On the Longitudinal and Scalar Photons in Lorentz Gauge and Lorentz Condition

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Quantization of the radiation field in Lorentz gauge is reexamined and it is shown that without using indefinite metric a consistent formulation is possible. Though it is not necessary to modify the usual supplementary condition, we should abandon to regard a vector potential \( A_\mu(x) \) as a Hermitian operator. It is suggested that the results of physical interests, however, are the same as those obtained from the ordinary formulation.

§ 1. Introduction

After the development of covariant formulation of quantum electrodynamics, it has been pointed out and emphasized\(^1\),\(^2\) that quantum theory of electromagnetic field in Lorentz gauge contains some inconsistencies originating from the subsidiary condition (the so-called Lorentz condition). The situation is represented directly by the contradiction between the Lorentz condition \( \partial_\mu A_\mu = 0 \) and the commutation relation \( \langle [\partial_\mu A_\mu, A_\nu] \rangle_0 \neq 0 \). It has been emphasized\(^5\) correctly that all the difficulties mentioned above\(^1\),\(^2\) originate from the infinite norm of state vectors satisfying the Lorentz condition. To avoid the difficulties, Gupta\(^3\) proposed to introduce indefinite metric into Hilbert space and to impose a condition weaker than the Lorentz one upon state vectors. This method is certainly a mathematically consistent formalism which gives correct answers for physical interests. However, the close correspondence to the classical theory is abandoned considerably. Moreover, it is not easy to apply this method to non-Abelian gauge vector field. The reason is the following. In electrodynamics, since the Lorentz condition satisfies the Klein-Gordon equation \( \Box A_\mu = 0 \) even in the interacting case, the positive frequency part \( (\partial_\mu A_\mu)_{(+)\chi} \) has a definite meaning. However, in non-Abelian gauge vector field theory, the Lorentz condition does not satisfy such an equation as that in electrodynamics.\(^4\) Therefore, it is very difficult how we can put consistently a weaker condition on state vectors. On the other hand, it has been shown\(^6\) that the usual form of the Lorentz condition \( \partial_\mu A_\mu = 0 \) is consistent formally in the non-Abelian case. Accordingly, it is desirable to formulate consistently the Lorentz gauge vector field theory in the usual form.
Utiyama et al.\textsuperscript{5} investigated this problem directly and showed that, if we regard the state vectors satisfying the Lorentz condition as a limiting case of suitably chosen normalizable states, we could obtain unique answers for physical processes. However, there still remains the elementary question whether the space spanned by free photon states cannot be treated as the ordinary Hilbert space. Moreover, as they pointed out, the photon propagator contains a divergent term though this has no physical importance.

From the discussion of reference 2), it is easily seen that we should give up either working in the ordinary Hilbert space or keeping the Hermitian character of the operator \( A_\mu(x) \) if we keep the Lorentz condition strictly. Contrary to the authors of reference 2) who took the former way, the present author wishes to propose the latter way and to show that this is also a mathematically consistent formulation. In this note, we investigate mainly the case of free photons and examine the mathematical structure of our formulation in detail. Of the interacting case, though the detailed examination is left for the forthcoming paper, we give a brief discussion on the equivalence between Dyson’s \( S \)-matrix\textsuperscript{5} and ours.

\section*{§ 2. Preliminary considerations\textsuperscript{(*)}}

Starting from the following Lagrangian:

\[
L = -\frac{1}{2} \frac{\partial A_\mu}{\partial x_\nu} \frac{\partial A^\nu}{\partial x^\mu},
\]

we usually obtain the quantized theory of radiation field and their characteristic relations are summarized as follows:

\[
[A_\mu(x), A_\nu(y)] = -i g_{\mu\nu} D(x-y), \quad (2.2)
\]

\[
\Box A_\mu(x) = 0. \quad (2.3)
\]

In order to get agreement with the Maxwell theory of radiation field, we should further impose the so-called Lorentz condition

\[
\partial_\mu A^\mu(x) = 0.
\]

However, as is well known in quantized theory, this condition cannot be regarded as operator relations. By following Fermi,\textsuperscript{6} it is usually regarded as a condition on the state vectors which are physically permissible. Therefore, in the quantum theory of radiation field, besides the operator relations (2.2) and (2.3), we have the subsidiary condition imposed on the state vectors

\[
\partial_\mu A^\mu(x) \langle \mathcal{F} \rangle = 0. \quad (2.4)
\]

