Energy Spectrum of Nucleon Isobars in Strong Coupling Meson Theory

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With the large cutoff the energy spectrum of isobar states of nucleons with \( y = 0 \), where the operator \( Y^i \) is defined by \( Y^i = L_0^i + T_0^i \), is obtained by taking into due account the quantum-mechanical effect. The newly discovered resonance \((T=1/2, J^P=1/2^+)\) with mass 1480 MeV in the pion-nucleon scattering process is explained as the first excited vibrational state. Since the isobar energies are not completely degenerate with respect to the quantum numbers of rotational motions in the strong coupling limit, the derivation of the "isobar energy condition" for the mass operator in the Cook, Goebel and Sakita work would have to be modified in its exact sense when one applies their procedure to the strong coupling theory with the Pauli-Dancoff representation.

The nature of solutions with \( y = 1 \) is very sensitive to the boundary condition to be applied, and it is shown that these states vanish. This will be an explanation of the reason why there is no evidence for low-lying resonance with \( T=1/2, J^P=3/2^+ \) or \( T=3/2, J^P=1/2^+ \) in the pion-nucleon scattering process.

§ 1. Introduction

Recently an attempt was made to use the non-compact algebra to explain the sequence of baryon resonaces by connecting adjacent submultiplets with the non-compact generators. The non-compact group \( U(6, 6) \) consists of 72 even-parity currents which generate the non-chiral \( U(6) \otimes U(6) \) algebra, and 72 negative parity operators which create and destroy the 36 lowest mesons. These currents and operators produce, in combination, the infinite dimensional representation just as in the strong coupling meson theory as clarified by Dothan et al. Under the ladder representations of \( U(6, 6) \) the mesons are put in the band containing the sub-multiplets, \((1, 1)^+, (6, 6)^-, (21, 21)^+, (56, 56)^-\), ... and the baryons are in band, \((56, 1)^+, (126, 6)^-, (252, 21)^+, \ldots\).

On the other hand with the same strong coupling limiting procedure the underlying group of this theory in static models, the unitary representations consisting of the isobar states of the pion-nucleon system, is obtained on the basis of the Chew-Low equation. The group is a non-compact \([ (SU(2))_2 \otimes (SU(2))_T ] \times T_u\) and representing the algebra of this group is supposed to be equivalent to solving the original equation, when the group representation is combined with the dynamical condition (Eqs. (II-1) and (II-2) of reference 2)).
which is also derived from the same equation. These two works which combine the group theory and the strong coupling theoretic idea, are, however, yet unsatisfactory in identifying their predictions with the actually observed baryon and meson excited states.

An example, which has so far not been successfully explained by the group theory arguments and which is one of the main subjects of our present study, is a newly discovered resonance, \( T=1/2 \) and \( J^P=1/2^+ \) with resonant mass value 1480 MeV in pion-nucleon scattering experiments. Group-theoretic analysis based on the quark model seems to show that there is no suitable place for the multiplet which includes this state as its member.

In this paper we shall prove (§ 3) that this new resonance is the first excited vibrational state of the pion-nucleon system, using the strong coupling method developed by Pauli and Dancoff and refined by Pais and Serber. From a previous work we might expect that this state could be identified with the rotational state of quantum numbers, \( T=1/2, J^P=1/2^+ \) and \( y=1 \). Here \( y \) is the eigenvalue of the operator \( Y^\iota \) defined by \( Y^\iota = L_\iota + T_\iota \), where \( L_\iota \) and \( T_\iota \) are the total ordinary and isotopic-spin angular momenta of the system referred to the collective coordinate frame. According to the work reported here, however, it turns out that the solutions of the differential equation for the states with \( y=1 \) are very sensitive to the boundary condition and its proper use shows no existence of such states contrary to the previous work.

This fact which is a purely quantum-mechanical effect, in turn, shows why we have no evidence for low-lying resonance state of the pion-nucleon system with \( T=1/2, J^P=3/2^+ \) or \( T=3/2, J^P=1/2^+ \). The newly discovered resonance is identified as the first excited vibrational state with \( y=0 \) as stated above.

The isobar energies with \( y=0 \) are given by \( E_{\iota=1}(\rho_{\mu\nu\kappa}) = -f^2/32(\kappa\Lambda)^2 \cdot \kappa + (2.9 + n_\kappa + n)/[\kappa\Lambda] \cdot \kappa + 12(\kappa\Lambda)/f^2 \cdot \kappa [\iota(\iota+1) - 3/4] \), and therefore are not completely degenerate with respect to the quantum numbers, \( t \) and \( \iota \) in the strong coupling limit. Therefore the derivation of the "isobar energy condition" in the Pauli-Dancoff representation would be rather complicated in its exact sense compared to the work of Cook et al. (derivations of their equations (II.1) and (II.2)) which depends heavily on the assumption that the Chew-Low equation has the isobar energies of the form \( M_\iota = M_\kappa + \Delta_\iota/f^2 \). Further we infer (§ 4) that in the Pauli-Dancoff representation it would be better to take a badly broken symmetry group, \( [SU(2)]_\iota \otimes [SU(2)]_\kappa \) \( \otimes [SU(3)]_\iota \), corresponding to the rotations in the ordinary and charge spaces and the vibrations in the three dimensional \( q_\iota \) space, as the underlying group for the spectrum-generating algebra.

