Electromagnetic Induction in Thin Conductors

A. T. Price and A. A. Ashour

(Received 1973 November 6)*

Summary

The various iteration methods that have been introduced for calculating electromagnetic induction effects in thin conductors are shown to be closely related to the two methods originally proposed by Price in 1949. In particular, one of the iteration methods recently given by Hutson et al., when expressed in terms of the current function is found to be identical with Price's first method, and their other method is shown to be a valuable extension of Price's first method.

With regard to Price's second method (for dealing with the case when self-induction is large), it has sometimes been stated that this method cannot be used because it is not convergent. Actually, it is shown that for certain problems it is strictly convergent, while for others it leads to an asymptotic expansion in powers of $\omega^{-1}$, which will give a good approximation to the solution when $\omega^{-1}$ is sufficiently small, provided the expansion is suitably truncated.

The procedures required for dealing with discontinuous distributions of conductivity, and the advantages and disadvantages of the integral equation method are briefly discussed.

1. General equations and methods of solution

The basic equation for the stream function $\psi$ of currents $\mathbf{J}$ induced by a varying magnetic field in a thin non-uniform conductor situated in a non-conducting medium was found by Price (1949) in the form

$$\kappa^{-1} \text{div} \ \text{grad} \ \psi + \kappa^{-1} \cdot \text{grad} \ \psi = \frac{\partial}{\partial t} \left( N^e + N^i \right)$$

(1)

where

$$\mathbf{J} = -n \wedge \text{grad} \ \psi$$

(2)

where $n$ is a unit vector normal to the conducting surface, $N^e$ and $N^i$ are the normal components of the inducing and induced magnetic fields at the surface, $\kappa$ is the conductivity in emu, and the permeability is taken as unity everywhere. If $\kappa$ is discontinuous across certain dividing lines in the conductor (including the edge, if it is a finite sheet), an equation of the above form holds in each open sub-domain in which $\kappa$ is non-zero, continuous and differentiable.

Only a few analytic solutions for special surfaces and distributions have been found, but useful methods using iteration procedures have been developed for obtaining the solution of practical geomagnetic problems. Two such procedures were

* Received in original form 1973 May 22.
originally suggested by Price from physical considerations. He argued that if the conditions are such that the self-induction field is small (e.g. \( \kappa \) small, and/or time rate of change small), we can get a first approximation \( \psi_1 \) to \( \psi \) by neglecting \( N^i \) in (1) compared with \( N^e \). We can then determine the \( N_{1,i} \) produced by the current distribution \( \psi_1 \), and use this \( N_{1,i} \) in (1) to get a second approximation \( \psi_2 \) and continually repeat this procedure. This can be represented by the iteration formula

\[
\kappa^{-1} \text{div grad} \psi_{n+1} + \text{grad} \kappa^{-1} \cdot \text{grad} \psi_{n+1} = \frac{\partial}{\partial t} (N^e + N_n^i).
\] (3)

On the other hand when the self-induction effect is large, the normal component \( N^i \) of the induced field will tend to cancel that of the inducing field. This suggests starting with the approximation \( N_{1} = -N^e \) and using the iteration

\[
\left( \frac{\partial N^i}{\partial t} \right)_{n+1} = -\frac{\partial N^e}{\partial t} + \kappa^{-1} \text{div grad} \psi_n + \text{grad} \kappa^{-1} \cdot \text{grad} \psi_n.
\] (4)

This procedure is found to be satisfactory for continuous distributions of \( \kappa \), but difficulties arise when \( \kappa \) is discontinuous, as at the edge of a spherical cap. These are associated with the fact that in (4) we are, in effect, iterating for the distribution of \( N^i \) instead of \( \psi \), and analytic solutions (Weidelt 1971) indicate that \( N^i \) has a logarithmic infinity at a discontinuity in \( \kappa \).