In order to see the character of (2.4) in detail, it is more convenient to use

\textsuperscript{(*)} Discussion of this section is based on reference 2).
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the Fourier expansion of the vector potential $A_{\mu}(x)$. This is given by

$$A_{\mu}(x) = \frac{1}{\sqrt{2(2\pi)^3}} \sum_{k} \frac{d^3 k}{\sqrt{k}} \times e_{\mu}^*(k) \{ a_{\mu}(k) e^{ikx} + a_{\mu}^*(k) e^{-ikx} \}, \quad (2.5)$$

where

$$kx = k_{\nu}x^\nu, \quad k_{\nu} = (k, -k) \quad k = |k|,$$

$$e_{\mu}^0(k) = n_{\mu}, \quad e_{\mu}^3(k) = \frac{k_{\mu} + n_{\mu}(nk)}{(nk)}, \quad (2.6)$$

and the commutation relation for $a$ and $a^*$ is

$$[a_{\mu}(k), a_{\mu}^*(k')] = \delta_{\mu\nu} \delta (k - k'). \quad (2.7)$$

From the commutation relation (2.7), the number operators are introduced as follows:

$$n_{\lambda}(k) = a_{\lambda}^*(k) a_{\lambda}(k) \quad \text{for} \quad \lambda = 1, 2, 3,$$

$$n_0(k) = b^*(k) b(k),$$

where we put

$$b^* = a_0 \quad \text{and} \quad b = a_0^*. \quad (2.8)$$

Now, the Lorentz condition (2.4) becomes the following:

$$(a_0(k) + b^*(k)) |\Psi \rangle = 0,$$

$$(a_0^*(k) + b(k)) |\Psi \rangle = 0. \quad (2.9)$$

With the following definition of the vacuum,**

$$a_0(k) |\psi_0 \rangle = 0,$$

$$b(k) |\psi_0 \rangle = 0, \quad (2.10)$$

we can construct the state vectors satisfying the conditions (2.8):

$$|\Psi \rangle = \exp[- \sum_k a^*(k) b^*(k)] |\psi_0 \rangle. \quad (2.10)$$

It is easily seen that the norm of $|\Psi \rangle$ given in (2.10) diverges. This difficulty is twofold. One is already investigated exhaustively and even if we ignore all photons except one with a special momentum $k$ the state vector is not normalizable. This difficulty can be avoided by introducing a suitable damping factor. For example, the state

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* The metric tensor $g_{\mu\nu}$ is defined as $-g_{00} = g_{11} = g_{22} = g_{33} = +1$ and $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

** Here, we ignore the transverse photons to avoid inessential complexity.
can be normalizable if $|\lambda_k| < 1$. The difficulty mentioned is related to the fact that the operator $\partial_\mu A^\mu(x)$ is unbounded* and, therefore, the states satisfying Lorentz condition are eigenstates of the unbounded operator. The situation is very similar to the two-body problem in non-relativistic quantum mechanics. In this case, the wave function is a function of the two coordinates $x_1$ and $x_2$, and because of translational invariance, the wave function can be written as follows:

$$\Psi(x_1, x_2) = \exp \left[ -i \frac{p \cdot (x_1 + x_2)}{2} \right] \Psi(x_1 - x_2).$$

Therefore, if we take an eigenstate of the total momentum which is obviously unbounded, the norm of the wave function diverges. The introduction of the damping factor $\lambda_k$ in the photon states corresponds to making a wave packet in this example.

It is, however, worthwhile to mention that there is another difficulty which originates from the infinite degrees of freedom of the radiation field. Since the norm of the state (2.11) is

$$\frac{1}{1 - |\lambda_k|^2},$$

the norm of the state (2.10) is given by

$$N = \prod_k \frac{1}{1 - |\lambda_k|^2}.$$  \hspace{1cm} (2.13)

To recover the Lorentz condition by taking the limit $\lambda_k \to 1$, we must keep all $|\lambda_k|^2$ to be non-zero. Therefore, the norm (2.13) obviously diverges because of infinite degrees of freedom. This fact shows that the state vectors satisfying the Lorentz condition are not the elements of a Hilbert space constructed from the vacuum (2.9).* Therefore, it is not enough only to introduce a damping factor in order to overcome the difficulties mentioned in this section.

