Smoothing supernova data to reconstruct the expansion history of the Universe and its age

Arman Shafieloo,1⋆ Ujjaini Alam,1⋆ Varun Sahni1⋆ and Alexei A. Starobinsky2⋆

1 Inter University Centre for Astronomy and Astrophysics, Pune, India
2 Landau Institute for Theoretical Physics, 119334 Moscow, Russia

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1 INTRODUCTION

The nature of dark energy has been the subject of much debate over the past decade (for reviews, see Sahni & Starobinsky 2000; Carroll 2001; Padmanabhan 2003; Peebles & Ratra 2003; Sahni 2005a). The supernova (SN) Type Ia data, which gave the first indications of the accelerated expansion of the Universe, are expected to throw further light on this intriguing question as their quality steadily improves. While the number of SNe available to us has increased two-fold over the past couple of years (at present, there are about 150 SNe between redshifts of 0 and 1.75, with 10 SNe above a redshift of unity; Riess et al. 1998; Perlmutter et al. 1999; Knop et al. 2003; Tonry et al. 2003; Riess et al. 2004), the SNe data are still not of a quality to firmly distinguish different models of dark energy. In this connection, an important role in our quest for a deeper understanding of the nature of dark energy has been played by the ‘reconstruction programme’. Commencing from the first theoretical exposition of the reconstruction idea (Starobinsky 1998; Huterer & Turner 1999; Nakamura & Chiba 1999; Sahni et al. 2000), which applied it to an early SN data set, there have been many attempts to reconstruct the properties of dark energy directly from observational data without assuming any particular microscopic/phenomenological model for the former. When using SNe data for this purpose, the main obstacle is the necessity (i) to differentiate the data once to pass from the luminosity distance $d_L$ to the Hubble parameter $H(t) \equiv \dot{a}(t)/a(t)$ and to the effective energy density of dark energy $\epsilon_{DE}$ and (ii) to differentiate the data a second time in order to obtain the deceleration parameter $q \equiv -\ddot{a}a/\dot{a}^2$, the dark energy effective pressure $p_{DE}$, and the equation of state parameter $w(t) \equiv p_{DE}/\epsilon_{DE}$. Here, $a(t)$ is the scalefactor of a Friedmann–Robertson–Walker (FRW) isotropic cosmological model, which we further assume to be spatially flat, as predicted by the simplest variants of the inflationary scenario of the early Universe and confirmed by observational cosmic microwave background (CMB) data.

To get around this obstacle, some kind of smoothing of $d_L$ data with respect to its argument – the redshift $z(t)$ – is needed. One possible way is to parametrize the quantity which is of interest [$H(z)$, $w(z)$, etc.] by some functional form containing a few free parameters, and then to determine the value of these parameters which produce the best fit to the data. This implies an implicit smoothing of $d_L$ with a characteristic smoothing scale defined by the number of parameters, and with a weight depending on the form of parametrization. Different parametrizations have been used for $d_L$ (Huterer & Turner 1999; Chiba & Nakamura 2000; Sahni et al. 2000), $H(z)$ (Sahni et al. 2003; Alam et al. 2004a; Alam, Sahni & Starobinsky 2004b), $w(z)$ (Chevallier, Polarski & Starobinsky 2001; Gerke & Efstathiou 2002; Maor et al. 2002; Weller & Albrecht 2002; Corasaniti & Copeland 2003; Linder 2003; Nesseris

ABSTRACT

We propose a non-parametric method of smoothing supernova data over redshift using a Gaussian kernel in order to reconstruct important cosmological quantities including $H(z)$ and $w(z)$ in a model-independent manner. This method is shown to be successful in discriminating between different models of dark energy when the quality of data is commensurate with that expected from the future Supernova Acceleration Probe (SNAP). We find that the Hubble parameter is especially well determined and useful for this purpose. The look-back time of the Universe may also be determined to a very high degree of accuracy ($\lesssim 0.2$ per cent) using this method. By refining the method, it is also possible to obtain reasonable bounds on the equation of state of dark energy. We explore a new diagnostic of dark energy – the ‘$w$-probe’ – which can be calculated from the first derivative of the data. We find that this diagnostic is reconstructed extremely accurately for different reconstruction methods even if $\Omega_{0\Lambda}$ is marginalized over. The $w$-probe can be used to successfully distinguish between $\Lambda$ cold dark matter and other models of dark energy to a high degree of accuracy.

Key words: methods: statistical – cosmological parameters – cosmology: theory.

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sity was expanded in a polynomial ansatz, the properties of which
sahni et al. (2003) proposed a slightly different approach in which the dark energy den-
sity was expanded in a polynomial ansatz, the properties of which
were examined in Alam et al. (2004a,b,c). See Alam et al. (2003),
Gong (2005b) and Basset, Corasaniti & Kunz (2004) for a summary
of different approaches to the reconstruction programme and for a
more extensive list of references. In spite of some ambiguity in the
form of these different parametrizations, it is reassuring that they
produce consistent results for the best-fitting curve over the range
0.1 \leq z \leq 1 where we have a sufficient amount of data (see, for
example, fig. 10 in Gong 2005b). However, it is necessary to point
out that the current SNe data are not of a quality that could allow us
to unambiguously differentiate ACDM from evolving dark energy.
This is why our focus in this paper is on better quality data – from the
Supernova Acceleration Probe (SNAP) experiment – which should be
able to successfully address this important issue.

A different, non-parametric smoothing procedure involves di-
rectly smoothing either \(d_L\), or any other quantity defined within
redshift bins, with some characteristic smoothing scale. Different
forms of this approach have been elaborated in Wang &
Loavelace (2001), Huterer & Starkman (2003), Saini (2003), Daly &
Djorgovski (2003, 2004), Wang & Teegmark (2005) and Espana-
Bonet & Ruiz-Lapuente (2006). One of the advantages of this ap-
proach is that the dependence of the results on the size of the smoo-
hing scale becomes explicit. We emphasize again that the present con-
sensus seems to be that, while the cosmological constant remains
a good fit to the data, more exotic models of dark energy are by
no means ruled out (although their diversity has been significantly
narrowed already). Thus, until the quality of data improves dra-
mically, the final judgement on the nature of dark energy cannot yet
be pronounced.

In this paper, we develop a new reconstruction method which for-
ormally belongs to the second category, and which is complementary
to the approach of fitting a parametric ansatz to the dark energy
density or the equation of state. Most of the papers using the non-
parametric approach cited above exploited a kind of top-hat smoo-
thing in redshift space. Instead, we follow a procedure which is well
known and frequently used in the analysis of large-scale structure
(Coles & Lucchin 1995; Martinez & Saar 2002); that is, we attempt
to smooth noisy data directly using a Gaussian smoothing function.
Then, from the smoothed data, we calculate different cosmolo-
gical functions, and thus extract information about dark energy. This
method allows us to avoid additional noise due to sharp borders be-
tween bins. Furthermore, because our method does not assume any
definite parametric representation of dark energy, it does not bias re-
sults towards any particular model. We therefore expect this method
to give us model-independent estimates of cosmological functions,
in particular, the Hubble parameter \(H(z) \equiv \dot{a}(t)/a(t)\). On the basis of
data expected from the SNAP satellite mission, we show that the
Gaussian smoothing ansatz proposed in this paper can successfully
distinguish between rival cosmological models and help shed light
on the nature of dark energy.