Two iteration methods for solving the same induction problems have also been given by Hutson, Kendall & Malin (1972). These are represented by the formulae (for periodic inducing fields)

\[
\lambda J_{n+1} = g + L J_n
\] (5)

\[
(\lambda - \alpha) J_{n+1} = g + L J_n - \alpha J_n
\] (6)

where

\[
\lambda = (i\omega \kappa_0 \alpha)^{-1}
\] (7)

\[
g = -\kappa (\kappa_0 \alpha)^{-1} \{ A^e_s + (i\omega)^{-1} \text{grad} \phi^e \}
\] (8)

\[
L J = -\kappa (\kappa_0 \alpha)^{-1} \{ A^i_s + (i\omega)^{-1} \text{grad} \phi^i \}
\] (9)

where \( \kappa_0 \) is a constant having the same dimensions and order of magnitude as \( \kappa \), so that \( \lambda \) is non-dimensional; \( A^e_s \) and \( A^i_s \) are the surface components of the vector potentials of the inducing and induced fields; \( \phi^e \) and \( \phi^i \) are the scalar potentials of corresponding electric fields; \( \alpha \) depends on the linear scale, \( \alpha \) is an arbitrary non-dimensional constant. \( A^j \) is expressed in terms of \( J \) by a well-known surface integral.

These two iteration methods, in the forms given in (5) and (6), are in general, rather complicated, because both components of \( J_{n+1} \), and the electric field grad \( \phi_n \) arising from the charge distributions produced by \( J_n \), have to be found at each step in the iteration. The complications can, however, be avoided by transforming the vector equations in \( J \) to scalar equations in \( \psi \).

2. Relations between iteration methods

If we divide (5) by \( \kappa \) and take the curl of each side, we get

\[
\lambda \, \mathbf{n} \cdot \text{curl} (\kappa^{-1} J_{n+1}) = -(\kappa_0 \alpha)^{-1} (N^e + N_n^i)
\] (10)

and using (2) and (7) this immediately reduces to (3) with \( \partial/\partial t \) replaced by \( i\omega \). Thus the first iteration method given by Hutson et al. (1972) is simply another form of Price’s (1949) first method, which is in fact, generally more convenient to use.

If we similarly transform the iteration formula (6), we obtain

\[
(\lambda - \alpha) P(\psi_{n+1}) = (\kappa_0 \alpha)^{-1} (N^e + N_n^i) - \alpha P(\psi_n)
\] (11)
where

\[ P(\psi) = n \cdot \text{curl} (\kappa^{-1} n \wedge \text{grad} \psi) = \kappa^{-1} \text{div grad} \psi + \text{grad} \kappa^{-1} \cdot \text{grad} \psi. \]  

(12)

In this form it may be regarded as a valuable extension of Price's first method, as indicated later.

While iteration methods are, of course, of main importance for direct numerical computation of the solution when analytic solutions are difficult or impossible to obtain, it is of interest to note that (3), or its equivalent (10), would give a formal solution as a series in ascending powers of \( \lambda^{-1} \), (4) as a series in ascending powers of \( \lambda \) and (11) as a series in ascending powers of \( (\lambda - \alpha)^{-1} \).

In using (11) it is desirable to examine as far as possible what value of \( \alpha \) will ensure rapid convergency. Clearly this will be obtained if we can make

\[ |\alpha P(\psi_n) - (\kappa_0 \alpha)^{-1} N_n| \]

small everywhere on the conductor at an early stage in the iteration. Now the equation

\[ \alpha \psi P(\psi) - (\kappa_0 \alpha)^{-1} N = 0 \]  

(13)

with

\[ \alpha = (-\beta \kappa_0 \alpha)^{-1} \]  

(14)

is the homogeneous equation for the modes of free decay with unchanging form, \( (\psi, e^{-\beta \psi}) \) of current in the conductor, and there is in general an infinite set of monotonically increasing positive constants \( \beta \) and corresponding distributions \( \psi \). There is a corresponding sequence of negative constants \( \alpha \) tending to zero as \( \nu \to \infty \), and it is possible to express the stream function \( \psi \) of any arbitrary distribution of current in the conductor in the form

\[ \psi = \sum A\psi \psi \psi. \]  

(15)

Usually this is a convergent series of discrete terms, but if, as sometimes happens, the free modes are not distinct, the series must be replaced by the integral form

\[ \psi = \int A(\nu) \psi(\nu) d\nu. \]  

(15a)

It follows from equation (13) that the normal component \( N_n \) of the magnetic field of the current distribution \( \psi \), is given by \( N_n = \kappa_0 \alpha \psi_\nu P_\nu(\psi); \) hence the normal component of a given arbitrary field can be formally expressed in the form

\[ N = \sum \nu A\nu \kappa_0 \alpha \psi_\nu P_\nu(\psi). \]  