§ 3. The modified vacuum for longitudinal and scalar photons

From the preceding discussion and the analysis given in reference 2), we can suppose that the operator $\partial_\mu A^\mu(x)$ is non-Hemitian in order to avoid the contradiction pointed out in the Introduction and § 2. In this viewpoint, the condition $\langle \Psi | \partial_\mu A^\mu = 0$ is not derived from the Lorentz condition $\partial_\mu A^\mu |\Psi\rangle = 0$.

* This is seen from the commutation relation 

$$[\partial_\mu A^\mu(x), A_\nu(y)] = -i \partial_\nu \delta(x-y),$$

if we assume $A_\mu (x)$ be Hermitian.
Then, we cannot say that \( <[\partial_\mu A^\mu, A_\lambda]> = 0 \) even if the Lorentz condition does hold. Generally speaking, it is not necessary that an operator corresponding to a real quantity in the classical theory should be Hermitian in quantized theory. Moreover, the vector potential \( A_\mu(x) \) is not observable even in the classical theory.

Now, let us define the following operators:

\[
A = \frac{1}{\sqrt{2}} (a + b^*), \quad B = \frac{1}{\sqrt{2}} (a^* + b).
\]

(3.1)*

In terms of these operators the Lorentz condition is written as

\[
A|\Psi\rangle = 0, \quad B|\Psi\rangle = 0.
\]

(3.2)

From (3.1) and the commutation relations (2.7), we can introduce the operators \( A^\dagger \) and \( B^\dagger \) which have the following properties:

\[
[A, A^\dagger] = 1, \quad [B, B^\dagger] = 1,
\]

\[
[A^\dagger, B^\dagger] = [A^\dagger, B] = [A, B^\dagger] = 0.
\]

(3.3)

These operators are given explicitly in the following:

\[
A^\dagger = \frac{1}{\sqrt{2}} (a^* - b),
\]

\[
B^\dagger = \frac{1}{\sqrt{2}} (-a + b^*).
\]

(3.4)

Without disturbing the commutation relation (3.3), we can add arbitrary \( c \)-numbers to \( A^\dagger \) and \( B^\dagger \) given in (3.4). As will be shown later, this corresponds to the freedom of the restricted gauge transformation.

In place of \( a, b, a^* \) and \( b^* \), we can use the new operators \( A, B, A^\dagger, \) and \( B^\dagger \) to describe longitudinal and scalar photons. The Hamiltonian of longitudinal and scalar photons is expressed in terms of these new operators as follows:

\[
H_{L.S.} = \sum_k |k| (a^*(k) a(k) - b^*(k) b(k))
\]

\[
= \sum_k |k| (A^\dagger(k) A(k) - B^\dagger(k) B(k)).
\]

(3.5)

Therefore, it seems to be very natural to define the vacuum state as follows:

\[
A|\Psi_0\rangle = 0, \quad B|\Psi_0\rangle = 0.
\]

(3.6)

*) Hereafter, we ignore the subscript \( k \) if confusions do not occur.
This definition is nothing but the Lorentz condition. From the new vacuum state \((3 \cdot 6)\) we can construct basic vectors in the usual way; that is,

\[
|n, m\rangle = \frac{1}{\sqrt{n! \cdot m!}} (A)^n (B')^m |\Psi_0\rangle.
\]

This is an ordinary occupation number representation of the new operators \(A\) and \(B\). To define the norm of the state vector, we introduce adjoint vectors as follows:

\[
\langle n, m | = \langle \Psi_0 | (A)^n (B')^m \frac{1}{\sqrt{n! \cdot m!}},
\]

\[
\langle \Psi_0 | A^* = \langle \Psi_0 | B^* = 0
\]

and

\[
\langle \Psi_0 | \Psi_0 \rangle = 1.
\]

By virtue of the commutation relations \((3 \cdot 3)\), we see that

\[
\langle n, m | n', m' \rangle = \delta_{nn'} \delta_{mm'}.
\]