2 Methodology

It is useful to recall that, in the context of structure formation, it is
often advantageous to obtain a smoothed density field \(\delta^s(x)\) from a
fluctuating ‘raw’ density field, \(\delta(x)\), using a low-pass filter \(F\) having
a characteristic scale \(R_t\) (Coles & Lucchin 1995)

\[
\delta^s(x, R_t) = \int \delta(x') F(|x - x'|; R_t) \, dx'.
\] (1)

Commonly used filters include (i) the ‘top-hat’ filter, which has a
sharp cut-off \(F_{TH} \propto \Theta(1 - |x - x'|/R_{TH})\), where \(\Theta\) is the Heaviside
step function \([\Theta(z) = 0\) for \(z \leq 0, \Theta(z) = 1\) for \(z > 0]\) and (ii) the
Gaussian filter \(F_G \propto \exp(-|x - x'|^2/2R_G^2)\). For our purpose, we
find it useful to apply a variant of the Gaussian filter to reconstruct
the properties of dark energy from SN data. In other words, we apply
Gaussian smoothing to SN data – which is of the form \([\ln d_L(z, C)]\) –
in order to extract information about important cosmological pa-
rameters such as \(H(z)\) and \(w(z)\). The smoothing algorithm calculates
the luminosity distance at any arbitrary redshift \(z\) to be

\[
\ln d_L(z, \Delta) = \ln d_L(z) + N(z) \sum_i \left[ \ln d_L(z_i) - \ln d_L(z_i)^g \right] \times \exp \left\{ -\ln^2 \left[ (1 + z) / (1 + z_i) \right] \right\} / 2\Delta^2
\]

\[
N(z)^{-1} = \sum_i \exp \left\{ -\ln^2 \left[ (1 + z) / (1 + z_i) \right] \right\} / 2\Delta^2.
\] (2)

Here, \(d_L(z, \Delta)\) is the smoothed luminosity distance at any redshift
\(z\) which depends on luminosity distances of each SNe event with
the redshift \(z_i\), and \(N(z)\) is a normalization parameter. Note that the
form of the kernel bears resemblance to the lognormal distribution
(such distributions find application in the study of cosmological
density perturbations; Sahni & Coles 1995). The quantity \(\ln d_L(z)^g\)
guessed background model which we subtract from the data before smoothing it. This approach allows us to smooth noise
only, and not the luminosity distance. After noise smoothing, we
add back the guess model to recover the luminosity distance. This
procedure is helpful in reducing noise in the results. Because we
do not know which background model to subtract, we may take a
reasonable guess that the data should be close to ACDM. We use
\(d_L(z)^g = d_L(z)_{\Lambda CDM}\) as a first approximation and then use a
boot-strapping method to find successively better guess models.
We discuss this issue in greater detail in Section 3. Having obtained
the smoothed luminosity distance, we differentiate once to obtain the
Hubble parameter \(H(z)\) and twice to obtain the equation of state of
dark energy \(w(z)\), using the formulae

\[
H(z) = \left( \frac{d}{dz} \left[ \frac{d_L(z)}{1 + z} \right] \right)^{-1},
\] (3)

\[
w(z) = \frac{[2(1 + z)/3] H'/H - 1}{1 - (H_0/H)^2 \Omega_m (1 + z)^3}.
\] (4)

The results will clearly depend upon the value of the scale \(\Delta\)
in equation (2). A large value of \(\Delta\) produces a smooth result, but the
accuracy of reconstruction worsens, while a small \(\Delta\) gives a more
Table 1. Expected number of SNe per redshift bin from the SNAP experiment.

<table>
<thead>
<tr>
<th>$\Delta z$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1–0.2</td>
<td>35</td>
</tr>
<tr>
<td>0.2–0.3</td>
<td>64</td>
</tr>
<tr>
<td>0.3–0.4</td>
<td>95</td>
</tr>
<tr>
<td>0.4–0.5</td>
<td>124</td>
</tr>
<tr>
<td>0.5–0.6</td>
<td>150</td>
</tr>
<tr>
<td>0.6–0.7</td>
<td>171</td>
</tr>
<tr>
<td>0.7–0.8</td>
<td>183</td>
</tr>
<tr>
<td>0.8–0.9</td>
<td>179</td>
</tr>
<tr>
<td>0.9–1.0</td>
<td>170</td>
</tr>
<tr>
<td>1.0–1.1</td>
<td>155</td>
</tr>
<tr>
<td>1.1–1.2</td>
<td>142</td>
</tr>
<tr>
<td>1.2–1.3</td>
<td>130</td>
</tr>
<tr>
<td>1.3–1.4</td>
<td>119</td>
</tr>
<tr>
<td>1.4–1.5</td>
<td>107</td>
</tr>
<tr>
<td>1.5–1.6</td>
<td>94</td>
</tr>
<tr>
<td>1.6–1.7</td>
<td>80</td>
</tr>
</tbody>
</table>

accurate, but noisy result. Note that, for $|z - z_i| \ll 1$, the exponent in equation (2) reduces to the form $-(z - z_i)^2/2\Delta z(1 + z)^2$. Thus, the effective Gaussian smoothing scale for this algorithm is $\Delta(1 + z)$. We expect to obtain an optimum value of $\Delta$ for which both smoothness and accuracy are reasonable.

The Hubble parameter can also be used to obtain the weighted average of $w$:

$$1 + \bar{w} = \frac{1}{\delta \ln(1 + z)} \int [1 + w(z)] \frac{dz}{1 + z} = \frac{1}{3} \ln \frac{\bar{\rho}_{DE}}{\bar{\rho}_C} \tag{5}$$

$\bar{\rho}_{DE}$ is the dark energy density $\bar{\rho}_{DE} = \rho_{DE}/\rho_C$ (where $\rho_C = 3H_o^2/8\pi G$). We show in Section 5 that $\bar{w}$, which we call the $w$-probe, acts as an excellent diagnostic of dark energy, and can differentiate between different models of dark energy with greater accuracy than the equation of state.

To check our method, we use data simulated according to the SNAP experiment. This space-based mission is expected to observe close to 6000 SNe, of which about 2000 SNe can be used for cosmological purposes (Aldering et al. 2006). We propose to use a distribution of 1998 SNe between redshifts of 0.1 and 1.7 obtained from Aldering et al. (2006). This distribution of 1998 SNe is shown in Table 1. Although SNAP will not be measuring SNe at redshifts below $z = 0.1$, it is not unreasonable to assume that, by the time SNAP comes up, we can expect high-quality data at low redshifts from other SN surveys such as the Nearby Supernova Factory.1 Hence, in the low-redshift region $z < 0.1$, we add 25 more SNe of equivalent errors to the SNAP distribution, so that our data sample now consists of 2023 SNe. Using this distribution of data, we check whether the method is successful in reconstructing different cosmological parameters, and also if it can help discriminate different models of dark energy.

We simulate 1000 realizations of data using the SNAP distribution with the error in the luminosity distance given by $\sigma_{ln d_L} = 0.07$ – the expected error for SNAP. We also consider the possible effect of weak lensing on high-redshift SNe by adding an uncertainty of $\sigma_{ln d_L}(z) = 0.46(0.00311 + 0.08687z - 0.00950z^2)$ (as in Wang & Tegmark 2005). Initially, we use a simple model of dark energy when simulating data – an evolving model of dark energy with $w = -a/a_0 = -1/(1 + z)$ and $\Omega_{DE} = 0.3$. It will clearly be of interest to see whether this model can be reconstructed accurately and discriminated from $\Lambda$CDM using this method. From the SNAP distribution, we obtain smoothed data at 2000 points taken uniformly between the minimum and maximum of the distributions used. Once we are assured of the efficacy of our method, we also attempt to reconstruct other models of dark energy. Among these, one is the standard cosmological constant ($\Lambda$CDM) model with $w = -1$. The other is a model with a constant equation of state, $w = -0.5$. Such models with constant equation of state are known as quiasence models of dark energy (Alam et al. 2003) and we refer to this model as the ‘quiasence model’ throughout the paper. These three models are complementary to each other. For the $\Lambda$CDM model, the equation of state is constant at $w = -1$, $w$ remains constant at $-0.5$ for the quiasence model and, for the evolving model, $w(z)$ varies rapidly, increasing in value from $w_0 = -1$ at the present epoch to $w \approx 0$ at high redshifts.