(16)

The actual distribution \( \psi \) at any instant in the required solution of the given induction problem will depend both on the distribution of \( \kappa \) in the conductor and on the given distribution of the inducing field. It can sometimes happen in a particular problem that the distribution of induced currents represented by \( \psi \) is identical with the distribution in some mode of free decay of currents in the conductor. In such a case equation (13) is satisfied for the corresponding \( \alpha \), and equation (11) reduces to

\[ P(\psi) = (\kappa_0 \alpha)^{-1} \frac{N}{(\lambda - \alpha)}. \]  

(17)

It is of interest to note that if \( |\lambda| < \alpha \), this solution could be expanded in ascending powers of \( \lambda/\alpha \), i.e. in powers of \( \omega^{-1} \). This is the form of solution given by Price's iteration (4), i.e. by his second method.

3. A general formal solution, and condition for applicability of iteration methods

A formal solution of the induction problem will now be obtained which, though not usually of practical value for calculations, is useful for examining the conditions under which the different iteration methods can be applied.
Let the \textit{inducing} field be expressed in the form (16) and let the stream function of the \textit{induced} currents be written as

\[ \psi = \sum_v B_v(\psi_v). \]  

(18)

Then since the equations are linear we can solve for each component in (18) separately and superpose the results. Using (12) and (15) this gives

\[ \psi = \sum_v \frac{\alpha_v A_v \psi_v}{\lambda - \alpha_v}. \]  

(19)

The values of \( \alpha_v \), given by (14) are real and negative and lie between \( \alpha_1 \) and zero. This is the formal solution of the problem.

(i) If \(|\lambda| > |\alpha_1|\) we can expand each term on the right of (17) as a power series in ascending powers of \( \lambda^{-1} \), and the solution (19) would then be expressed as a power series in \( \lambda^{-1} \). This is the form of the solution given by the iteration method (3) (or (5)), which is known to be convergent for \(|\lambda| > |\alpha_1|\) (Hutson et al. 1972).

(ii) If \(|\lambda| < |\alpha_1|\), and we write \( \lambda - \alpha_v = \lambda - \alpha + \alpha - \alpha_v \), then values of \( \alpha_v \) real or complex can be found such that \(|\lambda - \alpha| > |\alpha - \alpha_v|\) for every \( v \). Hence each term on the right of (19) can be expanded as a series in ascending powers of \( (\lambda - \alpha)^{-1} \). This is the form of solution obtained by iteration method (6) (or (11)) and our result is again in accordance with the proof of convergence of this method given by Hutson et al. If however \(|\lambda|\) is very small this method may become only slowly convergent because the condition \(|\lambda - \alpha| > |\alpha - \alpha_v|\) then requires \( \text{Re}(\alpha) \) negative and greater in magnitude than \( \frac{1}{2}(\alpha_1) \) so that \(|\lambda - \alpha| = \alpha + \epsilon \) where \( \epsilon < |\lambda| \), while \( \alpha - \alpha_v \to \alpha \) as \( v \to \infty \). Hence the terms in (19) corresponding to large \( v \) will give rise to correspondingly slowly converging power series in \( (\lambda - \alpha)^{-1} \).

(iii) The possible slow convergence in powers of \( (\lambda - \alpha)^{-1} \) raises the question whether we can as an alternative use the iteration method (4) which would give the solution in a form corresponding to a series in ascending powers of \( \lambda \) (and therefore of \( \omega^{-1} \)).

Suppose \(|\alpha_{s-1}| > |\lambda| > |\alpha_s|\); then it is easily shown that equation (19) can be written in the form

\[ \psi = -\sum_{p=0}^{\infty} \sum_{v=1}^{s-1} \frac{\lambda^p}{\alpha_v^p} A_v \psi_v + \sum_{v=s}^{\infty} \frac{\alpha_v A_v \psi_v}{\lambda - \alpha_v}. \]  

(20)

This separates the induced currents into two parts corresponding to the two parts of the inducing field for \( v < s \) and \( v \geq s \).

The convergence of (19) implies that the second part of (20) must tend to zero as \( s \to \infty \). This suggests that for sufficiently small \(|\lambda|\), which by the above inequality will make \( s \) large, an approximate solution can be found in ascending powers of \( \lambda \), corresponding to the first part of (20). This would be the form of solution given by the iteration method (4).

