Photoproduction of Pions at the Second Pion-Nucleon Resonance

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The resonant amplitudes for the single pion photoproduction from the nucleon are calculated at the second pion-nucleon resonance. A simple model is used, in which the second resonance and nucleon ground states are represented by solutions of the Klein-Gordon equation with effective potentials chosen suitably. The amplitudes for the electric dipole and the magnetic quadrupole transitions, $E_2$ and $M_2$, and the total cross section for the resonant photoproduction of neutral pions are computed. The main features of the resonant photoproduction process are well explained by this simple model.

§ 1. Introduction

In recent years numerous experiments on the photoproduction of pions have been made up to about 1.2 GeV and several quantities, for example, the differential cross sections and the spin polarizations of the recoil nucleons, were measured.

First the experimental results were interpreted phenomenologically in terms of the so-called second and third pion-nucleon resonances. Assuming a resonance model with adjustable parameters, Salin\cite{1} analysed the photoproduction of positive and neutral pions and gave a good fit of all photopion production data from threshold to 900 MeV. But later the differential cross sections of neutral pions were measured accurately at forward and backward angles,\cite{2} and revealed a disagreement with the result of Salin's calculation in the vicinity of 800 MeV. Around the second resonance, the measured angular distributions of neutral pions are approximately symmetric with respect to 90°, and small at forward and backward angles.

Beder\cite{3} presented a tentative fit of cross sections by taking appropriately the ratio of electric and magnetic transition amplitudes for the second $\pi$-$N$ resonance. If these ratios are in the vicinity of $E_2/M_2=3$, the angular distributions are well explained. Here the amplitudes $E_2$ and $M_2$ denote the electric and magnetic transition amplitudes, respectively, defined by Chew, Goldberger, Low and Nambu\cite{4} that is, the amplitudes $M_{\pi}$ and $E_{\pi}$ refer to transitions initiated by magnetic and electric radiation, respectively, leading to final $\pi$-$N$ states of orbital angular momentum $l$ and total angular momentum $j=l-\frac{1}{2}$. Also some other experimental evidences, for example, the transverse cross section for electrons scattered inelastically from protons\cite{5} or the positive-pion production asymmetry with polarized bremsstrahlung,\cite{6} support the existence
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of a considerable amount of the magnetic quadrupole transition.

Other attempts to predict the amplitudes were based on the dispersion relation approach. But in order to obtain a prediction one has to make drastic approximations. Using fixed-t-dispersion relations, Schmidt, Schwiderski and Wunder\textsuperscript{3} analysed the pion-photoproduction process in the region of the second resonance. They considered the contributions of the second resonance as small corrections to the amplitudes and phenomenologically concluded that a good fit is possible for $\Delta E_\gamma / \Delta M_\gamma \approx 3$. Bietti\textsuperscript{8} considered commutation relations between the non-relativistic electric dipole and magnetic quadrupole operators, and got sum rules. Then he used these sum rules for deriving the ratio between the $E_\gamma$ and $M_\gamma$ amplitudes in single-pion photoproduction, and obtained similar results.

Recently Yamaki\textsuperscript{9} made in detail a phenomenological analysis of the photoproduction amplitudes for neutral pions in the energy range from 300 MeV to 800 MeV, and concluded that the ratio $E_\gamma / M_\gamma \approx 3$ is favourable at the second resonance.

The main purpose of the present paper is to calculate the ratio of the electric and magnetic transition amplitudes at the second $\pi$-$N$ resonance on the basis of a simple model. In this model the pion-nucleon system is represented by the Klein-Gordon equation with effective potentials chosen suitably, and the second resonance and nucleon ground states are taken to be the solutions of this equation. The transition multipole moments are computed by using spherical Bessel functions. Feld\textsuperscript{10} used a similar model for describing the nucleon ground state and the first $\pi$-$N$ resonance, and also Otsuki and Sawada\textsuperscript{11} took an analogous model for systematizing the meson-baryon resonances.

In § 2 of this paper we deal with the multipole expansion of the radiation field. Section 3 is concerned with the description of the second resonance and nucleon ground states. In § 4 the formulas for photopion production are given. In § 5 the results of numerical computations are summarized and discussed.