§ 2. Brief review of strong coupling theory

The Hamiltonian of the system where the pion is coupled with the static

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\( ^* \) Of course this fact has been shown by Chew and Low a long time ago, but it is the first time in the strong coupling theory with due account of the quantum-mechanical effect.
extended nucleon source is given by

$$H = \frac{1}{2} \sum_a \left( \pi_a^\prime + \phi_a \left( \kappa^2 - \nabla^2 \right) \phi_a \right) dx - \sqrt{4\pi} \sum_\alpha \frac{f}{\kappa} \sum_a \phi_a \sigma \cdot \nabla U dx . \quad (2.1)$$

$U(r)$ is a spherically symmetric source function normalized to unity,

$$\int U(r) dx = 1 . \quad (2.2)$$

As the first step of the strong coupling method we introduce the bound meson field $\phi^0_{a\kappa}$ and their canonically conjugate momenta $\pi^0_{a\kappa}$. For the case of the physically interesting large cutoff where the source radius is small compared to the mesonic Compton wave length and the strong coupling treatment is a suitable method, they are given by the equations,

$$\int \phi^{a'}_a \nabla U dx = 0 , \quad (2.3)$$

$$\int \pi^{a'}_a \nabla \tilde{\pi} dx = 0 , \quad (2.4)$$

where

$$\tilde{\pi} = \frac{X}{I} , \quad (\kappa^2 - \nabla^2) X = 4\pi U \quad \text{and} \quad I \delta_{ij} = \int \frac{\partial U}{\partial x_i} \frac{\partial X}{\partial x_j} dx . \quad (2.5)$$

The external field operators $\phi^{a'}_a$ and $\pi^{a'}_a$ and the coordinates and momenta for bound meson fields $\phi^0_{a\kappa}$ and $\pi^0_{a\kappa}$ satisfy the following commutation relations,

$$[\phi^0_{a\kappa} , \pi^0_{b\lambda}] = i \delta_{a\beta} \delta_{b\lambda} , \quad (2.6)$$

$$[\phi^0_{a\kappa} , \phi^{b'}_{\alpha}] = [\phi^0_{a\kappa} , \pi^{b'}_{\alpha}] = 0 , \quad (2.7)$$

$$[\pi^0_{a\kappa} , \phi^{b'}_{\alpha}] = [\pi^0_{a\kappa} , \pi^{b'}_{\alpha}] = 0 , \quad (2.8)$$

$$[\phi^{a'}_a (x) , \pi^{b'}_{\alpha} (y)] = i \delta_{a\beta} \left[ \delta \left( x - y \right) \frac{1}{4\pi} \sum_k \frac{\partial \tilde{\pi}}{\partial x_k} \frac{\partial U}{\partial y_k} \right] . \quad (2.9)$$

Substituting Eqs. (2.3) and (2.4) into Eq. (2.1) we have the Hamiltonian expressed in terms of new dynamical variables,

$$H = \frac{1}{2} \sum_a \left( \pi^{a''}_a + \phi^{a'}_a \left( \kappa^2 - \nabla^2 \right) \phi^{a'}_a \right) dx + \frac{N}{2} \sum_{a,k} \left( \pi^0_{a\kappa} \right)^2 + \frac{1}{2I} \sum_{a,k} \left( \phi^0_{a\kappa} \right)^2$$

$$+ \sqrt{4\pi} \sum_{a,k} \frac{\partial U}{\partial x_k} \pi^{a'}_a dx - \frac{f}{\kappa} \sum_{a,k} \tau_a \sigma_b \phi^0_{a\kappa} , \quad (2.10)$$

where

$$N = \frac{4\pi}{3} \int \left( \nabla U \right)^2 dx . \quad (2.11)$$

From the expression (2.10) we see that the external meson field is not coupled
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directly with the nucleon source and that the isobar states of the approximated Hamiltonian obtained by neglecting the external meson field have the even parity in common.

