A similar procedure has been applied to other cases presented in Section 3. The values of the numerical coefficients ($W_p$, $W_e$) used in equations (1)–(2) obtained by comparing and matching the planet evolution calculated by hydrodynamic and N-body codes are given in Table 2.

In the case of two planets with masses $m_1 = 4 M_\oplus$ and $m_2 = 1 M_\oplus$ migrating towards a central star embedded in the disc with
Figure 17. The evolution of semimajor axes, eccentricities, angle between apsidal lines and resonant angle for two planets with masses $m_1 = 30 \, M_\oplus$ and $m_2 = 30 \, M_\oplus$ migrating towards a central star embedded in a disc with $\Sigma_0 = \Sigma_1$ (four upper panels). The mean surface density profile of the disc near the end of the simulation (solid line) together with the initial surface density profile (dashed line) and a surface density contour plot near the end of the simulation are given in lowest left- and right-hand panels, respectively.

$\Sigma_0 = \Sigma_2$, the situation is less stable. The 7:6 resonance found in the hydrodynamic simulations is maintained for around $2 \times 10^5$ yr; later there is a shift into a 10:9 commensurability. Finally, at around $6 \times 10^5$ yr scattering occurred. Also, the 5:4 commensurability in the lower surface density case ($\Sigma_0 = \Sigma_1$) was maintained only until $1.2 \times 10^5$ yr, then there is a transition to 6:5, which lasted until $2.1 \times 10^5$ yr. Next, the planets evolved until $6 \times 10^5$ yr locked in 7:6 resonance and after that continued in 8:7. That was the status of the evolution at the end of our calculations at $1.3 \times 10^6$ yr. The same pair of planets embedded in the disc with even lower surface density ($\Sigma_0 = \Sigma_{0.5}$) migrated together in 5:4 resonance until $1.15 \times 10^6$ yr, then they moved into a 6:5 commensurability. These outcomes are suggestive of the long-term instability of the high $p$ commensurabilities (Kary et al. 1993).
Table 1. This table lists some of the \(N\)-body calculations of interacting planets with imposed migration and eccentricity damping rates, assumed to result from interaction with a disc, which we performed, together with their outcomes. During these calculations, the migration has been followed for a period of time in which the inner planet changes its semimajor axis from \(a_2 = 1\) to \(a_2 = 0.385\) in dimensionless units. The outer planet is initially placed in circular orbit at \(r = 1.4\) in dimensionless units. Thus, the situation corresponds with appropriate scaling to the inner planet being initiated at 5.2 au and then migrating to 2 au. Column 1 gives the planet mass ratio; columns 2 and 3 give the outer and inner planet mass in Earth masses, respectively. Columns 4–7 indicate the outcome of the interaction with appropriate scaling to the inner planet being initiated at 5.2 au and then migrating to 2 au. Column 1 gives the planet mass ratio; columns 2 and 3 give the outer and inner planet mass in Earth masses, respectively. Columns 4–7 indicate the outcome of the interaction with appropriate scaling to the inner planet being initiated at 5.2 au and then migrating to 2 au.

We have noticed a strong sensitivity of outcomes to the fitting parameters (especially in the numerical coefficient \(W_m\)). This is symptomatic of chaotic motion. A very small change in the values of the numerical coefficients given in Table 2 can produce large variations in outcomes. The value \(W_m\) is difficult to fit for pairs of two equal-mass planets because of smaller and less regular changes in eccentricities obtained in the hydrodynamic simulations of such systems (see for example Fig. 13).

### Table 2. The fitting parameters obtained by comparing hydrodynamic and \(N\)-body calculations.

<table>
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<th>(m_1/m_2)</th>
<th>(m_1)</th>
<th>(m_2)</th>
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<th>(\Sigma_0 = \Sigma_1)</th>
<th>(\Sigma_0 = \Sigma_2)</th>
<th>(\Sigma_0 = \Sigma_4)</th>
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<td>10/3</td>
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4.2 \(N\)-body resonance survey

We applied the much faster \(N\)-body evolution, to be free parameters that we could choose to provide the best match between the hydrodynamic and \(N\)-body methods. In this way, we could test the applicability of the results of Tanaka et al. (2002) and Tanaka & Ward (2004) to this problem.