3 RESULTS

In this section we show the results obtained when our smoothing scheme is applied to data expected from the SNAP experiment. The first issue we need to consider is that of the guess model. As mentioned earlier, the guess model in equation (2) is arbitrary. Using a guess model will naturally cause the results to be somewhat biased towards the guess model at low and high redshifts where there is a paucity of data. Therefore, we use an iterative method to estimate the guess model from an initial guess.

3.1 Iterative process to obtain the guess model

To estimate the guess model for our smoothing scheme, we use the following iterative method. We start with a simple cosmological model, such as $\Lambda$CDM, as our initial guess model: $\ln d_L^{\Lambda CDM}$. The result obtained from this analysis, $\ln d_L^{\Lambda CDM}$ is expected to be closer to the real model than the initial guess. We now use this result as our next guess model ($\ln d_L^{\Lambda CDM}$) and obtain the next result $\ln d_L^{\Lambda CDM}$. With each iteration, we expect the guess model to become more accurate, thus giving a result that is less and less biased towards the initial guess model used. A few points about the iterative method should be noted here.

(i) Using different models for the initial guess does not affect the final result, provided the process is iterated several times. For example, if we use a $w = -1/(1 + z)$ ‘metamorphosis’ model to simulate the data and use either $\Lambda$CDM or the $w = -0.5$ quiasence model as our initial guess, the results for the two cases converge by $\geq 5$ iterations.

(ii) Using a very small value of $\Delta$ will result in an accurate but noisy guess model. Therefore, after a few iterations, the result will become too noisy to be of any use. Therefore, we should use a large $\Delta$ for this process in order to obtain smoother results.

(iii) The bias of the final result will decrease with each iteration, because with each iteration we become closer to the true model. The bias decreases non-linearly with the number of iterations $M$. Generally, after about 10 iterations, for moderate values of $\Delta$, the bias is acceptably small. Beyond this, the bias still decreases with the number of iterations but the decrease is negligible while the process takes more time and results in larger errors on the parameters.

(iv) It is important to choose a value of $\Delta$ which gives a small value of bias and also reasonably small errors on the derived cosmological parameters. To estimate the value of $\Delta$ in equation (2), we

1 See http://snfactory.lbl.gov.
consider the following relation between the reconstructed results, the quality and quantity of the data and the smoothing procedure. It can be shown that the relative error bars on $H(z)$ scale as (Tegmark 2002)

$$\frac{\delta H}{H} \propto \frac{\sigma}{N^{1/2} \Delta^{3/2}},$$

(6)

where $N$ is the total number of SNe (for an approximately uniform distribution of SNe over the redshift range) and $\sigma$ is the noise of the data. From the above equation we see that a larger number of SNe or larger width of smoothing, $\Delta$, will decrease the error bars on reconstructed $H$, but as we show in Appendix A, the bias of the method is approximately related to $\Delta^2$. This implies that by increasing $\Delta$ we will also increase the bias of the results. We attempt to estimate $\Delta$ such that the error bars on $H$ are of the same order as $\sigma$, which is a reasonable expectation.

If we consider a single iteration of our method, then for $N \simeq 2000$ we obtain $\Delta_0 \simeq N^{-1/3} \simeq 0.08$. However, with each iteration, the errors on the parameters will increase. Therefore, using this value of $\Delta$ when we use an iterative process to find the guess model will result in such large errors on the cosmological parameters as to render the reconstruction exercise meaningless. We show in Appendix A that at the $M$th iteration, the error on $\ln d_L$ will be approximately $\delta_{\ln d_L} \simeq \sqrt{M} \delta_0 (\ln d_L)$. The error on $\ln d_L$ scales as $1/\Delta$. We would like the errors after $M$ iterations to be commensurate with the optimum errors obtained for a single iteration, $\Delta_0$, so we require $\Delta_{\text{optimal}} \simeq \sqrt{M} \Delta_0$. Therefore, if we wish to stop the bootstrapping after 10 iterations, then $\Delta_{\text{optimal}} \simeq 3 \Delta_0 \simeq 0.24$. This is the optimal value of $\Delta$ we use for the best results for our smoothing procedure.

Considering all these factors, we use a smoothing scale $\Delta = 0.24$ for the smoothing procedure of equation (2) with an iterative method for finding the guess model (with $\Lambda$CDM as the initial guess). The bootstrapping is stopped after 10 iterations. We see that the results reconstructed using these parameters do not contain noticeable bias and the errors on the parameters are also satisfactory.

Fig. 1 shows the reconstructed $H(z)$ and $w(z)$ with 1σ errors for the $w = -1/(1 + z)$ evolving model of dark energy. From this figure we can see that the Hubble parameter is reconstructed quite accurately and can successfully be used to differentiate the model from $\Lambda$CDM. The equation of state, however, is somewhat noisier. There is also a slight bias in the equation of state at low and high redshifts. Because the $w = -1/(1 + z)$ model has an equation of state which is very close to $w = -1$ at low redshifts, we see that $w(z)$ cannot discriminate $\Lambda$CDM from the fiducial model at $z \lesssim 0.2$ at the 1σ confidence level.

### 3.2 Age of the Universe

We may also use this smoothing scheme to calculate other cosmological parameters of interest such as the age of the Universe at a redshift $z$:

$$t(z) = H_0^{-1} \int_z^{\infty} \frac{dz'}{(1+z')H(z')}.$$

(7)

In this case, because data are available only up to redshifts of $z \simeq 1.7$, it is not possible to calculate the age of the Universe. Instead, we calculate the look-back time at each redshift:

$$T(z) = t(z = 0) - t(z) = H_0^{-1} \int_0^z \frac{dz'}{(1+z')H(z')}.$$

(8)

Fig. 2 shows the reconstructed $T(z)$ with 1σ errors for the $w = -1/(1 + z)$ ‘metamorphosis’ model using the SNAP distribution. For this model, the current age of the Universe is about 13 Gyr and the look-back time at $z \simeq 1.7$ is about 9 Gyr for a Hubble parameter of $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$. We see that the look-back time is reconstructed extremely accurately. Using this method we may predict this parameter with a high degree of success and distinguish between the fiducial look-back time and that for $\Lambda$CDM even at the 1σ confidence level. Indeed any cosmological parameter which can be obtained by integrating the Hubble parameter will be reconstructed without problem, because integrating involves a further smoothing of the results.

Looking at these results, we draw the conclusion that the method of smoothing SN data can be expected to work quite well for...
future SNAP data as far as the Hubble parameter is concerned. Using this method, we may reconstruct the Hubble parameter and therefore the expansion history of the Universe accurately. We find that the method is very efficient in reproducing $H(z)$ to an accuracy of $\lesssim 2$ per cent within the redshift interval $0 < z < 1$, and to $\lesssim 4$ per cent at $z \approx 1.7$, as demonstrated in Fig. 1. Furthermore, using the Hubble parameter, we can expect to discriminate between different families of models such as the metamorphosis model $w = -1/(1 + z)$ and $\Lambda$CDM. This method also reproduces very accurately the look-back time for a given model, as seen in Fig. 2. It reconstructs the look-back time to an accuracy of $\lesssim 0.2$ per cent at $z \approx 1.7$.

**Figure 2.** The smoothing scheme of equation (2) is used to determine the look-back time of the Universe, $T(z) = t(0) - t(z)$, from 1000 realizations of the SNAP data set for a $w = -1/(1 + z)$ ‘metamorphosis’ model. The smoothing scale is $\Delta = 0.24$. The solid lines show the mean look-back time and the 1σ limits around it. The look-back time for the fiducial model, $T(z)$, shows the same number of parameters. However, repeated smoothing can make $H(z)$ less noisy than before, while using the same number of parameters. However, repeated smoothing can also result in the loss of information.