The total angular momentum of the system is given by

\[ J_{ij} = \frac{1}{4i} \left[ \sigma_i, \sigma_j \right] + L_{ij}^0 + L_{ij}^1, \] (2.12)

where \( L_{ij}^0 \) and \( L_{ij}^1 \) are the angular momenta of the bound and external meson fields respectively. They are given by

\[ L_{ij}^0 = \sum_\alpha \left( \phi_{\alpha i}^0 \pi_{\alpha j}^0 - \phi_{\alpha j}^0 \pi_{\alpha i}^0 \right) \] (2.13)

and

\[ L_{ij}^1 = -\sum_\alpha \int \pi_{\alpha i}' \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right) \phi_{\alpha i}' dx. \] (2.14)

Similarly the total isotopic spin of the system is given by

\[ T_{ij}^{a\beta} = \frac{1}{4i} \left[ \tau_{a_i}, \tau_{\beta j} \right] + T_{ij}^{a\beta} + T_{ij}^{a\beta}, \] (2.15)

where \( T_{ij}^{a\beta} \) and \( T_{ij}^{a\beta} \) are the isotopic spins of the bound and external meson fields respectively. They are given by

\[ T_{ij}^{a\beta} = \sum_k \left( \phi_{\beta k}^0 \pi_{\beta k}^0 - \phi_{\beta k}^0 \pi_{\alpha k}^0 \right) \] (2.16)

and

\[ T_{ij}^{a\beta} = \int \left( \phi_{\alpha i}^0 \pi_{\alpha j}^0 - \phi_{\beta i}^0 \pi_{\beta j}^0 \right) dx. \] (2.17)

The angular momenta and isotopic spins defined above, \( L_{ij}^0, L_{ij}^1, T_{ij}^{a\beta} \) and \( T_{ij}^{a\beta} \), satisfy the usual commutation relations separately and commute with one another.

As the second step we introduce the new coordinate frame which rotates together with the bound meson field by the Pauli-Dancoff representation such that

\[ \phi_{ij}^0 = \sum_r \Lambda_{ij}^r B_{ra} Q_r. \] (2.18)*

*) Here we have made a trivial change in the summation rule over the subscript of \( B_{a\beta} \) in comparison with our previous definition (2.22) of 1. Also in 1 we determined the variation domains of new operators by the condition that old and new variables must be in one-to-one correspondence. As its result we had to define the domains of the Euler angles for the transformations in the charge space differently from the usual convention. The extension of the variation domains of variable \( \theta \) and \( \Phi \) from \( \pi \) to \( 2\pi \) presents, however, no practical objection.**) Therefore we follow this definition while the variation domains of other variables remain as before.

**) See E. P. Wigner, Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra (Translated from German by J. J. Griffin, Academic Press, New York and London, (1959), p. 89). Erratum: The operation \( \phi \to \pi + \phi \) under the item (3) on p. 887 of 1 should read \( \phi \to -\phi + \pi \).
The orthogonal matrices $A_{x\beta}$ and $B_{\alpha\beta}$ are expressed in terms of six Euler angles, $(\theta, \phi, \psi)$ and $(\Theta, \Phi, \Psi)$ in the ordinary and charge spaces as follows:

$$
A = \begin{pmatrix}
\cos \theta \cos \phi \cos \psi - \sin \theta \sin \psi & \sin \theta \cos \phi \cos \psi + \cos \theta \sin \phi & -\sin \theta \cos \phi \\
-\cos \theta \cos \phi \sin \psi - \sin \theta \cos \psi & -\sin \theta \cos \phi \sin \psi & \sin \theta \cos \phi \\
\cos \theta \sin \phi & \sin \theta \sin \phi & \cos \phi
\end{pmatrix}
$$

and

$$
B = \begin{pmatrix}
\cos \Theta \cos \Phi \cos \Psi - \sin \Theta \sin \Psi & \sin \Theta \cos \Phi \cos \Psi + \cos \Theta \sin \Psi & -\sin \Theta \cos \Phi \\
-\cos \Theta \cos \Phi \sin \Psi - \sin \Theta \cos \Psi & -\sin \Theta \cos \Phi \sin \Psi & \sin \Theta \cos \Phi \\
\cos \Theta \sin \Phi & \sin \Theta \sin \Phi & \cos \Phi
\end{pmatrix}
$$

The momentum $P$, canonically conjugate to $Q$, is given by

$$
P = \sum_{\alpha, \beta} A_{\alpha k} B_{\alpha \beta} \pi_{\beta k}.
$$

The components $L_{\alpha}^{\mu}$ and $T_{\alpha}^{\mu}$ of $L_{\alpha k}^{\mu}$ and $T_{\alpha \beta}^{\mu}$ referred to this new coordinate frame are given by

$$
L_{\alpha}^{\mu} = \sum_{h, \xi} A_{\xi k} A_{\alpha k} L_{h k}^{\mu}
$$

and

$$
T_{\alpha}^{\mu} = \sum_{\alpha, \beta} B_{\alpha \beta} T_{\alpha \beta}^{\mu}.
$$