§ 2. Multipole radiation

We are concerned with the electromagnetic processes involved in the excitation of the second pion-nucleon resonance, $\gamma + N \rightarrow N^{*} \rightarrow \pi + N$. The conservation of angular momentum and parity limits the possible processes responsible for the photomeson production to the electric dipole and the magnetic quadrupole photon absorption. Rather than consider directly the photoproduction process, we first compute the cross section for the inverse reaction, $\pi + N \rightarrow N^{*} \rightarrow \gamma + N$, and then get the cross section for photopion production by the detailed-balancing argument.\textsuperscript{12} The cross section for the inverse process can be expressed in the center of mass system as\textsuperscript{3}

$$
\frac{d\sigma_{\gamma \gamma}}{d\Omega} = k \frac{\langle M \rangle^2}{4\pi W} |T|^2.
$$

(1)
Here $k$ and $q$ denote the momenta of the photon and pion, respectively, while $M$ and $W$ are the masses of the nucleon and the second $\pi$-$N$ resonance. The transition matrix $T$ is given by

$$T = \sqrt{4\pi} e \langle \Phi_i | j \cdot A | \Phi_f \rangle,$$

where $j$ and $A$ mean the electromagnetic current of the system and the vector potential of the radiation field, respectively, and we describe the initial and final states of the pion-nucleon system by the wave functions $\Phi_i$ and $\Phi_f$.

In the solenoidal gauge the vector potential of the plane wave can be represented as a superposition of multipoles in a following way:

$$A = \sqrt{2\pi} \sum_{L=0}^{\infty} \sum_{M=-L}^{L} i^L V^{2L+1} D_{0\phi}(\phi, 0, 0) \{ A_{L}^M(m) + i p A_{L}^M(e) \},$$

where $A_{L}^M(m)$ is the vector potential of the magnetic $2^L$ pole field and $A_{L}^M(e)$ that of the electric $2^L$ pole field, and these can be written explicitly as

$$A_{L}^M(m) = j_L V^{L}(L+1) Y_{L}^M, \quad L = -(r \times \nabla),$$

$$A_{L}^M(e) = -\frac{1}{k V^{L}(L+1)} \text{curl} (r \times \nabla) j_L Y_{L}^M.$$

Here $j_L$ is the spherical Bessel function. In Eq. (3) the matrix $D_{0\phi}$ means the usual representation of a rotation, and the polarization index $p = \pm 1$ corresponds to left and right circularly polarized waves, respectively. Then the amplitude of the electric radiation is given by

$$a_{L}^M(e) = \frac{-i}{k V^{L}(L+1)} \int dr j_L(kr) (LY_{L}^M)^* \cdot \text{curl} j.$$

Further, this expression is rewritten as

$$a_{L}^M(e) = \frac{1}{k V^{L}(L+1)} \int dr Y_{L}^M \{- (r \frac{dj_L}{dr} + j_L) \text{div} j - k^2 j_L (r \cdot j) \}.$$

Similarly the amplitude of the magnetic radiation is given by

$$a_{L}^M(m) = \frac{1}{k V^{L}(L+1)} \int dr j_L j \cdot (LY_{L}^M)^*$$

$$= \frac{i}{V^{L}(L+1)} \int dr j_L Y_{L}^{M*} \text{div} (r \times j).$$

Using the above formulas and taking angular integral and polarization sum with respect to the radiation in Eq. (1), we obtain the total cross section for the inverse process,

$$\sigma_{\pi} = 4\pi e^2 \frac{k}{q} \left( \frac{M}{W} \right)^2 \sum_{L,M} \{ | \langle \Phi_f | a_{L}^M(e) | \Phi_i \rangle |^2 + | \langle \Phi_f | a_{L}^M(m) | \Phi_i \rangle |^2 \}.$$
§ 3. Description of the second resonance and nucleon ground states

We assume that the pion-nucleon system is described in the centre of mass system by the Klein-Gordon equation with an effective potential \( V \),

\[
\{ A - \mu^2 + (E - V) \} \phi = 0, \tag{9}
\]

where \( E \) is the total energy of a pion, \( E^2 = q^2 + \mu^2 \). The second resonance state is defined as a \( d_{3/2} \) state, where the \( d_{3/2} \pi-N \) phase shift passes through \( \pi/2 \). For simplicity we shall take a square-well potential of range \( r_z \) for the \( d_{3/2} \) state,

\[
V = -V_2 \quad \text{for } r < r_z, \tag{10}
\]

\[
V = 0 \quad \text{for } r > r_z.
\]