If the inducing field were represented exactly by a finite number \((s-1)\) of terms of (16), i.e. \( A_v = 0 \) for \( v \geq s \), then the iteration would be strictly convergent for \(|\lambda| < |\alpha_{s-1}|\).

If, however, non-zero values of \( A_v \) for \( v \geq s \) occur, the formal expansion of the second part of (20) in positive powers of \( \lambda \) would cause the whole series for \( \psi \) to be divergent. Nevertheless we shall show that in many cases the iteration (4) will correspond to an asymptotic expansion in powers of \( \lambda \) or \( \omega^{-1} \) for \( \psi \).
We consider the series
\[ S_n(\lambda) = - \sum_{p=0}^{n} \sum_{v=1}^{\infty} \left( \frac{\lambda}{\alpha_v} \right)^p A_v \psi_v = \sum_{v=1}^{\infty} \frac{1 - (\lambda/\alpha_v)^{n+1}}{1 - \lambda/\alpha_v} A_v \psi_v \] (21)

By Poincaré's definition, \( S_n \) will be an asymptotic expansion for \( \psi \) if
\[ R_n = |\lambda^{-n}(S_n - \psi)| \to 0 \quad \text{as} \quad \lambda \to 0, \quad n \text{ fixed} \] (22)
even though this quantity may \( \to \infty \) as \( n \to \infty \).

It is easily shown that
\[ R_n = \left| \lambda \sum_{v=1}^{\infty} \frac{\alpha_v^{-n}}{\alpha_v - \lambda} A_v \psi_v \right| \] (23)
and since \( |\alpha_v| \) is small for large \( v \), \( R_n \to \infty \) as \( n \to \infty \). On the other hand, if the summation w.r. to \( v \) in (23) is finite or convergent, then, for any finite \( n \), \( R_n \to 0 \) as \( |\lambda| \to 0 \), and the above condition for \( S_n \) to be asymptotic to \( \psi \) is satisfied.

The values of \( \alpha_v \) and the functions \( \psi_v \) depend on the particular conductor considered, and the values of \( A_v \) depend on the applied inducing field. In practice we are seldom able to evaluate these analytically. However, since the expansion (16) for the inducing field must be convergent, it is likely that in many problems a finite number, \( \mu \) say, of terms of (16) would represent this field with sufficient accuracy. This would be equivalent to taking \( A_v = 0 \) exactly for \( v > \mu \). This finite summation in (23) would then make \( R_n \to 0 \) as \( \lambda \to 0 \) when \( n \) is fixed, and thus the expansion obtained is asymptotic.

It may also be noted that if \( \lambda \) is taken as exactly zero (corresponding to infinite conductivity, or infinite frequency) \( S_n \) reduces to \(-\sum A_v \psi_v \), which is the correct result, indicating that the normal component of the magnetic field of the induced currents exactly cancels that of the inducing field.

The Poincaré definition of an asymptotic expansion does not itself indicate when the expansion (or the corresponding iteration) should be stopped in order to give the closest approximation to \( \psi \). But it is clear that the contributions to the coefficient of \( \lambda^p \) in \( S_n \), from the formal (divergent) expansion in powers of \( \lambda \) coming from the terms in equation (20) corresponding to \( v \geq s \), will increase with \( p \) and ultimately outweigh the contributions from the terms corresponding to \( v < s \). The expansion should obviously be stopped before these divergent contributions become important. In practice each added term in the expansion corresponds to a step in the iteration and usually a stage will be reached when successively added terms are decreasing in magnitude, but at a later stage will start to increase. The iteration should be stopped before this second stage is reached.

4. Discontinuous distributions of \( K \)

The analytic solution obtained by Weidelt (1971) for the induction of currents in a thin composite sheet has shown that at a line of discontinuity of \( K \) the induced currents remain finite, but the normal component of the induced field \( N^i \) has a logarithmic infinity. Similar features were found by Parker (1968) at the edge of a thin uniform strip. It is necessary to examine whether such singularities in the induced field affect the above iteration procedures.