For the kinetic energy of the bound meson field $\sum_{\alpha, \xi} (\pi_{\alpha \xi}^{\mu})^2$ we have the following expression:

$$
\sum_{\alpha, \xi} (\pi_{\alpha \xi}^{\mu})^2 = -\sum_{\alpha, \xi} L_{\alpha}^{\mu} \frac{\partial}{\partial Q_{\alpha}} \frac{\partial}{\partial Q_{\xi}} + \frac{1}{4} \sum_{\alpha, \xi} \left[ \frac{(L_{\alpha}^{\mu} + T_{\alpha}^{\mu})^2}{(Q_{\alpha} + Q_{\xi})^2} + \frac{(L_{\alpha}^{\mu} - T_{\alpha}^{\mu})^2}{(Q_{\alpha} - Q_{\xi})^2} \right],
$$

where

$$
Q = (Q_{\alpha}^2 - Q_{\xi}^2) (Q_{\alpha}^2 - Q_{\xi}^2) (Q_{\alpha}^2 - Q_{\xi}^2).
$$

The main interaction term of (2·10) becomes

$$
-\sum_{\alpha, \xi} \tau_{\alpha} \sigma_{\beta} \phi_{\alpha \xi} = -\sum_{\alpha, \xi} \tau_{\alpha} \sigma_{\beta} A_{\alpha k} B_{\alpha \beta} Q_{\xi}.
$$

The last step is to diagonalize this interaction term. It is performed by two canonical transformations $S_1$ and $S_2$. They are defined by the following equations respectively:
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\[ S_i = \exp \left\{ i\phi \left( \frac{1}{2} \sigma_i + L_i' \right) \right\} \exp \left\{ i\theta \left( \frac{1}{2} \sigma_i + L_i' \right) \right\} \exp \left\{ i\phi \left( \frac{1}{2} \sigma_i + L_i' \right) \right\} \]
\[ \times \exp \left\{ i\phi \left( \frac{1}{2} \tau_i + T_i' \right) \right\} \exp \left\{ i\theta \left( \frac{1}{2} \tau_i + T_i' \right) \right\} \exp \left\{ i\phi \left( \frac{1}{2} \tau_i + T_i' \right) \right\} \]

and

\[ S_3 = \frac{1}{\sqrt{2}} (\sigma_3 - i\tau_3). \]  

(2.26)

The transformed Hamiltonian is, in the approximation of omitting both the external meson field and the nondiagonal terms, given by

\[ H = -\frac{N}{2} \sum_r \frac{\partial^2}{\partial^2 Q_r} + \frac{1}{2I} \sum_r Q_r^2 - \frac{f}{\kappa} \sum_r Q_r \]
\[ + \frac{N}{8} \sum_{r'\neq r} \frac{(L_{r'} + T_{r'}^{\pi})^2 - 1}{(Q_r - Q_{r'})^2} + \frac{N}{8} \sum_{r'\neq r} \frac{(L_{r'} - T_{r'}^{\pi})^2 - 1}{(Q_r + Q_{r'})^2}. \]  

(2.28)

Here we note that in deriving this expression we took the lowest energy state characterized by \( \tau_3 = \sigma_3 = 1 \) and that the state vector \( \Phi \) is transformed to \( \Phi' \) by a relation \( \Phi = \Phi'/Q^{3/2} \). Further we note that after the \( S_1 \) transformation \( L_0^i \) and \( T_0^i \) are the total angular momentum and the total isotopic spin of the system respectively. They satisfy the following anomalous commutation relations:

\[ [L_0^i, L_0^j] = -iL_0^{ij}, \]
\[ [T_0^i, T_0^j] = -iT_0^{ij}. \]  

(2.29)

and commute with each other.

§ 3. Eigenstates and eigenvalues

The average values of \( Q_r \) turn out to be quantities of magnitude, \( f \) (coupling constant) so that the fifth term becomes much small compared to the fourth term in Eq. (2.28). Therefore we neglect this term in the zeroth approximation so that the Hamiltonian (2.28) becomes

\[ H = -\frac{N}{2} \sum_r \frac{\partial^2}{\partial^2 Q_r} + \frac{1}{2I} \sum_r Q_r^2 - \frac{f}{\kappa} \sum_r Q_r + \frac{N}{8} \sum_{r'\neq r} \frac{(L_{r'} + T_{r'}^{\pi})^2 - 1}{(Q_r - Q_{r'})^2}. \]  

(3.1)

As we can see from this expression both the ordinary and charge spaces are no longer isotropic and further two spaces are coupled with each other. This reduction of the symmetry properties of the system is produced by the fact that the rotated coordinate frame is not a general frame but is the frame which rotates together with the bound meson field.

Mathematically this definition is stated in the Pauli-Dancoff representation,
$Q_r = \sum_{a, j} A_{aj} B_{ra} \phi^a_{ij}$ instead of $Q_{ar} = \sum_{b, j} A_{bj} B_{ar} \phi^b_{ij}$. As noted by Pais and Serber it is a principal axis transformation such that $Q_{ar} = Q_r \delta_{ar}$.

When we view the Pauli-Dancoff representation physically, the radial coordinates $Q_r$ are the weight functions in bringing two collectively rotated states defined for the ordinary and charge spaces into a one-to-one correspondence as explained in I. From such a situation it is very natural to introduce a new operator $Y^i$ which generates the simultaneous and same rotations of the rotated coordinate frames of two spaces. This new operator $Y^i$ is defined by

$$Y^i = L^i_0 + T^i_0$$

(3.2)

and satisfies the same anomalous commutation relations as Eq. (2.29).