**4 REDUCING NOISE THROUGH DOUBLE SMOOTHING**

As seen in the preceding section, the method of smoothing SN data to extract information on cosmological parameters works very well if we employ the first derivative of the data to reconstruct the Hubble parameter. It also works reasonably for the second derivative, which is used to determine $w(z)$, but the errors on $w(z)$ are somewhat large. In this section, we examine a possible way in which the equation of state may be extracted from the data to give slightly better results.

The noise in each parameter translates into larger noise levels on its successive derivatives. We have seen earlier that, using the smoothing scheme (2), we can obtain $H(z)$ from the smoothed $d_L(z)$ fairly successfully. However, small noises in $H(z)$ propagate into larger noises in $w(z)$. Therefore, it is logical to assume that if $H(z)$ were smoother, the resultant $w(z)$ might also have smaller errors. So, we attempt to smooth $H(z)$ a second time after obtaining it from $d_L(z)$. The procedure in this method is as follows. First, we smooth noisy data $\ln d_L(z)$ to obtain $\ln d_L(z)^g$ using equation (2). We differentiate this to find $H(z)^g$ using equation (3). We then further smooth this Hubble parameter by using the same smoothing scheme at the new redshifts

$$H(z, \Delta)^2 = H(z)^6 + N(z) \sum_i \left[ H(z)^6 - H(z_i)^6 \right]$$

$$\times \exp \left\{ -\frac{\ln^2 [(1 + z)/1 + z_i]}{2\Delta^2} \right\},$$

$$N(z)^{-1} = \sum_i \exp \left\{ -\frac{\ln^2 [(1 + z_i)/(1 + z)]}{2\Delta^2} \right\}. \quad (9)$$

We then use this $H(z, \Delta)^g$, to obtain $w(z)$ using equation (4). This has the advantage of making $w(z)$ less noisy than before, while using the same number of parameters. However, repeated smoothing can also result in the loss of information.

**Figure 3.** The double smoothing scheme of equations (2) and (9) has been used to obtain $H(z)$ and $w(z)$ from 1000 realizations of the SNAP data set. The smoothing scale is $\Delta = 0.24$. The dashed line in each panel represents the fiducial $w = -1/(1 + z)$ ‘metamorphosis’ model while the solid lines represent the mean and 1σ limits around it. The dotted line in both panels is $\Lambda$CDM. In the left-hand panel $H(z)$ for the fiducial model matches exactly with the mean for the smoothing scheme.
well as \( w(z) \). Thus, errors on the Hubble parameter decrease slightly and errors on \( w(z) \) also become smaller.

We now explore this scheme further for other models of dark energy. We first consider a \( w = -1 \) \( \Lambda \)CDM model. In Fig. 4, we show the results for this model. We find that the Hubble parameter is accurately reconstructed and even \( w \) is well reconstructed, with a little bias at high redshift. The next model we reconstruct is a \( w = -0.5 \) quiessence model. The results for double smoothing are shown in Fig. 5. There is a little bias for this model at the low redshifts, although it is still well within the error bars.

We note that in all three cases, a slight bias is noticeable at low or high redshifts. This is primarily because of edge effects, because at low (high) redshift, any particular point will have a fewer (greater) number of SNe to the left than to the right. Even by estimating the guess model through an iterative process, it is difficult to completely remove this effect. In order to remove this effect, it would be necessary to use a much larger number of iterations for the guess model, but this would result in very large errors on the parameters. However, this bias is so small as to be negligible and cannot affect the results in any way.

Looking at these three figures, we can draw the following conclusions. The Hubble parameter is quite well reconstructed by the method of double smoothing in all three cases while the errors on the equation of state also decrease. At low and high redshifts, a very slight bias persists. Despite this, the equation of state is reconstructed quite accurately. Also, because the average error in \( w(z) \) is less than that in the single smoothing scheme (Fig. 1), the equation of state may be used with better success in discriminating different models of dark energy using the double smoothing procedure.

5 THE \( w \)-PROBE

In this section we explore the possibility of extracting information about the equation of state from the reconstructed Hubble parameter.
by considering a weighted average of the equation of state, which we call the $w$-probe. An important advantage of this approach is that there is no need to go to the second derivative of the luminosity distance for information on the equation of state. Instead, we consider the weighted average of the equation of state (Alam et al. 2004a)

$$1 + \bar{w} = \frac{1}{\delta \ln(1 + z)} \int [1 + w(z)] \frac{dz}{1 + z}$$

which can be directly expressed in terms of the difference in dark energy density $\rho_{DE} = \rho_{DE}/\rho_0$ (where $\rho_0 = 3H_0^2/8\pi G$) over a range of redshift as

$$1 + \bar{w}(z_1, z_2) = \frac{1}{3 \delta \ln(1 + z)}$$

$$= \frac{1}{3} \ln \left[ \frac{H^2(z_1) - \Omega_{DE}(1 + z_1)^2}{H^2(z_2) - \Omega_{DE}(1 + z_2)^2} \right] \ln \left( \frac{1 + z_1}{1 + z_2} \right),$$

where $\delta$ denotes the total change of a variable between integration limits. Thus, even if the equation of state is noisy, the $\bar{w}$ parameter may be obtained accurately provided the Hubble parameter is well constructed.

The parameter $\bar{w}$ has the interesting property that for the concordance $\Lambda$CDM model, it equals $-1$ in all redshift ranges while for other models of dark energy it is non-zero. For (non-$\Lambda$CDM) models with constant equation of state, this parameter is a constant (but not equal to $-1$), while for models with a variable equation of state, it varies with redshift. The fact that $\Lambda$CDM is a fixed point for this quantity may be utilized to differentiate between the concordance $\Lambda$CDM model and other models of dark energy. Therefore, the parameter $\bar{w}$ may be used as a new diagnostic of dark energy which acts as a discriminator between $\Lambda$CDM and other models of dark energy. We call this diagnostic the $w$-probe.

We now calculate the $w$-probe for the three models described above using the method of double smoothing. In Table 2, we show the values of $\bar{w}$ obtained in different redshift ranges after applying double smoothing on SNAP-like data. The ranges of integration are taken to be approximately equally spaced in $\ln(1 + z)$. Two points of interest should be noted here: (i) $\bar{w}$ is very close to $\bar{w}_{\text{exact}}$ at all redshifts for all three models of dark energy; (ii) as expected, this parameter is good at distinguishing between $\Lambda$CDM and other dark energy models.

5.1 Robustness of the $w$-probe to observational uncertainties in $\Omega_{0m}$

In the above analysis, we have assumed that the matter density is known exactly, $\Omega_{0m} = 0.3$. Studies of large-scale structure and the CMB have resulted in very tight bounds on the matter density, but still some uncertainty remains regarding its true value. As noted in Maor et al. (2002), a small uncertainty in the value of $\Omega_{0m}$ may affect the reconstruction exercise quite dramatically. The Hubble parameter is not affected to a very high degree by the value of matter density, because it can be calculated directly as the first derivative of the luminosity distance, which is the measured quantity. However, when calculating the equation of state of dark energy, the value of $\Omega_{0m}$ appears in the denominator of the expression (4), and hence any uncertainty in $\Omega_{0m}$ is bound to affect the reconstructed $w(z)$.