An equation similar to (9) was used by Otsuki and Sawada\(^{11} \) for systematizing the meson-baryon resonances. Then the wave function \( \phi_i \) can be written as

\[
\phi_i = \chi(1/2) \Psi(d_{3/2}) R_z(r), \tag{11}
\]

in which \( \chi(1/2) \) refers to the isotopic spin 1/2 state, \( \Psi(d_{3/2}) \) the angular momentum \( d_{3/2} \) state, and \( R_z(r) \) the radial wave function which satisfies the above equation (9). For example, the \( m = 1/2 \) member of the \( \Psi(d_{3/2}) \) state, \( \Psi_{3/2, 1/2} \), is

\[
\Psi_{3/2, 1/2} = -\sqrt{\frac{2}{5}} \alpha Y^0_z + \sqrt{\frac{3}{5}} \beta Y^1_z, \tag{11'}
\]

and \( R_z(r) \) is expressed explicitly in terms of the spherical Bessel functions \( j_n \) and \( n_z \),

\[
R_z(r) = N_z j_z(\alpha_z r) \quad \text{for } r < r_z, \tag{12}
\]

\[
R_z(r) = \sqrt{8\pi n_z(qr)} \quad \text{for } r > r_z, \tag{12'}
\]

where the constant \( N_z \) and the pion wave number \( \alpha_z \) inside the range \( r_z \) are chosen so as to satisfy the boundary conditions at \( r = r_z \). The range of the potential, \( r_z \), is a parameter which, for the time being, we may leave open.

The nucleon ground state is approximated as a \( p_{3/2} \) bound state of the nucleon core and a single pion, which satisfies Eq. (9). Feld\(^{10} \) called this model the “atomic model” of the physical nucleons. The interaction between the nucleon core and the pion is described by a square-well potential \(- V_0\) of range \( r_0 \) similar to Eq. (10). This represents phenomenologically a modification of the vertex part \( A \) in Fig. 1, accordingly a nucleon form factor. In this way, the wave function of the nucleon ground state, \( \phi_f \), can be written as

\[
\phi_f = \chi(1/2) \Psi(p_{3/2}) R_z(r), \tag{13}
\]
where \( \mathcal{W}(p_{\nu_2}) \) refers to the angular momentum \( p_{\nu_2} \) state, and the radial wave function \( R_\alpha(r) \) is given explicitly as follows:

\[
R_\alpha(r) = N_\alpha j_1(\alpha_0 r) \quad \text{for } r<r_0, \tag{14}
\]

\[
R_\alpha(r) = V^3 fJ(1 - \frac{1}{\mu r^2}) e^{-\mu r} \quad \text{for } r>r_0, \tag{14'}
\]

The form of the expression (14') is determined by a condition that outside the range \( r_0 \) it should coincide with the static solution of the Yukawa equation

\[
\Box \Phi - \mu^2 \Phi = -\frac{V}{4\pi} \int \tau(\mathbf{\sigma} \cdot \mathbf{v}) \delta(r), \tag{15}
\]

that is,

\[
\Phi = \frac{1}{V4\pi} \int \tau(\mathbf{\sigma} \cdot \mathbf{v}) e^{-\mu r}. \tag{16}
\]

As before, the constant \( N_\alpha \) and the wave number \( \alpha_0 \) are settled by the boundary condition at \( r=r_0 \), once the parameter \( r_0 \) is fixed.