The mechanism of the coupling of two subsystems is, however, not so simple as in the case of spin-orbit coupling in the atomic spectroscopy. The Hamiltonian (2.28) does not commute with $\sum_i (Y^i)^2$ as well as with $Y^i$. With the approximated Hamiltonian (3.1) does $\sum_i (Y^i)^2$ commute and this operator becomes a constant of motion together with the two other Casimir operators $\sum_i (L^i_0)^2$ and $\sum_i (T^i_0)^2$. Thus with the Hamiltonian (3.1) we work in the representation in which these operators are diagonal and are given by $l(l+1)$, $\ell(\ell+1)$ and $y(y+1)$ respectively. At this point we note that the operators $L^i_0$ and $T^i_0$ commute with $L^i_0$ and $T^i_0$ and so with the Hamiltonian (3.1). Therefore the two operators $L^i_0$ and $T^i_0$ can be diagonalized together with the Casimir operators $\sum_i (L^i_0)^2$, $\sum_i (T^i_0)^2$, $\sum_i (Y^i)^2$ and the Hamiltonian (3.1).

Now we consider the states characterized by $y=0$. Then the Hamiltonian (3.1) becomes as follows:

$$H = -\frac{N}{2} \sum \frac{\partial^2}{\partial Q^2} + \frac{1}{2I} \sum Q^2 - \frac{f}{\kappa} \sum Q, \quad -\frac{N}{4} \left[ \frac{1}{(Q_1 - Q_2)^2} + \frac{1}{(Q_1 - Q_3)^2} + \frac{1}{(Q_2 - Q_3)^2} \right].$$

(3.3)

Performing a coordinate transformation given by

$$Q_0 = Q_1 + Q_2 + Q_3, \quad Q = Q_3 - \frac{1}{2} (Q_1 + Q_2),$$

$$Q' = (3/8)^{1/2} (Q_1 - Q_2),$$

(3.4)

we have the following expression for the Hamiltonian (3.3):

$$H = -\frac{3N}{2} \left( \frac{\partial^2}{\partial Q_0^2} + \frac{1}{2} \frac{\partial^2}{\partial Q^2} + \frac{1}{4} \frac{\partial^2}{\partial Q'^2} \right) + \frac{N}{6} (Q_1^2 + 2Q_2^2 + 4Q_3^2)$$

$$- \frac{f}{\kappa} Q_0 - \frac{N}{4} \left[ \frac{3}{8Q'^2} \left( Q - (2/3)^{1/2} Q' \right)^2 + \frac{1}{(Q + (2/3)^{1/2} Q')^2} \right].$$

(3.5)
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We separate the equation of motion for $Q_0$ from those for $Q$ and $Q'$ and we have

\[
\left( -\frac{3N}{2} \frac{d^2}{dQ_0^2} + \frac{1}{6I} Q_0^3 - \frac{f}{k} Q_0 \right) \phi_\omega (Q_0) = E_\omega \phi_\omega (Q_0) \quad (3.6)
\]

and

\[
\left[ -\frac{3N}{2} \left( \frac{1}{2} \frac{\partial^2}{\partial Q^2} + \frac{1}{4} \frac{\partial^2}{\partial Q'^2} \right) + \frac{1}{3I} (Q^2 + 2Q'^2) \right. \\
\left. \quad - \frac{N}{4} \left( \frac{3}{8Q'^2} + \frac{1}{(Q - (2/3)^{1/2}Q')^2} + \frac{1}{(Q + (2/3)^{1/2}Q')^2} \right) \right] \phi_\delta (Q, Q') = E_\delta \phi_\delta (Q, Q'). \quad (3.7)
\]

To solve Eq. (3.7) we introduce the following new variables $r$, $\theta$ and $\lambda$:

\[
r \cos \theta = (4/9NI)^{1/4} Q, \\
r \sin \theta = (16/9NI)^{1/4} Q', \\
\lambda = 2(I/N)^{1/4} E,
\]

where $\theta$ varies over the region between 0 and $\tan^{-1}\sqrt{3}$. Then we have the equation given by

\[
\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - r^2 + \frac{1}{r^2} \left[ \frac{\partial^2}{\partial \theta^2} + \frac{1}{4} \frac{1}{\sin^2 \theta} + \frac{1/3}{(\cos \theta - \sqrt{3} \sin \theta)^2} \right. \\
\left. + \frac{1/3}{(\cos \theta + \sqrt{3} \sin \theta)^2} \right] + \lambda \right\} \phi_\lambda = 0. \quad (3.9)
\]

