The latter result follows from equations (36) and (37), the combination of which yields

$$P' = q_i' < q_i < q_i^*$$  \(\text{(38)}\)

Equations (27), (28), (36), and (37) were all programmed on an IBM 7044 digital computer to give values of \(q_i, q_i', q_i^*\), and \(P'\), respectively, for the independent variables \(\gamma, \kappa, E\) each ranging between zero and unity. The program and complete results, of which the present discussion has presented a small sample, are available from the authors on request.

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References


DISCUSSION

Jan Kruyer

The present paper by Bentwich, Kelly, and Epstein is purely theoretical in nature and therefore presents no experimental verification for the computed results. Some experimental measurements which verify these theoretical solutions are here presented for the case of zero viscosity ratio (free-flowing solid cylindrical core).

For convenience of comparison between theory and experiment a theoretical pressure ratio \((PR)\), velocity ratio \((VR)\), and clearance \((C)\) may be defined from the flow rates and the eccentricity ratio of equations (31), (32), and (35) by Bentwich, et al.:

$$PR = \frac{1}{q_i + q_i}$$  \(\text{(1)}\)

$$VR = \frac{q}{k(q_i + q_i)}$$  \(\text{(2)}\)

$$C = (1 - k - \epsilon)/2000$$  \(\text{(3)}\)

These three equations were derived by Kruyer, Redberger, and Ellis (reference [3] of Bentwich, et al.). The experimental pressure ratio is obtained by comparing two sections of a pipeline; the first contains a moving cylinder with its annular liquid while the second contains liquid only, with the same volumetric flow ratio in both sections. The pressure ratio then is the ratio of the pressure drop per unit length of cylinder (section 1) divided by the pressure drop per unit length of liquid (section 2). The velocity ratio is the ratio of the cylinder velocity to the average velocity of the cylinder and its annular liquid combined. 

Fig. 7 is a plot of the pressure ratio versus the velocity ratio for a variety of clearances. The chain line is a plot of the theoretical prediction by Bentwich, et al., and was obtained by selecting a number of clearances and calculating pressure ratios and velocity ratios for these. For each clearance the pressure ratio was then plotted versus the velocity ratio. The dots on the chain line and the numerical values beside these indicate the clearances, in thousandths of pipe diameter, selected for this plot. The larger symbols in the figure are experimental values of pressure ratio versus velocity ratio. No attempt was made to measure the clearances experimentally. A hollow steel cylinder 12 in. long was used in these experiments and was loaded to three specific gravities from 2.09 to 11.64 as indicated by \(\sigma\) in the figure. The pipeline used was made of 1/4-in. reamed copper pipe giving an internal diameter of 0.53 in. and resulting in a cylinder to pipe diameter ratio \((K)\) of 0.826. The liquid used was a heavy lubricating oil with a 36 cp viscosity and 0.86 specific gravity. 

The figure reveals that the theory as presented by Bentwich, et al., agrees very closely with experimentally measured values if plotted on this basis. An interesting observation here is that while the theory is for cylinders of infinite length it compares well with measurements for cylinders of low length/diameter ratio.

This ratio is about 27 for Fig. 7 but Fig. 8 of Kruyer, et al., shows that this agreement still holds at a length/diameter ratio of 7.

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\(\text{Fig. 7 A comparison between theory and experiment for solid cylinders flowing in oil in a pipe. The theory is presented by the chain line. Scales of experimentally measured average velocity of cylinder and annular liquid combined are inserted for the three cylinder specific gravities here presented. (Cylinders: } \kappa = 0.826 \text{ polished steel tubing, } 5.0. \kappa = 2.09, \kappa = 4.98; \Delta = 11.64 \text{ pipe: reamed copper, 0.53-in. ID; oil: imperial oil "Faxam 40" 36 cp S.G. = 0.86)\)}