§ 4. Cross section for photopion production

The cross section for photopion production, \( \sigma_{\gamma\pi} \), is obtained from that for the inverse reaction, \( \sigma_{\pi\gamma} \), by the detailed-balancing condition,\(^{15}\)

\[
\sigma_{\gamma\pi} = \frac{1}{2} \frac{q^2}{\mu^3} \sigma_{\pi\gamma}. \tag{17}
\]

The resonant cross section, \( \sigma_{\gamma\pi} \), is divided into two parts, one for the electric dipole transition, \( \sigma(E1) \), and the other for the magnetic quadrupole transition, \( \sigma(M2) \);

\[
\sigma_{\gamma\pi} = \sigma(E1) + \sigma(M2). \tag{18}
\]

In the calculation of the \( T \) matrix, the spin sum over the initial nucleon state is simplified, if the \( z \)-axis is chosen to lie along the direction of the incident pion. Then, the sum over initial states contains just two terms, corresponding to the two possible polarizations of the target nucleons and, if the target nucleons are initially unpolarized, these two terms are equal. It suffices, then, to compute just two transition amplitudes corresponding to the spin non-flip \( (M=0) \) and the spin-flip \( (M=1) \) transitions for each multipole in Eq. (8).

In this paper we take only the contributions of the \( \pi \) meson current into account, so that we assume that the current density operator \( j \) is given by

\[
j = -i(\phi^* \cdot \nabla \phi - \nabla \phi^* \cdot \phi), \tag{19}
\]

and it is a third component of an isovector. In Eq. (19) \( \phi \) is the usual complex pion field. Of course, our effective potential \( V \) depends on both the isotopic
spin of the system and the momentum of the pion. In other words the potential \( V \) contains exchange and velocity-dependent interactions. Therefore, besides the meson and nucleon currents the gauge invariance requires some additional couplings to the radiation field. Since these couplings are directly dependent on the pion-nucleon interactions which at present we do not know very well, we shall omit the contributions of the above additional interaction currents as well as the nucleon current.

We compute the cross section for the process \( \gamma + p \rightarrow N^{**} \rightarrow \pi^0 + p \), \( \sigma_p^{\pi^0} \). In this case the expression (17) must be multiplied by a factor of \( 1/3 \), which takes account of the weight in isotopic spin. Introducing Eq. (19) into Eqs. (6') and (7') and using Eqs. (8), (17) and (18), the partial cross sections \( \sigma(E_1) \) and \( \sigma(M_2) \) are calculated. The ratio of the electric dipole transition amplitude \( a(E_1) \) and the magnetic quadrupole one \( a(M_2) \) is usually connected with the ratio of the corresponding cross sections \( \sigma(E_1) \) and \( \sigma(M_2) \) by a relation

\[
a(E_1) = \sqrt{\frac{\sigma(E_1)}{\sigma(M_2)}}.
\]

But the ratio of the amplitudes \( E_2 \) and \( M_2 \) is different from the above value by a factor of \( \sqrt{3} \),

\[
\frac{E_2}{M_2} = \sqrt{3} \frac{a(E_1)}{a(M_2)}.
\]

In order to check the reliability of the radial wave functions, we also calculate the root-mean-square radius of the distribution of charge in proton, \( \langle r^2 \rangle_p^{1/2} \), and the width of the second resonance, \( \Gamma \). The former is given in this model by

\[
\langle r^2 \rangle_p^{1/2} = \sqrt{\frac{2}{3}} \langle r^2 \rangle_0,
\]

in which the subscript 0 denotes the expectation value with respect to the nucleon ground state. The width \( \Gamma \) is estimated by the following expression,\(^{10}\)

\[
\Gamma = \frac{2 \cos^2 \beta_2}{(q_r) n_r^2 (q_r) ((d\Phi_0/dE) + (d\beta_2/dE))_{E=E_r}},
\]

where \( E_r \) refers to the resonance energy and the quantities \( \Phi_2 \) and \( \beta_2 \) are defined by the relations

\[
\tan \Phi_2 = (\alpha_2 r_2) \frac{j'_2(\alpha_2 r_2)}{j_2(\alpha_2 r_2)} \quad (24),
\]

\[
\tan \beta_2 = -(q_r) \frac{n'_2(q_r)}{n_2(q_r)} \quad (25),
\]

\[
\tan \beta_2 = -(q_r) \frac{n'_2(q_r)}{n_2(q_r)} \quad (25).
\]
§ 5. Results and discussion

In the practical application of our model we shall, for simplicity, take the two parameters \( r_0 \) and \( r_z \) to be equal,

\[ r_0 = r_z = b. \]  

(26)

For four values of the parameter \( b \), the quantities mentioned hitherto are computed. The numerical results are summarized in Table I, in which the values \( e^2 = 1/137 \), \( f' = 0.08 \), \( W = 1518 \text{ MeV} \), \( M = 938 \text{ MeV} \) and \( \mu = 140 \text{ MeV} \) are employed.