One of the main results of this paper is that, although the equation of state $w(z)$ may be reconstructed badly if $\Omega_{0m}$ is not known accurately, the uncertainty in $\Omega_{0m}$ does not have such a strong effect on the reconstruction of the $w$-probe $\bar{w}$. This is because $\bar{w}$ in equation (11) is the difference of two terms, both involving $\Omega_{0m}$. As a result, uncertainty in $\Omega_{0m}$ does not affect $\bar{w}$ as much as it affects $w(z)$. Therefore, even when $\Omega_{0m}$ is not known to a high degree of accuracy, the $w$-probe may be reconstructed fairly accurately.

We now demonstrate this by showing the results obtained using our smoothing scheme after marginalizing over the matter density. We simulate SNAP-like data for two models: (i) $\Lambda$CDM and (ii) a $w = -1/(1 + z)$ ‘metamorphosis’ model. When applying the smoothing scheme, we assume that $\Omega_{0m}$ follows a Gaussian probability distribution with mean $\bar{\Omega}_{0m} = 0.3$ and variance $\sigma = 0.07$ (the error being commensurate to that expected from the current CMB and large-scale structure data; Percival et al. 2001). In Fig. 6 and Table 3, we show the results for the $w$-probe calculated for the two models. We find that the $w$-probe ($\bar{w}$) is determined to a high degree of accuracy for both the models even when we marginalize over $\Omega_{0m}$. The value of $\bar{w}$ for the $\Lambda$CDM model is approximately equal to $-1$, while that for the metamorphosis model shows a clear signature of evolution. Thus, even if the matter density of the Universe is known uncertainly, this uncertainty does not affect the accuracy of the reconstructed $w$-probe significantly. This is a powerful result because it indicates that unlike the equation of state, the $w$-probe is not overtly sensitive to the value of $\Omega_{0m}$ for SNAP-quality data.

5.2 (In)Dependence of the $w$-probe on the method of reconstruction

In the previous section we calculated the $w$-probe for the smoothing scheme proposed in this paper. However, looking at equation (11), we surmise that any reconstruction method that accurately calculates either the dark energy density or the Hubble parameter may also be used to determine $\bar{w}$. In this section we test this idea. For this purpose we consider a complementary reconstruction ansatz that uses a polynomial fit to the dark energy density (Sahni et al. 2003)

$$H^2(z) = H_0^2 \left[ \Omega_{DE}(1 + z)^2 + A_0 + A_1(1 + z) + A_2(1 + z)^2 \right],$$

where $A_0 = 1 - \Omega_{DE} - A_1 - A_2$ for a flat universe. This ansatz is known to give accurate results for the dark energy density (Alam et al. 2004a,b).

We now test whether the $w$-probe remains insensitive to the value of matter density for this reconstruction scheme as well. As in the

<table>
<thead>
<tr>
<th>$\Delta z$</th>
<th>$w = -1/(1 + z)$</th>
<th>$\bar{w}$</th>
<th>$\bar{w}_{\text{exact}}$</th>
<th>$w$</th>
<th>$\bar{w}$</th>
<th>$\bar{w}_{\text{exact}}$</th>
<th>$w = -0.5$</th>
<th>$\bar{w}$</th>
<th>$\bar{w}_{\text{exact}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–0.414</td>
<td>$-0.839 \pm 0.019$</td>
<td>$-0.845$</td>
<td>$-1.001 \pm 0.017$</td>
<td>$-1.0$</td>
<td>$-0.489 \pm 0.025$</td>
<td>$-0.5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.414–1</td>
<td>$-0.595 \pm 0.033$</td>
<td>$-0.598$</td>
<td>$-1.009 \pm 0.038$</td>
<td>$-1.0$</td>
<td>$-0.506 \pm 0.039$</td>
<td>$-0.5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1–1.7</td>
<td>$-0.471 \pm 0.069$</td>
<td>$-0.432$</td>
<td>$-1.017 \pm 0.087$</td>
<td>$-1.0$</td>
<td>$-0.493 \pm 0.075$</td>
<td>$-0.5$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 6. The \( w \)-probe is reconstructed for the unevolving \( \Lambda \)CDM model with \( w = -1 \) (left-hand panel) and an evolving DE model with \( w = -1/(1+z) \) (right-hand panel). 1000 realizations of SNAP-like data have been used. The thick dashed line in both panels indicates the exact value of \( \bar{w} \) for the fiducial model, the dark grey boxes in each panel indicate the 1\( \sigma \) confidence levels on \( \bar{w} \) reconstructed for the two models using the double smoothing scheme with \( \Delta = 0.24 \) and marginalizing over \( \Omega_{\text{om}} = 0.3 \pm 0.07 \). This figure illustrates that the \( w \)-probe works remarkably well for both \( \Lambda \)CDM (left-hand panel) and for evolving DE (right-hand panel). The details for this figure are given in Table 3.

Table 3. The reconstructed \( w \)-probe \( \bar{w} \) (equation 11) over specified redshift ranges (and its 1\( \sigma \) error) is shown for 1000 realizations of SNAP data. Two fiducial models are used: the \( w = -1/(1+z) \) ‘metamorphosis’ model and \( w = -1 \) (\( \Lambda \)CDM). We deploy the method of double smoothing with \( \Delta = 0.24 \) and marginalize over \( \Omega_{\text{om}} = 0.3 \pm 0.07 \).

<table>
<thead>
<tr>
<th>( \Delta_z )</th>
<th>( w = -1/(1+z) )</th>
<th>( \bar{w}_{\text{exact}} )</th>
<th>( \bar{w} )</th>
<th>( \bar{w}_{\text{exact}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-0.414</td>
<td>-0.837 ± 0.025</td>
<td>-0.845</td>
<td>-1.003 ± 0.021</td>
<td>-1.0</td>
</tr>
<tr>
<td>0.414-1</td>
<td>-0.618 ± 0.042</td>
<td>-0.598</td>
<td>-1.018 ± 0.052</td>
<td>-1.0</td>
</tr>
<tr>
<td>1-1.7</td>
<td>-0.461 ± 0.127</td>
<td>-0.432</td>
<td>-1.051 ± 0.147</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

Figure 7. The reconstructed equation of state \( w(z) \) (left-hand panel) and the \( w \)-probe \( \bar{w} \) (right-hand panel) are shown for 1000 realizations of a \( \Omega_{\text{om}} = 0.3 \), \( w = -1 \) model. We assume an incorrect value for the matter density, \( \Omega_{\text{om}} = 0.2 \) in the reconstruction exercise. This is done to study the effect if the observed uncertainty in \( \Omega_{\text{om}} \) on the reconstructed quantities. The double smoothing scheme of equations (2) and (9) has been used with a smoothing scale of \( \Delta = 0.24 \). The dashed line in each panel represents the fiducial \( \Lambda \)CDM model with \( w = -1.0, \bar{w} = -1.0 \). The solid lines in panel (a) represent the mean \( w(z) \) and the 1\( \sigma \) limits around it. In panel (b) the dark grey boxes represent the 1\( \sigma \) confidence levels on \( \bar{w} \). Comparing the left and right-hand panels, we see that while an incorrect value of \( \Omega_{\text{om}} \) results in a completely wrong reconstruction of \( w(z) \), the \( w \)-probe (right-hand panel) is quite robust to small changes in \( \Omega_{\text{om}} \) and provides correct information about the equation of state.
of $\bar{w} = -1$. Fig. 8 shows the results using the polynomial expansion of dark energy density (12). Here, also, we see that the equation of state reconstructed for this ansatz, with the wrong value of $\Omega_{0m}$, has very different characteristics from the true model; it appears to vary with redshift $w(z) \geq -1$, whereas the true equation of state is $w = -1$. Here, too, the value of $\bar{w}$ is commensurate with the actual value. This result addresses a key issue regarding the reconstruction exercise raised in Maor et al. (2002): if $\Omega_{0m}$ is known inaccurately, reconstruction may result in completely wrong information about the nature of dark energy when we use cosmological parameters such as the equation of state. We see, however, that the $w$-probe can yield precious information about the averaged equation of state for SNAP-quality data even when the value of $\Omega_{0m}$ is not known precisely.