Let us consider the separated equation for $\theta$,

\[
d^2y(\theta) \frac{d\theta}{d\theta} \left[ \frac{1/4}{\cos^4 \left( \theta - \pi \frac{6}{6} \right)} + \frac{1/4}{\cos^4 \left( \theta + \pi \frac{6}{6} \right)} + \frac{1/4}{\cos^4 \left( \theta + 3\pi \frac{6}{6} \right)} + \rho \right] y(\theta) = 0. \quad (3.10)
\]

Changing the independent variable from $\theta$ to $z = \cos \theta$ we have

\[
(1-z^2) \frac{d^2y}{dz^2} - \frac{dy}{dz} + \left[ \frac{9/64}{(z^2 - \frac{1}{2})^2(1-z^2)} + \rho \right] y = 0, \quad (3.11)
\]

where $z$ varies between $1/2$ and 1. Inside this variation domain we have two singular points $1/2$ and 1. Approximate analytic solutions of Eq. (3.11) which are valid near these singular points are given below.

\[
y_{1} = a_{0}(1-z)^{1/4} \left[ 1 - \left( \frac{1}{3} + \frac{1}{2} \rho \right) (1-z) \right. \\
\left. - \left( \frac{857}{2304} + \frac{1}{3} \rho \right) (1-z^2) \right] \text{ for } z \leq 1,
\]

\[
y_{1/2} = b_{0} \left( z - \frac{1}{2} \right)^{1/4} \left[ \frac{1}{2} + \frac{1}{6} \left( z - \frac{1}{2} \right) - \left( \frac{1}{24} + \frac{1}{3} \rho \right) (z - \frac{1}{2})^2 \right] \text{ for } \frac{1}{2} \leq z. \quad (3.12)
\]
Joining two functions smoothly at \( z = 3/4 \), we obtain \( \rho(\rho_{\text{lowest}}) = 2.1 \) and \( b_0 = 0.92a_0 \). Therefore the energies of states with \( y = 0 \) are given, from Eqs. (3.6), (3.9) and the rotational energy represented by the last term of Eq. (2.28), through the following equation:

\[
E_{l=1}(y=0, \rho_{\text{lowest}}) = -\frac{3IF^3}{2\kappa^2} + (2.9 + n_0 + n) \left( \frac{N}{I} \right)^{1/2} + N \left( \frac{\kappa}{IF} \right)^2 \left[ t(t+1) - \frac{3}{4} \right] ,
\]

(3.13)

where \( n_0 \) and \( n \) are the running quantum numbers characterizing the states for dynamical variables \( Q_0 \) and \( r \) respectively.

For the \( y=1 \) states we examine the analysis performed in I by the proper use of boundary conditions. The Hamiltonian for this case is given by

\[
H = -\frac{N}{2} \sum Q_i^2 + \frac{1}{2I} \sum Q_i^2 - \frac{F}{\kappa} \sum Q_r - \frac{N}{4} \left( Q_1 - Q_2 \right)^2 .
\]

(3.14)

To solve the Schrödinger equation for this Hamiltonian we introduce the polar coordinates for \( q_r = Q_r - IF/\kappa \) such that

\[
q_1 = q \sin \alpha \cos \beta ,
q_2 = q \sin \alpha \sin \beta ,
q_3 = q \cos \alpha ,
\]

(3.15)

where we assume that \( q_r \) takes any value positive as well as negative though they suffer from the condition \( q_3 \geq q_2 \geq q_1 \). Then the separated Schrödinger equations for the Hamiltonian (3.14) become as follows:

\[
\left[ -\frac{3IF^3}{2\kappa^2} - \frac{N}{2q^2} \frac{d}{dq} \left( q^2 \frac{d}{dq} \right) + \frac{1}{2I} q^2 - \frac{N}{2q^2} \lambda \right] R(q) = \lambda R(q) \quad (3.16)
\]

and

\[
\left[ \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} \left( \sin \alpha \frac{\partial}{\partial \alpha} \right) + \frac{1}{\sin^2 \alpha} \left( \frac{\partial^2}{\partial \beta^2} + \frac{1/2}{(\sin \beta - \cos \beta)^2} \right) \right] y(\alpha, \beta) = \lambda y(\alpha, \beta) .
\]

(3.17)

In the allowed domain for \( \alpha \) and \( \beta \) Eq. (3.17) has one singular point \( \alpha = 0 \) in \( \alpha \), so that \( \lambda \) can take arbitrary values. Therefore Eq. (3.16) allows arbitrary negative values for \( E \) and guarantees no stable lowest state. This fact would explain the experiments that there exist no low-lying resonance state of \( l = 1/2, \ t = 3/2 \) or \( l = 3/2, \ t = 1/2 \) with positive parity. In other words quantum effects

\[\text{in determining the eigenvalue } \lambda \text{ of Eq. (3.9) we replaced the potential, } V(r) = r^2 + (\rho - 1/4)r^2 \text{ for the function } R_\lambda(r) = \sqrt{r} \phi_\lambda(r) \text{ by a simple harmonic oscillator potential around the extremum point. Therefore the apparent degeneracy of } n_0 + n = 1, \text{ namely } n_0 = 0 \text{ and } n_1 = 1 \text{ or } n_0 = 1 \text{ and } n_1 = 0 \text{ disappear in an exact treatment. To assess this delicate point of our present theory we must wait for more accurate results of experiments.}\]
Energy Spectrum of Nucleon Isobars in Strong Coupling Meson Theory

prohibit the existence of the $\gamma=1$ states.