Thus we can see that the norms of our basic vectors \((3 \cdot 5)\) are positive definite. Moreover, the \(A^*\)’s and \(B^*\)’s are Hermitian conjugate operators of the \(A\)’s and \(B\)’s in this Hilbert space. Employing these variables, we can write the electromagnetic four potential \(A_\mu (x)\) as follows:

\[
A_\mu (x) = \frac{1}{\sqrt{2} (2\pi)^3} \frac{d^3 k}{k} \left\{ \sum_{\lambda=0}^2 e_\lambda (k) [a_\lambda (k) e^{ikx} + a_\lambda^* (k) e^{-ikx}] + [A (k) e^{ikx} e^{(+)} (k) + A^* (k) e^{-ikx} e^{(-)} (k)] + [B (k) e^{ikx} e^{(+)} (k) - B^* (k) e^{ikx} e^{(-)} (k)] \right\},
\]

where

\[
e^{(+)}_\mu (k) = \frac{1}{\sqrt{2} (nk)} (k_\mu + 2n_\mu (nk)) = \frac{1}{\sqrt{2}} (e^{(0)}_\mu + e^{(3)}_\mu),
\]

\[
e^{(-)}_\mu (k) = \frac{1}{\sqrt{2} (nk)} k_\mu = \frac{1}{\sqrt{2}} (-e^{(0)}_\mu + e^{(3)}_\mu),
\]

\[
e^{(+)}_\mu e^{(+)} (k) = e^{(-)} e^{(-)} = 0; \quad e^{(+)} e^{(-)} = 1.
\]

From the expression \((3 \cdot 10)\), it is obviously seen that the vector potential \(A_\mu (x)\)

\(^*)\) For defining polarization of transversal photons one usually uses a tetrad \(e^{(\lambda)}_\mu (\lambda=1, 2, 3, 4)\) defined in Eq. \((2 \cdot 6)\). However, the definition given in Eq. \((3 \cdot 10)\) is different and it is defined in such a way that the two space-like vectors \(e^{(\mu)}_\mu (k^2 = 0)\) and \(e^{(\mu)}_\mu (k^2 = 0)\) are called polarization vectors; i.e. \(e^{(+)}_\mu (k^2 = 0), e^{(-)}_\mu (k^2 = 0), e^{(+)}_\mu e^{(+)} (k^2 = 0), e^{(-)}_\mu e^{(-)} (k^2 = 0)\). This definition was used in reference 10.)
is not Hermitian. The commutation relations of \( A_\mu(x) \) is, however, the same as (2.2); i.e.

\[
\{ A_\mu(x), A_\nu(y) \} = -ig_\mu D(x-y).
\]  

(3.12)

The Lorentz condition is, now, the following:

\[
\begin{align*}
\partial_\mu A^\mu(x) |_\Psi = 0, \\
\langle \Psi | \partial_\mu A^\mu \rangle = 0, \\
\langle \Psi | \partial_\mu A^\mu \rangle = 0.
\end{align*}
\]  

(3.13)

It is important that, even though the \( A_\mu(x) \)'s are not Hermitian, the free Hamiltonian (3.5) is Hermitian. As was already mentioned, \( A, B \) and the commutation relations (3.3) do not determine uniquely \( A^1 \) and \( B^1 \). We can easily see that the arbitrariness is additive \( c \)-number; i.e. the following transformation:

\[
\begin{align*}
A^1 &\rightarrow A^1 + \lambda, \\
B^1 &\rightarrow B^1 + \lambda'
\end{align*}
\]

\( A, B \)

does not change the commutation relations. From (3.10) and (3.11), however, this arbitrariness just corresponds to invariance under the restricted gauge transformation

\[
A_\mu \rightarrow A_\mu + \partial_\mu A,
\]

\( A, B \)

§ 4. Vacuum expectation functions

Since the vacuum expectation value of time ordered product of two vector potentials plays an important role, let us examine explicitly various vacuum expectation functions. As already stated in the preceding section, commutation relations are not influenced by the modified definition of the vacuum. However, the vacuum expectation value of the following quantity:

\[
K^{(1)}_{\nu} (x,y) = \langle A_\mu (x) A_\nu (y) + A_\nu (y) A_\mu (x) \rangle_0
\]

depends essentially on all the characteristics of the vacuum state. By using (3.6) and (3.9), this is easily calculated and the result is as follows:

\[
K^{(1)}_{\nu} (x,y) = g_\mu D^{(1)} (x-y)^{\nu}\]

\[\begin{align*}
&\left[\frac{\partial^k}{k} \left[ k_k k_\nu + (n_k k_\nu + n_\nu k_\mu) \right] \right]\times \\
&-\frac{1}{2(2\pi)^3} \left[ \frac{d^k}{k} e^{ikx} + e^{-iky} \right].
\end{align*}\]  