From the above results, we see that the $w$-probe is very effective as a diagnostic of dark energy, especially in differentiating between $\Lambda$CDM and other models of dark energy. We summarize some important properties of the $w$-probe below.

(i) $\bar{w}(z_1, z_2)$ is determined from the first derivative of the luminosity distance. Its reconstructed value is therefore less noisy than the equation of state $w(z)$, which is determined after differentiating $d_L(z)$ twice.

(ii) $\bar{w}(z_1, z_2) = -1$ uniquely for concordance cosmology ($\Lambda$CDM). For all other dark energy models $\bar{w} \neq -1$. This remains true when $\bar{w}$ is marginalized over $\Omega_{0m}$.

(iii) $\bar{w}$ is robust to uncertainties in the value of the matter density. As seen earlier, this uncertainty can induce large errors in determinations of the cosmic equation of state $w(z)$ (see also Maor et al. 2002). The exceedingly weak dependence of $\bar{w}$ on the value of $\Omega_{0m}$ in the range currently favoured by observations ($0.2 \leq \Omega_{0m} \leq 0.4$) implies that the $w$-probe can cope very effectively with the existing uncertainty in the value of the matter density for SNAP-quality data.

(iv) Our results in Figs 7 and 8 have demonstrated that $\bar{w}$ is very weakly dependent on the ansatz used for cosmological reconstruction. We therefore feel that because $\bar{w}$ is constructed directly from $\rho_{DE}$, any method which determines either the dark energy density or the Hubble parameter from observations can be used to also determine $\bar{w}$.

Thus, we expect that the $w$-probe may be used as a handy diagnostic for dark energy, especially in discriminating between $\Lambda$CDM and other models of dark energy, for SNAP-like data sets. Its efficacy lies in the fact that it is not very sensitive to both the value of the present matter density and the reconstruction method used.

### 6 COSMOLOGICAL RECONSTRUCTION APPLIED TO OTHER PHYSICAL MODELS OF DARK ENERGY

In this section we draw attention to the dangers encountered during cosmological reconstruction of a typical dark energy model. There are currently two plausible ways of making the expansion of the Universe accelerate at late times. The first approach depends on changing the matter sector of the Einstein equations. Examples of this approach are the quintessence fields. A completely different approach has shown that it is possible to obtain an accelerating universe through modifying the gravity sector (see, for instance, Dvali, Gabadadze, & Porrati 2002; Freese & Lewis 2002; Capozziello, Carloni & Troisi 2003; Carroll, Hoffman & Trodden 2003; Dolgov & Kawasaki 2003; Nojiri & Odintsov 2003; Sahni & Shtanov 2003; Sahni 2005; and references therein). In these models, dark energy should not be treated as a fluid or a field. Instead, it may be better dubbed as ‘geometric dark energy’. Indeed the Dvali–Gabadadze–Porrati (DGP) model can cause the Universe to accelerate even in the absence of a physical dark energy component. As pointed out in Alam et al. (2003) and Sahni (2005), the equation of state is
not a fundamental quantity for geometric dark energy. For example, using \( u(z) \) in the reconstruction of such models may result in very strange results, including, for instance, singularities in the equation of state.\(^2\)

As an example, we consider the braneworld dark energy model proposed in Sahni & Shtanov (2003), described by the following set of equations for a flat universe:

\[
\frac{H^2(z)}{H_0^2} = \Omega_{\text{om}}(1+z)^3 + \Omega_\sigma + 2\Omega_l - 2\sqrt{\Omega_l}\sqrt{\Omega_{\text{om}}(1+z)^3 + \Omega_\sigma + \Omega_l + 2\Omega_\Lambda}
\]

\[
\Omega_\sigma = 1 - \Omega_{\text{om}} + 2\sqrt{\Omega_l(1 + \Omega_\Lambda)}.
\]

Here, the densities \( \Omega \) are defined as

\[
\Omega_{\text{om}} = \frac{\rho_{\text{om}}}{3m^2H_0^2}, \quad \Omega_\sigma = \frac{\sigma}{3m^2H_0^2},
\]

\[
\Omega_l = \frac{1}{l^2H_0^2}, \quad \Omega_\Lambda = -\frac{\Lambda_0}{6H_0^2},
\]

where \( l = m^2/M^4 \) is a new length-scale (\( m \) and \( M \) refer, respectively, to the four- and five-dimensional Planck masses), \( \Lambda_0 \) is the bulk cosmological constant and \( \sigma \) is the brane tension. In this section we have used \( \hbar = c = 1 \). For short length-scales \( r \ll l \), and at early times, we recover general relativity, whereas for large length-scales \( r \gg l \) and at late times brane-related effects become important and may lead to the acceleration of the Universe. The ‘effective’ equation of state for this braneworld model is given by

\[
\rho = \frac{3H^2}{8\pi G}[1 - \Omega_{\text{om}}(z)], \quad p = \frac{H^2}{4\pi G}[q(z) - 1/2]
\]

\[
w_{\text{eff}} = \frac{p}{\rho} = \frac{q(z) - 1/2}{3[1 - \Omega_{\text{om}}(z)]}.
\]

It is obvious that the effective equation of state in this braneworld model may become singular if \( \Omega_{\text{om}} = \Omega_{\text{om}}(1+z)^3 H_0^2/H^2(z) \) becomes unity. This does not signal any inherent pathologies in the model, however. We should remember that the acceleration of the Universe in this model is a result of modification of the expansion of the Universe at late times due to extradimensional effects. Hence, it is not very appropriate to describe dark energy by an equation of state for such a model. However, it would be interesting to see if the singularity in the effective \( w \) for this model can be recovered by our smoothing method.

We attempt to reconstruct an \( \Omega_{\text{om}} = 0.3, \Omega_l = 1, \Omega_\Lambda = 0 \) braneworld model which is a good fit to the current SN data (Alam & Sahni 2002). We simulate data according to SNAP and obtain results for the double smoothing method with \( \Delta = 0.24 \). In Fig. 9, we show the reconstructed Hubble parameter for this reconstruction. We see that the Hubble parameter is very well reconstructed and shows no pathological behaviour.

We now obtain the equation of state of dark energy for this model. For this purpose, we also use an ansatz for the equation of state as suggested by Chevallier et al. (2001) and Linder (2003) (the CPL fit)

\[
w(z) = w_0 + \frac{w_1z}{1+z}.
\]

The results are shown in Fig. 10. We find, as expected, that it is impossible to catch the singularity in the equation of state at \( z \approx 0.8 \) using an equation of state ansatz. Of course, we can try and improve upon this somewhat dismal picture by introducing fits with more free parameters. However, it is well known that the presence of more degrees of freedom in the fit leads to a larger degeneracy (between parameters) and hence to larger errors of reconstruction (Weller & Albrecht 2002). In contrast to this approach, when we reconstruct the equation of state using the smoothing scheme (which does not presuppose any particular behaviour of the equation of state), the Hubble parameter is reconstructed very accurately and hence the ‘effective’ equation of state for this model is also reconstructed well, as shown in Fig. 10. From this figure we see clear evidence of the singularity at \( z \approx 0.8 \). Thus, to obtain maximum information about the equation of state, especially in cases where the dark energy model is very different from the typical quintessence-like models, it may be better to reconstruct the Hubble parameter or the dark energy density first.