When we chose \(W_m = 0.3704\) in equation (1) and \(W_c = 0.289\) in equation (2) for the case when the planets have equal masses \((m_1/m_2) = 1\), their initial separation in circular orbits in dimensionless units is such that \(r_1/r_2 = 1.4\) and they are embedded in a disc, which extends from \(r = 0.3\) to \(r = 1.4\), then the most likely outcome of their orbital evolution caused by the disc–planet interaction, in agreement with the hydrodynamic simulations, is attainment of a 3:2 commensurability. This is the case for a wide range of disc surface density scalings. For all values of \(\Sigma_0\) considered here, this commensurability was attained. A transition from the 3:2 to the 4:3 commensurability for a given mass ratio (bigger than 1) is expected to occur when either the surface density scaling and/or the mass ratio is large enough. The value of \(p\) for the resonance that might be established accordingly increases.

When \( m_1/m_2 > 6 \), scattering is likely to occur rather than stable resonance trapping, especially for higher surface densities. This is clearly indicated in Table 1.

5 CONCLUSIONS

We have investigated migration induced resonant capture in a system of two planets for the most part in the mass range 1–4 \( M_\oplus \), embedded in a gaseous disc. We also considered a system with equal masses of 30 \( M_\oplus \). As the disc was laminar, apart from in the 30-\( M_\oplus \) case, the planets underwent type I migration, which occurs through the linear response of the disc to perturbation by the planets. We considered disc surface densities in the range 0.5–4 times that expected for the minimum mass solar nebula at 5.2 au. Thus, migration rates should be typical of those expected for standard protoplanetary discs.

Our conclusions are based on the results of two surveys in which we studied the evolution of a pair of planets with a range of planetary masses (1–30 \( M_\oplus \)). The disc was laminar, apart from in the 30-\( M_\oplus \) case, the planets underwent type I migration, which occurs through the linear response of the disc to perturbation by the planets. We considered disc surface densities in the range 0.5–4 times that expected for the minimum mass solar nebula at 5.2 au. Thus, migration rates should be typical of those expected for standard protoplanetary discs.

The numerical calculations are supplemented with an analytic model describing two planets migrating in resonance with arbitrary eccentricities. We found a very good agreement between numerical and analytical results for an established commensurability.

Both our hydrodynamic and \( N \)-body approaches give clear evidence of an interesting relation between the mass ratio in a planetary pair and the type of the resonance that is expected to be established as a consequence of tidally induced orbital migration. When both planets have similar masses, provided the disc surface density at their locations is very similar, the rate of convergent or relative migration is small. This favours attainment of low \( p \) commensurabilities.

An example of this occurred in the equal 4 \( M_\oplus \) case we considered. There, the planets became locked in the nearest first-order commensurability lying between them. Thus, if they were assembled with \( 1.55 > r_1/r_2 > 1.31 \), the migration led to trapping in 3:2 resonance.

Here, we comment that the very well known example of such a system, namely the two largest mass planets orbiting a millisecond radio pulsar PSR B1257+12, is close to a 3:2 commensurability. The most recent determination of the masses of these planets gives them as 4.3 ± 0.2 and 3.9 ± 0.2 \( M_\oplus \) respectively, (Konacik & Wolszczan 2003). The mass ratio is thus very close to unity and, according to our findings, if migration in a standard quiescent protoplanetary gaseous disc is responsible for the evolution, then a possible outcome could be attainment of a 3:2 commensurability. The orbital periods for both planets are 66.5419 and 98.2114 d (Konacki & Wolszczan 2003), which places them near 3:2 resonance. Such a resonance could be formed and maintained through the processes discussed in this paper, with the planets later moving slightly out of resonance. More specific detailed modelling will be presented elsewhere.
For more disparate mass ratios, our simulations indicate the attainment of first-order commensurabilities with higher \( p \) (for example 4:3, 5:4, 7:6 and 8:7). The dynamics develops a stochastic character and these may be maintained for different periods of time before becoming unstable. Then, often a passage from one resonance to another takes place. The short duration of the hydrodynamic simulations precluded extensive exploration of stochastic behaviour or the long-term stability of resonances. However, we could investigate these issues employing the simplified \( N \)-body approach.