Now turning to the quantititative discussion of the above results we assume a simple exponential function for the nucleon source,

$$U(r) = \frac{1}{8\pi A^3} \exp\left(-\frac{r}{A}\right).$$  \hspace{1cm} (3.18)

Then the two model-dependent quantities $N$ and $I$ are given in $I$ such that

$$N = \frac{1}{48A^3},$$

$$I \approx \frac{1}{48A^3}.$$  \hspace{1cm} (3.19)

The energy expression (3.13) becomes

$$E_{l=1}(\gamma=0, \rho_{\text{lowest}}) = \frac{f^2}{32(\kappa A)^3} \kappa + \frac{(2.9 + n_0 + n)}{(2\lambda A)} \kappa + \frac{12(\kappa A)}{f^3} \kappa \left[ t(t+1) - \frac{3}{4} \right].$$  \hspace{1cm} (3.20)

Since $(\kappa A) \ll 1$, we see from this expression that the strong coupling theory is an appropriate method. Further we obtain the first excited rotational level $(l=1/2, n_0 + n = 0$, positive parity) with resonance mass $300\text{MeV}$ when the unrenormalized coupling constant $f$ is $1.6/\sqrt{m}$, where the cutoff parameter $A$ is $1/m$ times the Compton wave length of nucleons (0.210 F) and $\kappa$ is the rest mass of pions (140 MeV). Also from Eq. (3.20) we have a relation between the vibrational and rotational states:

$$E_{l=1/2}(\gamma=0, n_0 + n = 1, \rho_{\text{lowest}}) - E_{l=1/2}(\gamma=0, n_0 + n = 0, \rho_{\text{lowest}}) = 940 \cdot m \text{ MeV}. \hspace{1cm} (3.21)$$

Identifying this expression to the experimental value, 540 MeV for the newly discovered pion-nucleon resonance state ($T=1/2$, $J^P=1/2^+$, Mass=1480 MeV) we have $m=0.57$ and so $f=2.1$. Further we expect to have another isobar state ($T=3/2$, $J^P=3/2^+$) with resonant mass 1780 MeV though in this energy region the predictions of the static strong coupling meson theory are less reliable.

\section*{§ 4. Group-theoretic analysis}

In this section we clarify the implications of the strong coupling theory based on the Pauli-Dancoff representation for the group-theoretic analysis recently made by Cook, Goebel and Sakita. First for simplicity we write down the Hamiltonian for the case of small cutoff where the source radius is large compared to the mesonic Compton wave length. In this case the bound meson fields and their canonically conjugate momenta are defined, instead of Eqs. (2.3) and (2.4), through
\[ \phi_a = \phi_a' + \sqrt{4\pi N} \sum_k \phi_{ak}^0 \frac{\partial U}{\partial x_k}, \quad \int \phi_a' \nabla U dx = 0, \quad (4.1) \]
\[ \pi_a = \pi_a' + \sqrt{4\pi N} \sum_k \pi_{ak}^0 \frac{\partial U}{\partial x_k}, \quad \int \pi_a' \nabla U dx = 0. \quad (4.2) \]

Then the Hamiltonian (2.1) becomes
\[ H = \frac{1}{2} \sum \left( \pi_a'^2 + \phi_a' (k^2 - \nabla^2) \phi_a' \right) dx + \frac{N}{2} \sum_{a,k} (\pi_{ak}^0)^2 + \frac{1}{2K} \sum_{a,k} (\phi_{ak}^0)^2 \]
\[ + \frac{\sqrt{4\pi}}{N} \sum_{a,k} \phi_{ak}^0 \int \phi_a' (k^2 - \nabla^2) \frac{\partial U}{\partial x_k} dx - \int \sum K \sum_{a,k} \sigma_k \sigma_k \phi_{ak}^0, \quad (4.3) \]
where
\[ K = \frac{N^2}{3} \frac{\nabla U \cdot (\nabla^2 - \nabla^2) \nabla U dx}{4\pi}. \quad (4.4) \]