Therefore, we find that the smoothing scheme, which performs reasonably when reconstructing quintessence models of dark energy models, can be also applied to models which show a departure from

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\(^2\) A very simple model which has a well-behaved \( q(z) \) but singular \( u(z) \) is a model which has, in addition to the cosmological constant, a second dark energy component disguised as a spatial curvature term: \( H^2(z)/H_0^2 = \Omega_{\text{om}}(1+z)^3 + \Omega_\sigma(1+z)^2 + \Lambda_0 \). If we assume that \( \Omega_{\text{om}} = 0.3, \Omega_\sigma = -0.05, \Omega_\Lambda = 0.75 \), then \( u(z) \) becomes singular when \( \Omega_\sigma(1+z)^2 + \Lambda_0 = 0 \), i.e. at \( z \approx 2.8 \). Although this property of \( u(z) \) can be easily understood physically and rests in the fact that it is an ‘effective’ equation of state for the combination of dark energy fluids, nevertheless any reasonable parametrization of \( u(z) \) will clearly experience difficulty in reproducing this behaviour. An effective equation of state with a similar ‘pole-like’ divergence is frequently encountered in braneworld models of dark energy (Sahni & Shtanov 2003, 2005) as well as in holographic models (Linder 2004).
general relativistic behaviour at late times.\footnote{Note, however, that most reconstruction methods including the present one may have problems in reproducing the rapidly oscillating equation of state predicted to arise in some models of dark energy (Sahni & Wang 2000).} This section illustrates the fact that, in general, reconstructing $H(z)$ and its derivatives such as the deceleration parameter $q(z)$ may be less fraught with difficulty than a reconstruction of $w(z)$, which, being an effective equation of state and not a fundamental physical quantity in some dark energy models, can often show peculiar properties.

7 CONCLUSION

In this paper we have presented a new approach to analysing SN data and have used it to extract information about cosmological functions, such as the expansion rate of the Universe $H(z)$ and the equation of state of dark energy $w(z)$. In this approach, we deal with the data directly and do not rely on a parametric functional form for fitting any of the quantities $d_L(z)$, $H(z)$ or $w(z)$. Therefore, we expect the results obtained using this approach to be model-independent. A Gaussian kernel is used to smooth the data and to calculate cosmological functions including $H(z)$ and $w(z)$. The smoothing scale used for the kernel is related to the number of SNe, the errors of observations and the derived errors of the parameters by a simple formula (equation 6). For a given SN distribution, the smoothing scale determines both the errors on the parameters and the bias of the results (see Appendix A). $\Delta$ cannot be increased arbitrarily as this would diminish the reliability of the results. We use a value of $\Delta$ which gives results that have reasonably small bias as well as acceptable errors of $H(z)$ for the SNAP-quality data used in our analysis (see Section 3). As can be seen from equation (6), when the data improve (i.e. the number of data points increases and/or measurement errors decrease), we expect that the same value of $\Delta$ would result in smaller errors on $H(z)$.

We demonstrate that this method is likely to work very well with future SNAP-like SNe data, especially in reconstructing the Hubble parameter, which encodes the expansion history of the Universe. Moreover, our successful reconstruction of the Hubble parameter can also be used to distinguish between cosmological models such as $\Lambda$CDM and evolving dark energy. The method can be further refined, if one wishes to reconstruct the cosmic equation of state to greater accuracy, by double smoothing the data: smoothing the Hubble parameter, after it has been derived from the smoothed luminosity distance, so as to reduce noise in $w(z)$ (as in Section 4). The results obtained using the smoothing scheme compare favourably to results obtained by other methods of reconstruction. Another quantity which may be reconstructed to great accuracy is the look-back time of the Universe.

An important result of this paper is the discovery that the $w$-probe (originally proposed in Alam et al. 2004a) provides us with an excellent diagnostic of dark energy. We summarize some of the attractive features of this diagnostic below.

(i) The $w$-probe defined in equations (10) and (11) is obtained from the luminosity distance by means of a single differentiation. Therefore, it avoids the pitfalls of $w(z)$ which is obtained from the luminosity distance through a double differentiation (see equation 4), and hence is usually accompanied by large errors (see also Maor, Brustein & Steinhardt 2001).

(ii) The $w$-probe is robust to small uncertainties in the value of $\Omega_{\text{m}}$. This attractive property allows us to get around observational uncertainties in the value of $\Omega_{\text{m}}$ currently known to an accuracy of about 30 per cent. Indeed, when marginalized over $\Omega_{\text{m}}$, the $w$-probe can be used to great advantage to distinguish between $\Lambda$CDM and other dark energy models for SNAP-quality data.

We therefore conclude that the proposed reconstruction method by smoothing the SN data appears to be sufficiently accurate and, when applied to SNAP-type observations, should be able to distinguish between evolving dark energy models and a cosmological constant.

The method proposed by us can also be used for other forms of data which deliver the luminosity (or angular size) distance.

ACKNOWLEDGMENTS

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APPENDIX A: SMOOTHING ERRORS AND BIAS

In this appendix we explore the errors on the cosmological parameters as a result of the smoothing scheme, and also the bias which enters the results.

A1 Smoothing errors

The smoothing scheme used in this paper is of the form

\[ y(z) = y_G(z) + \sum_{i=1}^{N} [y_G(z_i) - y_G(z)] S(z_i, \Delta) / \sum_{i=1}^{N} S(z_i, \Delta) \]  \hfill (A1) \n
where the quantity \( S(z_i; \Delta) \) represents the smoothing function with a scale \( \Delta \) and \( y_G(z) \) is the subtracted guess model. The quantity being smoothed (in this case \( \ln d_L \)) is represented by \( y \), while \( y' \) represents the smoothed result. Let the errors in the data at any redshift \( z_i \) be given by \( \sigma_i(z_i) \) and the errors in the guess model be \( \sigma_{G_i}(z_i) \). If we look at the second term on the right-hand side of equation (A1), we see that the errors on this term would be approximately given by the errors on \( y \) weighted down by the smoothing scale \( \Delta \) and the number of data points \( N \). Therefore, the error on the smoothed result is

\[ \sigma_{y_i}(z) \approx \sigma_{G_i}(z) + \frac{\sigma_i(z_i)^2}{\Delta N}. \]  \hfill (A2) \n
We now consider the errors for an iterative method. The first guess is an exact model, \( \Lambda \text{CDM} \). Therefore, the error on the result of the first iteration is simply

\[ \sigma_{y_1}(z) \approx \frac{\sigma_i(z)}{\Delta N}. \]  \hfill (A3) \n
The next guess model is \( y_1(z) \). Therefore, the error on the result is

\[ \sigma_{y_2}(z) \approx \left( 2 + \frac{1}{N} \right) \sigma_{y_1}(z) N/\Delta. \]  \hfill (A4) \n
From this we can show that the error on the result for the \( M \)th iteration is

\[ \sigma_{y_M}(z) \approx \left[ 1 + \sum_{i=1}^{M-1} \left( 1 + \frac{1}{N} \right)^i \right] \sigma_{y_1}(z) N/\Delta. \]  \hfill (A5) \n
The second term on the right-hand side is small for a reasonable number of iterations, because \( N \approx 2000 \) and \( \Delta > 0.01 \) usually.

Therefore, we may approximate the errors on the log luminosity distance after $M$ iterations for the guess model as

$$
\sigma^M_{\ln d_L}(z) = \sqrt{M} \sigma^0_{\ln d_L}(z),
$$

(A6)

where $\sigma^0_{\ln d_L}(z)$ is the error for a simple smoothing scheme where the data are smoothed without using a guess model.