In this context, we comment on a result we have obtained for an 8:7 commensurability, being the stable end state of the evolution of two planets in circular orbit with the initial separation ratio \( r_1/r_2 = 1.2 \) and mass ratio \( m_1/m_2 = 4 \) and the highest disc surface density scaling considered here with \( \Sigma_0 = \Sigma_4 \). The stable configuration was maintained until \( 1.4 \times 10^6 \) yr at which point our calculations were stopped. Of course, this result should be considered in the context of extreme sensitivity to initial conditions and other parameters. None the less, the final value of the inner planet eccentricity is in good agreement with the prediction of our analytic model for established commensurabilities.

Thus, while stochastic behaviour tends to disrupt higher \( p \) commensurabilities, the calculations presented here do not enable us to rule them out entirely.

Simple \( N \)-body calculations allowed us to extend our investigations not only to consider longer term behaviour but also a wider range of initial conditions. These calculations strengthen the conclusions derived from the hydrodynamic simulations.

Finally, we comment further on the onset of stochastic behaviour, which we encountered both in our hydrodynamic simulations and \( N \)-body calculations. The reason for this phenomenon can be related to resonance overlap. According to our estimates, we might expect isolated resonances in which a system of planets in the mass range considered can be locked and migrate together to be such that the integer \( p \) in the expression \( (p + 1):p \) for a first-order commensurability is \( \lesssim 8 \). For larger \( p \), as we can see from Table 1, a system is likely to undergo a scattering and exchange of orbits of the two planets. However, stochastic behaviour may extend to values as small as 4. Stable commensurabilities with sufficiently small \( p \) correspond to physically possible planetary configurations and they should be the centre of further investigation.

Among all planned future space missions COROT will be the first with the capability to observe planets in the mass range relevant to our investigation. Studies of commensurabilities have already been applied successfully to examine the motion of the pulsar planets and they have proved to be a powerful tool in that context. They have the potential to be similarly useful in our search for Earth-like planets in other systems. Detection of such resonances can also yield useful information about orbital migration as a process operating during planet formation.

**ACKNOWLEDGMENTS**

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APPENDIX A: TWO-PLANET SYSTEMS IN GENERAL FIRST-ORDER COMMENSURABILITIES

A1 A simple model

The equations of motion for a system consisting of two planets and a primary star, modelled as point masses moving under their mutual gravitational attraction, are conveniently expressed in Hamiltonian form using Jacobi coordinates (see Sinclair 1975). Then the radius vector \( r_1 \) of the inner planet of reduced mass \( m_2 \) is measured from the primary star and inner planet. The Hamiltonian can be written correct to second order in the planetary masses as

\[
H = \frac{1}{2} \left( m_1 r_1^2 + m_2 r_2^2 \right) - \frac{GM_1 m_1}{r_1} - \frac{GM_2 m_2}{r_2} - \frac{GM_1 m_2}{r_{12}} + \frac{Gm_1 m_2 r_1 \cdot r_2}{r_{12}^3}.
\] (A1)

Here, \( M_1 = M_1^* + m_1, \) \( M_2 = M_2^* + m_2, \) and \( r_{12} = r_2 - r_1. \) The Hamiltonian can be expressed in terms of the osculating semimajor axes, eccentricities and longitudes of periastron \( a_i, e_i, \) \( \sigma_i, \) \( i = 1, 2, \) respectively as well as the longitudes \( \lambda_i \) and the time \( t. \) We recall that \( \lambda_i = n_i (t - t_0) + \sigma_i, \) with \( n_i = \sqrt{GM_i/a_i^3}, \) being the mean motion and \( t_0 \) giving the time of periastron passage. The energy is given by \( E_i = -Gm_i M_1^*/(2a_i), \) and the angular momentum \( h_i = m_i \sqrt{GM_i a_i (1 - e_i^2)} \), which may be used to describe the motion instead of \( a_i \) and \( e_i. \)

The Hamiltonian may quite generally be expanded in a Fourier series involving linear combinations of the three angular differences \( \lambda_i - \sigma_i, \) \( i = 1, 2, \) and \( \sigma_1 - \sigma_2 \) (e.g. Brouwer & Clemence 1961).

Near a first-order \( (p + 1):p \) resonance, we expect that both \( \psi = (p+1)\lambda_1 - p\lambda_2 - \sigma_1 \) and \( \psi = (p+1)\lambda_1 - p\lambda_2 - \sigma_2 \) will be slowly varying. When resonances are non-overlapping so that the \( (p + 1):p \) resonance may be isolated, in its neighbourhood, terms in the Fourier expansion involving only linear combinations of \( \phi \) and \( \psi \) as an argument are expected to produce the largest perturbations, while the remainder, which will have rapidly varying argument, may be averaged out.