From Eq. (4.3) we see that the sources of external mesons \( \phi_{ak}^0 \) are the bound meson field operators \( \phi_{ak}^0 \) except for a numerical factor and these operators commute with each other. Therefore we could think of a non-invariant extended group \([SU(2)]_J \otimes [SU(2)]_T \times T_s\) which is the semi-direct product of the invariance group \([SU(2)]_J \otimes [SU(2)]_T\) and the nine tensor operators \( \phi_{ak}^0 \) for the bound mesons in orbital \( \rho \) state. The special meaning attributed to this group is that the kinetic energy term of meson fields, \( (N/2) \sum (\pi_{ak}^0)^2 \), could be neglected in the strong coupling limit and in this limit the group becomes the invariance group. However, as we have seen in the Hamiltonian (2.28), this term introduces a strong interaction between the rotational and the vibrational motions. Therefore we retain this term and regard the algebra for this group as the spectrum-generating algebra \( \mathcal{S} \) of the Hamiltonian (4.3). This situation is clearly shown up in the Pauli-Dancoff representation. In fact the invariant \( \sum (\phi_{ak}^0)^2 \) of this group becomes \( \sum Q_{ij} \) which does not commute with the Hamiltonian and tends to a constant \( 3 (I f/\kappa)^2 \) only in the strong coupling limit.

Next we look at the eigenvalues of the Hamiltonian (4.3) which are given by
\[ E_{l-t}(y = 0, \rho_{lower}) = -\frac{f^2}{32(\kappa A)^3} \kappa + (2.9 + n_0 + n) \kappa + \frac{(\kappa A)^3}{4f^2} \kappa \left[ t(t+1) - \frac{3}{4} \right], \]
\[ n_0, n = 0, 1, 2, \ldots, \]
\[ l = t = 1/2, 3/2, 5/2, \ldots. \quad (4.5) \]

Here we have assumed again an exponential function (3.18) for the nucleon source. Since \( (\kappa A) \) is much greater than unity so that the expansion in powers of \( (1/f^2) \) is not a suitable method, and since the excited states with \( (n_0 + n) \geq 1 \) are very low contrary to the experimental results we must realize that for the
case of the small cutoff the strong coupling treatment is not a good method when we apply the Pauli-Dancoff representation and also probably when we use the Chew-Low equation even if this equation does not give vibrational excited states. Therefore we turn to the case of a large cutoff.

In this case where the results of the strong coupling method are reliable, the sources of the external meson fields $\pi_{rh}$ are given by the following expression except for a numerical factor:

$$\sum_{r,s} A_{rh} B_{rs} \left[ (P_r + i Q_r \sum_{\ell(q_r)} \frac{1}{Q^2_r - Q^2_i}) \delta_{rs} + \frac{1}{2} \frac{L^r_s + T^r_s}{Q_r - Q_s} - \frac{1}{2} \frac{Q^r_s - Q^r_i}{Q_r + Q_s} \right].$$

(4.6)

and commute with each other. Therefore also in this case as the group for the spectrum-generating algebra we can take the same extended non-invariant group $[(SU(2))_r \otimes (SU(2))_r] \times T_9$, as the group for the case of a small cutoff. Energies of isobar states are given by

$$E_{l-1}(y=0, \rho_{lowen}) = -\frac{f^2}{32(\kappa A)^2} \varepsilon + \frac{(2.9 + n_b + n)}{(\kappa A)} \frac{12(\kappa A)}{f^2} \left[ t(t+1) - \frac{3}{4} \right].$$

(4.7)

From this expression we see that the isobar energies are characterized by two sets of quantum numbers related to the rotational and vibrational motions and are not completely degenerate with respect to the rotational motion, the energy contribution of which has the form $\Delta_l/f^2$. For this reason which is supported by a successful explanation of the new resonance ($T=1/2$, $J^P=1/2^+$, 1480 MeV), the derivation of the "isobar energy condition" which is supposed to be equivalent to solving the Schrödinger equation when we combine the group representation with this condition would become rather complicated in its exact sense compared to the method of Cook et al.'s derivations of their equations (II·1) and (II·2) based on the Chew-Low equation which was assumed to have the isobar energies of the form, $M_l = M_0 + \Delta_l/f^2$ in their work.

In this way the underlying group of the strong coupling theory was assumed to be $[(SU(2))_r \otimes (SU(2))_r] \times T_9$, but it would be better in the Pauli-Dancoff representation to take the group $[(SU(2))_r \otimes (SU(2))_r] \otimes (SU(3))_{qr}$ corresponding to rotations in the ordinary and charge spaces and vibrations in the three dimensional $q_r$ space where $q_r = Q_r - (I_f/\kappa)$, as the group for the spectrum-generating algebra as seen from the Hamiltonian (2·28), though it is a badly broken symmetry. The unitary irreducible representation of this broken symmetry group is characterized by a set of numbers $(n_b, n, \rho; l, t, y)$, where $y = \lfloor l - t \rfloor, \cdots, l + t$, in comparison with the $(\infty, \lambda_r, \lambda_s)$ representation of $SU(4)$ for the former group. The general expression for the isobar energies of this characterization can be written as follows:
From this formula we can well expect that the transition matrix elements between states of different values of \( n_0 \), \( n \) and \( \rho \) become small and therefore only in this limitation\(^{11} \) a method similar to the Cook et al. derivation of the “isobar energy condition” can be applied to the present case as well.

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