**A2 Smoothing bias**

In any kind of smoothing scheme for the luminosity distance, some bias is introduced both in it and in derived quantities such as $H(z)$ and $w(z)$. To illustrate the effect of this bias, we calculate it for the simplest Gaussian smoothing scheme for $\ln d_L(z)$ with the width $\Delta(z) \ll 1$

$$
\ln d_L(z)^\prime = N(z) \sum_{i=1}^{M} \ln d_L(z_i) \exp \left[ -\frac{(z-z_i)^2}{2\Delta^2} \right],
$$

(A7)

$$
N(z)^{-1} = \sum_{i=1}^{M} \exp \left[ -\frac{(z-z_i)^2}{2\Delta^2} \right],
$$

(A8)

where $M$ is the total number of SNe data points. The bias at each redshift $[B(z) = \ln d_L(z)^\prime - \ln d_L(z)]$ is the difference between the smoothed $\ln d_L(z)$ and the exact value of $\ln d_L(z)$:

$$
B(z) = N(z) \sum_{i=1}^{M} \left[ \ln d_L(z_i) - \ln d_L(z) \right] \exp \left[ -\frac{(z-z_i)^2}{2\Delta^2} \right].
$$

(A9)

Expanding $\ln d_L(z_i)$ in terms of $\ln d_L(z)$ and its derivatives by Taylor expansion, we obtain

$$
B(z) = N(z) \sum_{i=1}^{M} \times \left\{ \ln d_L(z_i) \left[ (z_i - z) + \ln d_L(z_i) \right] \frac{(z_i - z)^2}{2} \right\} \exp \left[ -\frac{(z-z_i)^2}{2\Delta^2} \right],
$$

(A10)

where the prime denotes the derivative with respect to $z$ and we neglect higher derivatives. To see the effect of this bias at low and high redshifts where the numbers of SNe on both sides of each $z$ are not equal, we rewrite equation (A10) in another way. Let $\delta$ be the spacing between two neighbouring data points, so that $z = m\delta$. For $m < M/2$, we have

$$
B(z) = N(z) \sum_{i=1}^{2m} \left\{ \ln d_L(z_i) \right\} \frac{\delta^2(i - m)^2}{2} \exp \left[ -\frac{\delta^2(i - m)^2}{2\Delta^2} \right] + N(z) \sum_{i=2m+1}^{M} \times \left\{ \ln d_L(z_i) \delta(i - m) + \ln d_L(z_i) \right\} \frac{\delta^2(i - m)^2}{2} \exp \left[ -\frac{\delta^2(i - m)^2}{2\Delta^2} \right].
$$

(A11)

The first term in the above equations is the general bias of the method, while the second term is the bias arising because of an asymmetric number of data points around each SN. For $m = M/2$, the number of data points is the same from both sides and we have

$$
B(z) = N(z) \sum_{i=1}^{M} \left\{ \ln d_L(z_i) \right\} \frac{\delta^2(i - m)^2}{2} \exp \left[ -\frac{\delta^2(i - m)^2}{2\Delta^2} \right].
$$

(A12)

In the continuous limit where $x = i - m$ is assumed, we obtain

$$
B(z) = N(z) \int \left\{ \ln d_L(z) \right\} \frac{\delta^2 x^2}{2} \exp \left[ -\frac{\delta^2 x^2}{2\Delta^2} \right] dx.
$$

(A13)

Therefore, the bias has the simple form

$$
B(z) = \frac{1}{2} \frac{\Delta^2}{\delta^2} \ln d_L(z)^{\prime\prime}.
$$

(A14)

This is a good analytical approximation for the bias at redshifts in the middle range, where we do not encounter the problem of data asymmetry. To see the effect of this bias, let us assume that the real model is the standard $\Lambda$CDM, add the bias term to this model and then calculate the biased $H(z)$ and $w(z)$. The result from this analytical calculation can be compared to the result of smoothing the exact $\Lambda$CDM model using our method. Fig. A1 simply illustrates that the results obtained using Gaussian smoothing and by the use of equation (A12) are in good agreement in the middle range of redshifts. However, we do not expect equation (A12) to work properly at very low ($z < 0.1$) and high ($z > 1$) redshifts where the above-mentioned asymmetry of points adds a further bias.

Also, it appears that the smoothing bias has a tendency to decrease $w(z)$ below its actual value in the middle range of $z$. Thus, $\Lambda$CDM may appear to be a ‘phantom’ ($w < -1$) if too large a smoothing scale is chosen.

**APPENDIX B: EXPLORING SMOOTHING WITH VARIABLE WIDTH $\Delta(z)$**

In order to deal with the problem of data asymmetry and paucity at low and high redshifts we may consider using a variable $\Delta(z)$.

(i) Low $z$ ($z_s < 1$). In this case, there are many more SNe at $z > z_s$, than there are at $z < z_s$. The error bars are also small in the low-redshift region. Therefore, a smaller value of $\Delta$ appears to be more appropriate at low $z$. 

B1 \( \Delta(z) = \Delta_0 z/(1 + z)^2 \)

In Section 2 we mentioned that, for \(|z - z_*| \ll 1\), the exponent in equation (2) reduces to the form \((-z - z_*)^2/2\Delta^2(1 + z)^2\) and the effective Gaussian smoothing scale becomes \(\Delta(1 + z)\). So if we use a variable \(\Delta(z) = \Delta_0 z/(1 + z)^2\), then the effective Gaussian smoothing scale approaches a constant at large \(z\) and tends to a small value at small \(z\). The results obtained using this method are shown in Fig. B1 for SNAP data, using the model \(w = -0.5\). We find that the result for the Hubble parameter does not change much. However, the equation of state is better reconstructed, but noisier at low redshift because of the small width of smoothing.

B2 The tan-hyperbolic form of \(\Delta(z)\)

The tangent hyperbolic form for \(\Delta(z)\) is another form of the variable \(\Delta(z)\), which can simultaneously satisfy both the low- and high-\(z\) requirements. It has a small value at low redshifts and a bigger value at higher redshifts. An additional important property of this function is that it changes smoothly from low to high \(z\), which translates into a smoother second derivative \(w(z)\) (see equations 2–4).

A drawback of this method is that the tangent hyperbolic function introduces a number of free parameters into the problem. However, the role of these parameters can be understood as follows. The tangent hyperbolic function can be written in the general

\[ w(z) = \frac{w_0}{1 + e^{b(z - z_0)}}. \]
Probing cosmic expansion using Sn data

Figure B2. The smoothing scheme of equation (2) is used with a tangent hyperbolic form of variable $\Delta(z)$ to obtain smoothed $H(z)$ and $w(z)$ from 1000 realizations of the SNAP data set. Panel (a) represents the form of $\Delta(z)$ used, while panels (b) and (c) represent the reconstructed $H(z)$ and $w(z)$. The dashed lines in panels (b) and (c) represent the fiducial $w = -0.5$ ‘metamorphosis’ model while the solid lines represent the mean and 1σ limits around it. The dotted line is $\Lambda$CDM.

The results obtained using this method are shown in Fig. B2 for SNAP data for the fiducial model $w = -0.5$. We find that this variable form of $\Delta(z)$ leads to a slight improvement of results at low redshifts by removing the small bias which remains in the bootstrap iterative process. This improvement of the results is expected, especially for cosmological models whose equation of state at low redshift is very different compared to the $\Lambda$CDM model, which is our initial guess model.

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Figure B2: The smoothing scheme of equation (2) is used with a tangent hyperbolic form of variable $\Delta(z)$ to obtain smoothed $H(z)$ and $w(z)$ from 1000 realizations of the SNAP data set. Panel (a) represents the form of $\Delta(z)$ used, while panels (b) and (c) represent the reconstructed $H(z)$ and $w(z)$. The dashed lines in panels (b) and (c) represent the fiducial $w = -0.5$ ‘metamorphosis’ model while the solid lines represent the mean and 1σ limits around it. The dotted line is $\Lambda$CDM.

\begin{align}
\Delta(z) &= a \tanh \left( \frac{b + z}{c} \right). \\
\Delta(z) &= a \tanh \left( \frac{b + z}{c} \right).
\end{align}