In any piecewise continuous distribution of sheet conductivity, the fundamental equations (1) and (2) will hold good within each open sub-domain in which \( K \) is non-zero and differentiable. At a line of discontinuity of \( K \), the stream function \( \psi \) will, from its definition, be continuous on crossing the line and in any region in which \( K \) is zero, \( \psi \) will be constant and equal to its value on the boundary of the region.
In the iteration formulae (3) and (11), $\psi_{n+1}$ is calculated from $N_n^i$, where $N_n^i$ is the normal component of the magnetic field due to the current distribution $\psi_n$, found in the previous step. $N_n^i$ can always be calculated by means of a surface integral over the surface of the conductor only, whether this is a closed shell or an open sheet. The integration will in general be done numerically by dividing the surface up, by some suitable network, into small elements and evaluating the corresponding finite sum. This numerical method of integration will necessarily have some smoothing effect which will tend to hide the singularities of $N_n^i$ at the discontinuities of $\kappa$. Also $N_n^i$ (for finite $n$) is not generally singular at these discontinuities but will tend to $\infty$ as $n \to \infty$. Hence if the network is taken fine enough, the value of $N_n^i$ near the discontinuity should increase greatly as the iteration proceeds. We may conclude that iteration methods (3) and (11) will always be applicable, but a check on the accuracy of the resulting solution should be made by examining the value obtained for $N_n^i$ near the discontinuities of $\kappa$.

In considering the iteration represented by (4), two complications arise which do not appear in the other methods when discontinuous distributions of $\kappa$ are considered. The iteration formula determines $N_{n+1}^i$ from $N_n^i$, and $\psi_n$ has to be found from the previous determination of $N_n^i$. In the case of a closed (e.g. spherical) shell, this can be done by a suitable surface integral formula (cf. Hobbs & Price 1970) but for a finite sheet a more elaborate calculation is required. If the finite sheet is regarded as a part of a smooth closed surface $S$ (e.g. a sphere), in order to use (4) to calculate $N_{n+1}^i$, it is necessary to find the $\psi_n$ which accounts for the already calculated $N_n^i$, and at the same time is zero on the remaining part of $S$. This is analytically possible for finite sheets with simple geometry, such as the circular disc, the spherical cap and the infinite strip (Rikitake 1962; Ashour 1965; Doss & Ashour 1971; Ashour 1971a).

The other complication relates to the starting approximation $N_1^i$. Although when $|\lambda|$ is small we can assume $N^i \sim -N^e$ at places well removed from any discontinuity in $\kappa$, this approximation is unsatisfactory near a discontinuity because as mentioned earlier $N^i$ has a logarithmic infinity, though $\psi$ and $J$ remain finite there. In the case of infinite conductivity we have $\lambda = 0$, and $N^i = -N^e$ exactly over the entire sheet right up to the edge, but in the non-conducting region beyond $N^i \to O(\delta^{-3})$ as the distance $\delta$ from the edge is reduced. The physical picture is that the magnetic field lines cannot in this case penetrate the conductor but are deflected parallel to it and slip over the edge where they are densely crowded. When $\kappa$ is large but still finite some of the field lines can penetrate the conductor, thereby reducing the order of the singularity at the edge, but not removing it.

It should also be noted that the effect of ignoring $\kappa^{-1}$ in (4) for a first approximation when $\kappa$ is not actually infinite is equivalent to assuming that $J$ becomes infinite at the edge (Ashour 1965; Wiedelt 1971). Hence the approximation $N_1^i = -N^e$ would introduce an incorrect singularity in $J$ at the edge which subsequent iterations would not remove. It is clearly necessary therefore to find some alternative first approximation $N_1^i$ where there are discontinuities in $\kappa$.

In the important case when the sheet is finite and $\kappa$ is continuous and tends to zero at the edge, the current density is zero at the edge and the induced field is finite there (Ashour 1971a). Hence, by regarding the sheet as a part of a suitable closed surface iteration procedures (3) and (4) can be applied for $|\lambda|$ large and small respectively. However, it is often more convenient in this case, and more especially in cases involving discontinuities of $\kappa$, to use alternative forms of the iteration formulae expressed in terms of the vector potential $A$, because these do not involve derivatives of $\kappa$.