The resulting Hamiltonian \( (m_1 m_2) \) may then be written in the general form

\[
H_{12} = -\frac{Gm_1 m_2}{a_1} \sum C_{k,l} (a_1/a_2, e_1, e_2) \cos(k\phi + l\psi),
\] (A2)

where, in the above and similar summations below, the sum ranges over all positive and negative integers \( (k, l) \) and the dimensionless coefficients \( C_{k,l} \) depend on \( e_1, e_2 \) and the ratio \( a_1/a_2. \) Only here, we shall not need to specify this further. We also make the inconsequential simplification of replacing \( M_1^* \) by \( M_2. \)

A2 Basic equations

We take the Hamiltonian system derived from equation (A2) and modify it by incorporating torques and rates of change of energy that act on each of the protoplanets and that are presumed to be derived from interaction with the disc. The equations of motion are very similar to those given in Papaloizou (2003). They differ only in that the additional forces derived from disc interaction are allowed to act on both protoplanets rather than only the outer one in this case:

\[
\frac{dE_i}{dt} = -n_i \frac{\partial H_{12}}{\partial \lambda_i} + \left( \frac{n_i T_i}{\sqrt{1 - e_i^2}} + D_i \right),
\]

\[
\frac{dh_i}{dt} = -\frac{\partial H_{12}}{\partial \sigma_i} - \frac{\partial H_{12}}{\partial \sigma_i} - T_i,
\]

\[
\frac{d\lambda_i}{dt} = n_i + n_i \frac{\partial H_{12}}{\partial E_i} + \frac{\partial H_{12}}{\partial h_i},
\]

\[
\frac{d\sigma_i}{dt} = \frac{\partial H_{12}}{\partial h_i}.
\]
The external disc torque acting on the planet $m_i$ is $-T_i$. Associated with this, we remove orbital energy at a rate $n_iT_i/\sqrt{1-e_i^2}$ from $m_i$, which would correspond to the action of a disc density perturbation rotating with a pattern speed $n_i/\sqrt{1-e_i^2}$ (see also Snellgrove et al. 2001; Nelson & Papaloizou 2002). We include an additional energy loss rate $D_i$ for $m_i$, which leads to an orbital circularization time for $m_i$ given by

$$t_{ci} = \frac{n_i m_i c_i^3 \sqrt{GM_{*} a_{i}}}{D_i \left(1 - e_i^2\right)}.$$  

This circularization time is such that, if the other planet was absent such that $m_i$ was affected only by the central star and disc, $e_i$ would decay according to

$$\frac{de_i}{dt} = -\frac{e_i}{t_{ci}}$$  

We presume that $T_i$ and $D_i$ are given as functions of the orbital parameters $a_i$, $e_i$ associated with $m_i$. In addition, we suppose that they lead to orbital evolution on a time-scale sufficiently long compared to other effects that they may always be considered to act as small perturbations to the equations of motion. However, we do not assume an expansion in terms of small eccentricities.

Thus, we obtain to lowest order in the perturbing masses:

$$\frac{dn_1}{dt} = \frac{3(p + 1)n_1^2 m_1^2}{M_{*}} \sum C_{k,l}(k + l) \sin(k\phi + l\psi) + \frac{3n_1 a_1}{GM_{*} m_1} \left(-n_1 T_1 \left(1 - e_i^2\right) + D_i\right),$$  

(A5)

$$\frac{dn_2}{dt} = -\frac{3n_2 m_1 a_2}{M_{*} a_1} \sum C_{k,l}(k + l) \sin(k\phi + l\psi) + \frac{3n_2 a_2}{GM_{*} m_2} \left(n_1 T_2 \left(1 - e_i^2\right) + D_2\right),$$  

(A6)

$$\frac{de_1}{dt} = -\frac{e_1}{t_{ci}} - \frac{m_1 a_1}{e_1 M_{*}} \sum C_{k,l}(k + l) \left[k - (p + 1)(k + l) \left(1 - \sqrt{1 - e_i^2}\right)\right],$$  

(A7)

$$\frac{de_2}{dt} = -\frac{e_2}{t_{ci}} - \frac{m_2 a_1}{e_1 a_2 M_{*}} \sum C_{k,l}(k + l) \left[l + p(k + l) \left(1 - \sqrt{1 - e_i^2}\right)\right],$$  