(4.1)

\( g_\mu 
\)
Putting symbolically
\begin{equation}
\mathcal{D}(x) = \frac{-i}{2(2\pi)^3} \int \frac{d^3k}{k} \left( \frac{1}{(nk^2)} \left[ e^{i\mathbf{k} \cdot \mathbf{x}} - e^{-i\mathbf{k} \cdot \mathbf{x}} \right] \right),
\end{equation}

we can rewrite
\begin{equation}
\mathcal{D}^{(1)}(x) = \frac{1}{2(2\pi)^3} \int \frac{d^3k}{k} \left( \frac{1}{(nk^2)} \left[ e^{i\mathbf{k} \cdot \mathbf{x}} + e^{-i\mathbf{k} \cdot \mathbf{x}} \right] \right),
\end{equation}

From the above result and the commutation function, the propagation function is easily obtained and the result is given as follows:
\begin{equation}
K^{(1)}_{\mu\nu}(x-y) = \gamma_{\mu} D^{(1)}(x-y) + \frac{\partial}{\partial x_\mu} \mathcal{D}^{(1)}(x-y) - (n_\mu \partial_\nu + n_\nu \partial_\mu) (n_\rho \partial_\rho) \mathcal{D}(x-y).
\end{equation}

Contrary to the procedure proposed by Utiyama et al.,\textsuperscript{2) all the vacuum expectation functions do not involve any infinite quantity. Non-Hermitian character of the electromagnetic potential \( A_\mu(x) \) reflects on the expression (4·4) where \( K^{(1)}_{\mu\nu}(x) \) is not real.

§ 5. Remarks on the interacting case and unitarity of the \( S \) matrix

We have so far discussed the case of free photons and showed that the modified treatment of the Lorentz condition is a consistent mathematical formulation. However, there still remains the essential question whether our formalism assures unitarity of the \( S \) matrix when we apply it to the interacting photon case. Though the investigation and examination in detail of our formalism are left for forthcoming papers, we wish to remark briefly, on this problem.

\textsuperscript{2) \( D_\mu(x) = D^{(1)}(x) + i\mathbf{e}(x) D(x) \).}
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Starting from the Heisenberg representation, we can formally obtain an interaction representation. In this course, it is not necessary to assume the transformation function $U(t, t')$ be unitary. Then, we have

$$i \frac{\partial U(t, t')}{\partial t} = H(t) U(t, t')$$

$$= -\int j_\mu(t, x) A_\mu(t, x) d^3x U(t, t'),$$

(5·1)

$$\mathcal{T}(t) = U(t, t_0) \Phi(t_0).$$

By making use of this equation, the general perturbation expansion formulae for the $S$ matrix is obtained and, in this case, the internal photon line is replaced by $\langle T(A_\mu A_\nu) \rangle_s$ given by (4·5). However, the contribution of the last three terms in (4·5) gives rise to no contribution to the $S$ matrix. The proof of this fact can be given in the same way as in reference 2). Therefore, our procedure gives exactly the same $S$ matrix as that of Dyson$^5$ and, hence, there is no difficulty in the problem of unitarity of the $S$ matrix even though the interaction Hamiltonian is not Hermitian.

Finally, we wish to comment the prescription proposed recently by Schwinger.$^9$ To overcome difficulties originating from constraints, he proposed that some of variables should be left not to be integrated if we normalize the states satisfying constraints. He called such variables group parameters. His proposition seems to be very reasonable if constraints are based on variational principle or derived inevitably from the generalized canonical formalism. However, in the usual Lorentz gauge formalism studied in this note, the Lorentz condition is not based on variational principle, but we put aposteriori the condition to recover the Maxwell theory of radiation field. Therefore, variables describing longitudinal and scalar photons are not regarded as group parameters, but they are independent dynamical variables. Moreover, in the covariant perturbation theory, we treat all the components of vector potential $A_\mu$ as dynamical variables.

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