<table>
<thead>
<tr>
<th>( b ) (10^{-13} \text{ cm})</th>
<th>( \sqrt{2/3} \langle r^2 \rangle_{10} ) (10^{-13} \text{ cm})</th>
<th>( \Gamma ) (MeV)</th>
<th>( V_6 ) (MeV)</th>
<th>( V_z ) (MeV)</th>
<th>( \sigma p^0 ) (10^{-30} \text{ cm}^2)</th>
<th>( \alpha(E1)/\alpha(M2) )</th>
<th>( E_{1-}/M_{1-} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.75</td>
<td>32.4</td>
<td>1069</td>
<td>985</td>
<td>128.5</td>
<td>4.76</td>
<td>8.25</td>
</tr>
<tr>
<td>0.7</td>
<td>0.82</td>
<td>51.9</td>
<td>914</td>
<td>769</td>
<td>48.2</td>
<td>3.52</td>
<td>6.10</td>
</tr>
<tr>
<td>0.8</td>
<td>0.88</td>
<td>78.0</td>
<td>805</td>
<td>610</td>
<td>20.1</td>
<td>2.68</td>
<td>4.65</td>
</tr>
<tr>
<td>0.9</td>
<td>0.95</td>
<td>113.1</td>
<td>721</td>
<td>491</td>
<td>9.0</td>
<td>2.08</td>
<td>3.61</td>
</tr>
</tbody>
</table>

Here the presence of inelastic channels is not taken into account. Therefore, the above width \( \Gamma \) corresponds to the elastic width, \( \Gamma_{el} \). Experimental values of the r. m. s. radius \( \langle r^2 \rangle_{1/2} \) and the width \( \Gamma_{el} \) are given by \( \langle r^2 \rangle_{1/2} = (0.813 \pm 0.01) \times 10^{-13} \text{ cm} \) and \( \Gamma_{el} = 120 \times 75/100 = 90 \text{ MeV} \), respectively. From columns 2 and 3 in Table I one sees that the values \((0.6-0.7) \times 10^{-13} \text{ cm}\) for the parameter \( b \) fit the observed r. m. s. radius \( \langle r^2 \rangle_{1/2} \), while the values \((0.8-0.9) \times 10^{-13} \text{ cm}\) are good for the width \( \Gamma_{el} \). This fact can be understood qualitatively by the meson theory. That is, the potential \( V_z \) corresponds to the exchange of at least two pions, while the potential \( V_6 \) to three pions. According to a phenomenological analysis of the experimental data by Yamaki,\(^9\) the partial photoproduction cross section \( \sigma p^0 \) for the resonant \( d_{3/2} \) state seems to be between 20 and 30 \( \mu \) barns. Column 6 fits these values for the parameter \( b \sim (0.7-0.8) \times 10^{-13} \text{ cm} \). Then the ratio \( E_{1-}/M_{1-} \) is expected to be between the values of 4 and 6 from the last column. These figures are somewhat larger than the value of 3 mentioned in § 1. However, considering the ambiguities involved in the multipole analysis of the experimental data and the crudeness of our model, we can conclude that the main features of the resonant photoproduction process are well explained by our simple model.

We neglect the absorption effects of inelastic processes and the contributions of the nucleon current and the additional interaction currents. Though we should take these effects into account, our results will not change essentially even if we consider them. An investigation of this point is now in progress.

Finally we remark on the works based on the quark model. Moorhouse\(^7\) showed that the quark model makes no remarkable statements about the photoproduction of the second pion-nucleon resonance, that is, the allowed electro-
magnetic multipole transitions are also allowed by the quark model. Recently Fujimura, T. H. Kobayashi and T. R. Kobayashi have investigated the photoproduction of the second resonance more precisely. Taking account of only the contributions of one quark excitation and neglecting those of the exchange current, they calculated the ratio of the magnetic and electric transition widths, $\Gamma_m/\Gamma_e$, and obtained a value $\Gamma_m/\Gamma_e \approx 1/10$. This value corresponds to the ratio $a(E1)/a(M2) \approx \sqrt{10}$ and is in good agreement with our results.

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