(A8)

$$\frac{d\phi}{dt} = (p + 1)n_1 - p n_2 - \sum (D_{k,l} + E_{k,l}) \cos(k\phi + l\psi),$$  

(A9)

$$\frac{d\psi}{dt} = (p + 1)n_1 - p n_2 - \sum (D_{k,l} + F_{k,l}) \cos(k\phi + l\psi).$$  

(A10)

Here,

$$D_{k,l} = \frac{2(p + 1)n_1 a_1^2 M_{*}}{M_{*}} \frac{\partial}{\partial a_1} (C_{k,l}/a_1) - \frac{2n_2 a_2 m_1}{M_{*}} \frac{\partial}{\partial a_2} (C_{k,l}/a_1),$$  

(A11)

$$E_{k,l} = \frac{n_1 m_2 \left[(p + 1)(1 - e_i^2) - p \sqrt{1 - e_i^2}\right]}{e_1 M_{*}} \frac{\partial C_{k,l}}{\partial e_1} + \frac{n_2 a_2 m_1 \left(\sqrt{1 - e_i^2} - 1 + e_i^2\right)}{a_1 e_2 M_{*}} \frac{\partial C_{k,l}}{\partial e_2},$$  

(A12)

and

$$F_{k,l} = \frac{(p + 1)n_1 m_2 \left(1 - e_i^2 - \sqrt{1 - e_i^2}\right)}{e_1 M_{*}} \frac{\partial C_{k,l}}{\partial e_1} + \frac{n_2 a_2 m_1 \left[(p + 1) \sqrt{1 - e_i^2} - p(1 - e_i^2)\right]}{a_1 e_2 M_{*}} \frac{\partial C_{k,l}}{\partial e_2}.$$  

(A13)

We further note that $\phi - \psi = \sigma_2 - \sigma_1$ is the angle between the two apsidal lines of the two planetary orbits.

### A3 Time-independent solutions

When no external disc torques or dissipation act ($T_i = D_i = 0$), time-independent or stationary solutions may occur for which $\psi$, $\phi$ and each of $n_1$, $n_2$, $e_1$, $e_2$ are constant. In principle, any stationary values of $\psi = \psi_0$ and $\phi = \phi_0$ might be possible. Obvious possibilities, as also indicated in our numerical simulations, are values of $\phi_0$ and $\psi_0$ that are either 0 or $\pi$. In that case, it follows directly from equations (A5)–(A8) with $T_i = D_i = 0$ that there are stationary solutions with $n_1$, $n_2$, $e_1$, $e_2$ being constant. In other cases, consideration of equations (A5)–(A10) indicates that we should regard $\phi_0$ and $\psi_0$ as functions of $e_1$, $e_2$ and the ratio $a_2/a_1$.

A relation between $e_1$, $e_2$ and the ratio $a_2/a_1$ for steady solutions follows by subtracting equations (A9) and (A10) in the form

$$\sum E_{k,l} \cos(k\phi_0 + l\psi_0) + \sum F_{k,l} \cos(k\phi_0 + l\psi_0) = S_0.$$  

(A14)

This condition in fact matches the precession rates of the orbits of the two planets such that they maintain a fixed orientation. It then additionally follows from equations (A9) and (A10) that

$$(p + 1)n_1 - p n_2 = \sum D_{k,l} \cos(k\phi_0 + l\psi_0) + S_0.$$  

(A15)

This gives a further relation between $e_1$, $e_2$ and the ratio $a_2/a_1$. We could for example specify $e_1$ and then both $e_2$ and $a_2/a_1$ would be specified, but in any case the quantity $[(p + 1)n_1]/(p n_2) - 1$, supposing $m_1$ and $m_2$ to be comparable, is at least of order of smallness of the mass ratio $m_1/M_{*} \ll 1$. Thus, in a perturbation quantity that is already first order in the perturbing masses, we may set $a_2/a_1 = [(p + 1)/p]^{2/3}$, which gives the condition for a $(p + 1):p$ commensurability.