In terms of the vector potential $A$, the basic equation corresponding to (1) can be
Electromagnetic induction in thin conductors

written in the case of axial symmetry in the form

$$\frac{\partial A^i}{\partial n} = 4\pi \kappa \frac{\partial (A^i + A^t)}{\partial t}, \quad (24)$$

where $\partial/\partial n$ denotes differentiation in the direction of the normal to the sheet and $F$ the change in a function $F$ on crossing the sheet. The equations corresponding to (3), (4) and (5) after removing the suffix $s$, are

$$(4\pi \kappa)^{-1} \frac{\partial A^i_{n+1}}{\partial n} = \frac{\partial (A^e + A^t)}{\partial t} \quad (25)$$

$$\frac{\partial A^i_{n+1}}{\partial t} = -\frac{\partial A^e}{\partial t} + (4\pi \kappa)^{-1} \frac{\partial A^i_n}{\partial n} \quad (26)$$

$$(\lambda - \rho) \frac{\partial A^i_{n+1}}{\partial n} = (\kappa_0 a)^{-1} (A^e + A^t) - \rho [\partial A^i_n/\partial n]. \quad (27)$$

In case of small $|\lambda|$, the first approximation may be obtained by taking the entire closed surface to be of infinite conductivity, i.e. $A^i_n = -A^e$ over all the closed surface. In this way the infinity at the edge of a finite sheet is avoided. A test of the accuracy of the iteration in this case is the gradual decrease to zero of the intensity of the current density outside the conducting sheet.

The integral equation method (Ashour 1950; Hutson et al. 1972) obtains successive approximations for the current density in the sheet. This is convenient because the current density is finite everywhere in the sheet whether it is closed or not. However, it has the disadvantage that a new calculation must be made to obtain the components of the induced field. Also when the problem is not axisymmetric, the gradient of the electric potential of the charge distribution appears in the equations and complicates the process. Moreover, for large values of $a\omega \kappa$ (e.g. for periods of variations less than one hour and for an ocean of global dimensions), the method will become increasingly slowly convergent. It is worth mentioning here that analytic solutions of Price's basic equation (1) can be found for certain non-uniform closed sheets (Price 1949; Ashour & Price 1948). The problem is reduced to the solution of a difference equation in the coefficients of the series for the potential of the induced field. Solutions can be obtained for any period and to any required degree of accuracy. These solutions have been used to test the accuracy of the results given by various iteration methods. Similar solutions have been obtained by Ashour (1971a, b) for finite sheets with conductivity decreasing to zero at the edge using the equivalent equation (24).

5. General conclusions

The above results show that the two iteration methods discussed by Hutson et al. (1972) for solving electromagnetic induction problems relating to thin sheet conductors are closely allied to one of the two iteration methods introduced by Price (1949). In fact, when their iteration formulae are transformed to deal with the current stream line function $\psi$ instead of the vector current density $J$, their method for dealing with cases of small $a\omega \kappa_0$ is seen to be identical with Price's first method. Also, their method for dealing with large values of $a\omega \kappa_0$ is an important and valuable extension of Price's first method because it extends the range of convergence of this method, though for very small values of $(a\omega \kappa_0)^{-1}$ it may converge rather slowly. These two variants of Price's first method are applicable to finite sheets as well as to closed shells.

Price's second method, devised for dealing with large values of $a\omega \kappa_0$, leads effectively to a power series in $(a\omega \kappa_0)^{-1}$. It can be used for closed shells of continuous non-uniform $\kappa$, for finite sheets with $\kappa \to 0$ at the edge, and for problems involving discontinuities in $\kappa$, provided suitable precautions are taken. For some problems this method is strictly convergent, but in the general case it leads to an asymptotic expansion in $(a\omega \kappa_0)^{-1}$. For sufficiently small values of $(a\omega \kappa_0)^{-1}$, this
can give a satisfactory approximation to the accurate solution of the problem, provided the expansion is suitably truncated.

Iterative calculations of the integral equation solution of the electromagnetic induction problem, as used by Ashour and by Hutson et al., lead to successive approximations for the current density in the sheet. This method has the advantage that the current density is finite everywhere on the sheet, whether it is closed or open. However, it has the disadvantage that a further calculation is then necessary to determine the induced field, and it is the field rather than the current density that is usually observed in geophysical studies.

Albert T. Price:
47, Whidborne Close,
Torquay, Devon TQ1 2PF.

Attia A. Ashour:
Faculty of Science,
Cairo University,
Cairo, Egypt.

References