A4 Time-dependent solutions with migration and circularization

We now generalize the above solution to incorporate the effects of small non-zero disc torques \( T \) and dissipation \( D \). In this case, the two planets migrate inwards locked in resonance with \( n_1/n_2 \), maintained nearly equal to \( p/(p+1) \). In the absence of any tidal effects, which could act to circularize the orbits, the eccentricities are found to increase monotonically with time. We note that equations (A5) and (A6) define inward migration time-scales or e-folding rates for \( n_i \) as 

\[
\tau_{\text{inj}} = GM_m m_i \sqrt{1 - e_i^2}/(3T_d a_m).
\]

We suppose that, in order to accommodate small values of \( T \) and \( D \), the angles \( \psi_i = \psi - \psi_0 \) and \( \phi_i = \phi - \phi_0 \), which measure the departures from the values appropriate to the steady-state solution, take on non-zero values of small magnitude, which can also change very slowly with time. These magnitudes are presumed sufficiently small that we may employ a first-order Taylor expansion so that the perturbation to \( \sin (k\phi + l\psi) \) is \( (k\phi_1 + l\psi_1) \cos (k\phi_0 + l\psi_0) \) and similarly the perturbation to \( \cos (k\phi + l\psi) \) is \( -(k\phi_1 + l\psi_1) \sin (k\phi_0 + l\psi_0) \).

When the migration time is very long compared to any other time-scale in the problem, equations (A9) and (A10) are of the same form as in the time-independent case, apart from small corrections proportional to the migration rate, which we neglect. These then lead to the same conclusions as in the time-independent case. Accordingly, we conclude that there are relationships between \( e_1 \), \( e_2 \) and \( a_2/a_1 \) that specify the latter two once \( e_1 \) is specified, but that always \( a_2/a_1 \) satisfies the condition for \((p+1)p\) resonance with a correction of the order of the perturbing masses.

Equations (A5)–(A8) then become

\[
\frac{dn_1}{dt} = \frac{3(p+1)n_1^2m_2}{M_*} \sum C_{k,l}(k+l)(k\phi_1 + l\psi_1) + \frac{3n_1a_1}{GM_m m_1} \left( \frac{n_1T_1}{\sqrt{1 - e_1^2}} + D_1 \right),
\]

(A16)

\[
\frac{dn_2}{dt} = -\frac{3n_2^2m_2a_2}{M_* a_1} \sum C_{k,l}(k+l)(l\psi_1) + \frac{3n_2a_2}{GM_m m_2} \left( \frac{n_2T_2}{\sqrt{1 - e_2^2}} + D_2 \right).
\]

(A17)

\[
\frac{de_1}{dt} = -\frac{e_1}{\tau_{\text{inj}}} - \frac{m_2n_1}{e_1 M_*} \sum C_{k,l} \left[ \frac{e_1}{\sqrt{1 - e_1^2}} \sum C_{k,l} \left[ \frac{n_1T_1}{\sqrt{1 - e_1^2}} + D_1 \right] \right],
\]

(A18)

\[
\frac{de_2}{dt} = -\frac{e_2}{\tau_{\text{inj}}} - \frac{m_2n_2}{e_2 M_*} \sum C_{k,l} \left[ \frac{e_2}{\sqrt{1 - e_2^2}} \sum C_{k,l} \left[ \frac{n_2T_2}{\sqrt{1 - e_2^2}} + D_2 \right] \right].
\]

(A19)

Here, \( C_{k,l} = C_{k,l}(k\phi_1 + l\psi_1) \).

Taking the ratio of equations (A16) and (A17) gives

\[
\frac{dn_1}{dn_2} = \frac{(p+1)n_1^2m_2a_2}{pm_2^2m_1a_2} \frac{\sum C_{k,l}(k+l) \left[ \frac{n_1T_1}{\sqrt{1 - e_1^2}} + D_1 \right]}{\sum C_{k,l}(k+l) \left[ \frac{n_2T_2}{\sqrt{1 - e_2^2}} + D_2 \right]},
\]

(A20)

with \( C_{k,l} = C_{k,l}(k\phi_1 + l\psi_1) \).

The ratio of equations (A18) and (A19) gives another relation of the form

\[
\frac{m_2e_2a_1n_1}{m_1e_1a_2n_2} \frac{de_2}{de_1} = \frac{1}{\sqrt{1 - e_1^2}} \frac{\sum C_{k,l} \left[ l + p(k+l) \left( 1 - \sqrt{1 - e_1^2} \right) \right]}{\sum C_{k,l} \left[ k - (p+1)(k+l) \left( 1 - \sqrt{1 - e_1^2} \right) \right]} + \frac{\sqrt{1 - e_1^2} M_* a_1}{\sqrt{1 - e_1^2} m_2a_2n_2}.
\]

(A21)

Using \( pm_2 = (p+1)n_1 \), equation (A20) gives

\[
\phi_1 \sum kC_{k,l}(k+l) + \psi_1 \sum lC_{k,l}(k+l) = \frac{-m_1a_2^2 \left[ \frac{n_1T_1}{\sqrt{1 - e_1^2}} + D_1 \right] + pm_2a_1a_2 \left[ \frac{n_2T_2}{\sqrt{1 - e_2^2}} + D_2 \right]}{GM_m m_2 (p+1)n_1^2m_2a_1 + p^2n_2^2m_2a_2}.
\]

(A22)

Given that we may regard \( e_2 \) and \( a_2/a_1 \) as functions of \( e_1 \), we may also regard equations (A20) and (A21) as determining \( \phi \) and \( \psi \) as functions of \( a_1 \) and \( e_1 \). Using this information, equations (A16) and (A18) may be used to determine \( n_1 \) and \( e_1 \) as functions of time, so completing the solution.

One readily finds that these satisfy

\[
\frac{1}{n_1} \frac{dn_1}{dt} = 3 \frac{m_1a_2 \left[ \frac{1}{37_{\text{inj}}^2} + \frac{e_1^2}{(1 - e_1^2) M_3} \right] + a_1 m_2 \left[ \frac{1}{37_{\text{inj}}^2} + \frac{e_2^2}{(1 - e_2^2) M_2} \right]}{(m_2a_1 + m_1a_2)},
\]

(A23)

\[
\frac{de_1}{dt} = -\frac{e_1 \tau_{\text{inj}}}{\Lambda t_1} - \frac{e_1 e_2(A - 1)}{\Lambda t_1} \frac{de_1}{dt} + \left\{ \frac{1}{37_{\text{inj}}^2} + \frac{e_1^2}{(1 - e_1^2) M_3} - \frac{1}{37_{\text{inj}}^2} + \frac{e_2^2}{(1 - e_2^2) M_2} \right\} \frac{T}{\Lambda (m_2a_1 + m_1a_2)},
\]

(A24)
where

\[ \Lambda = 1 + \frac{e_2}{e_1} \frac{d e_2}{d e_1} \left( \frac{\sqrt{1 - e_1^2 m_2 n_1 a_1}}{\sqrt{1 - e_2^2 m_1 n_2 a_2}} \right) \]  

(A25)

and

\[ T = \frac{m_2 a_1 \sqrt{1 - e_1^2}}{p + 1} \left[ 1 - (p + 1) \left( 1 - \sqrt{1 - e_1^2} \right) - p \sqrt{1 - e_2^2 + p} \right]. \]  

(A26)

Equation (A23) implies that \( n_1 \) always increases with time corresponding to inward migration. Notably, this equation does not depend on the order of the resonance.

Equation (A24) governs the evolution of the eccentricities. It indicates that when \( e_1 = e_2 = 0 \), given that the derivative \( \frac{d e_2}{d e_1} \) remains finite, the eccentricities will increase with time provided that \( 1/t_{\text{mig}1} > 1/t_{\text{mig}2} \). This is just the condition for the migration of the uncoupled two-planet system to be convergent, in the sense that the relative separation should decrease. When the circularization times are finite, there is the possibility of steady-state eccentricities found by equating the right-hand side of equation (A24) to zero. Thus, in the limit that they are small, the equilibrium eccentricities must satisfy

\[ \frac{e_1}{t_{c1}} + \frac{e_2}{t_{c2}} \frac{m_2 n_1 a_1}{m_1 n_2 a_2} = \left( \frac{e_1}{t_{c1}} - \frac{e_2}{t_{c2}} \right) f = \left( \frac{1}{t_{\text{mig}1}} - \frac{1}{t_{\text{mig}2}} \right) \frac{f}{3}, \]  

(A27)

where \( f = m_2 a_1 / [(p + 1)(m_2 a_1 + m_1 a_2)] \). Note that, when the migration and circularization times \( t_{\text{mig}} \) and \( t_c \) scale in the same way as the mean motions at an orbital location, the equilibrium eccentricities are independent of orbital location.