journal of applied mechanics discussion

Approximate Solutions in Linear, Coupled Thermoelasticity[1]

O. W. DILLON, JR.[2] This is a very nice paper. It illustrates very nicely what the combination of the computer and some thinking can do. I would like to ask four questions:

1 Have the authors considered the second variation? It is always positive or does it indicate (when combined with equations (21) of the paper) a finite time for the applicability of the approximation? If so this would give a means of estimating proper increments.

2 In view of Fig. 3 of the paper, how would the authors expect the present technique (especially equations (21) of the paper) to work for oscillatory boundary conditions?

3 Do the authors anticipate good long-time results for small (say, 0.005) values of the coupling parameter?

4 How would the authors compare their numerical technique when used for the case of small values of the coupling parameter with the perturbation method of Soler and Brull?[3]

R. E. JONES.[4] The authors have derived a finite element formulation for problems of dynamic thermoelasticity, in which the coupling effect of internal heat generation due to deformation is included. They have demonstrated the applicability of the finite element method to this problem by showing good numerical agreement with analytical solutions for several problems. Their principal contribution in this paper lies in the inclusion of coupling effects, since finite element applications to problems of heat flow, thermal stress analysis, and wave motions have been made previously. A valuable feature of the work is the authors' use of a variational principle, rather than an intuitive approach, as the basis of their derivations.

This discussion is aimed at certain characteristics of finite element wave responses and a problem of thermal stress analysis, both of which appear in the authors' paper, with which the writer has had experience. The first item concerns wave motion and the spatial mesh.

Finite element models behave as dispersive, low pass filters in the propagation of stress waves. Long, low-frequency waves propagate at the elastic-wave velocity, higher-frequency waves at reduced velocities, and waves of frequency above a certain, cutoff value do not propagate at all. The result is that stress pulses are smoothed, or rounded, with distance of propagation, and are trailed by gradually decaying oscillatory motions. In addition, motions are observed ahead of the wave fronts, due to a phenomenon which is nonwavelike in nature. Mesh oscillations with a wavelength equal to twice the mesh size take place at the mesh cutoff frequency and cannot propagate as elastic waves. They spread through the mesh due to the finite-difference integration procedure, however. Their apparent velocity is roughly equal to the mesh spacing divided by the integration step time. The authors' numerical results appear to show these effects. The use by the authors of a variable mesh size may also contribute to the oscillations of their results. The coarser mesh cannot propagate the higher frequencies contained in the motions of the finer mesh and acts as a sort of fixed boundary in reflecting these frequencies. This produces a pattern of reflected oscillations in the model. For this reason, usually a constant mesh size is preferable in applications.

The second discussion item concerns the interrelationship between the assumed elemental functions for temperature and displacement. In the authors' numerical work both are linear functions between nodal points. The variational principle provides equilibrium requirements based on the work done by the stresses in variations of the strains which are caused by varied nodal motions. The strain variations are constants over the elements. Hence only the constant components of the stresses are subject to the equilibrium requirements. The equilibrium state is therefore one in which the finite element model has a discontinuous rather than a continuous temperature field, in which each element has a constant average temperature, and there are temperature jumps between the elements. It is noted that this comment applies only to the stresses and the elastic portion of the motions. For the heat-flow calculations the linearity of the temperatures is preserved.

The use of element displacement functions of the same degree as the temperature function characteristically produces oscillations in computed results, particularly in the stresses. Improved accuracy is obtainable with higher-degree displacement functions.

Authors' Closure

The authors would like to thank both O. W. Dillon, Jr., and R. E. Jones for their discussion and comments. Both have raised relevant questions about the accuracy and applicability of the extended Ritz method for this class of initial-boundary-value problems. In responding to Dillon's questions we would like to emphasize that the method is a discrete variable approach, having characteristic limitations in this regard. Approximate methods of this kind cannot hope to yield accurate quantitative information at or near surfaces of discontinuity unless the jump conditions are incorporated into the mathematical model; the frequency content of the signal exceeds the descriptive capacity of a finite degree-of-freedom system, as Jones has pointed out in his discussion. As the order of the discontinuity increases, the results near the wave fronts significantly improve, indicating a more moderate frequency content.

We have not considered the second variation of the generating variational principle nor its implications with regard to estimating optimum time increment size. Choice of the increment size is usually made from considerations of numerical stability; however, preliminary investigations on this point have been inconclusive. Complications arise because the method is not a "two-step" forward integration procedure in the classical finite difference sense, but instead is a "multistep" procedure, even retaining some knowledge of the initial conditions. In our case the time increment size was generally chosen in order to adequately define the boundary conditions. It therefore seems likely that oscillatory boundary conditions whose frequency can be reasonably represented by the discrete spatial variables and which can be adequately described through the choice of time increment size can be successfully treated. Reiterating comments made by Jones in his

[1] By R. E. Nickell and J. L. Sackman, published in the June, 1968, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 35, TRANS. ASME, Vol. 90, Series E, pp. 255–266.

[2] Professor and Chairman of the Engineering Mechanics Department, University of Kentucky, Lexington, Ky.

[3] Soler, A. I., and Brull, M. A., "On the Solution to Transient Coupled Thermoelastic Problems by Perturbation Techniques," JOURNAL OF APPLIED MECHANICS, Vol. 32, No. 2, TRANS. ASME, Vol. 87, Series E, June 1965, pp. 389–399.

[4] Research Specialist, Missile and Information Systems Division, The Boeing Company, Seattle, Wash.

discussion, it should be noted that oscillatory boundary conditions characterized by frequencies above a certain value will not elicit a response from the media.

There is no evidence to indicate that the magnitude of the coupling parameter has any effect on the accuracy of the approximate solution after (say) several hundred time steps. The dominant criteria in this regard are accumulated round-off error and discretization error. If long-time results are sought, it is unlikely that a forward integration procedure would suffice; asymptotic techniques would be more feasible. It should also be pointed out that the extended Ritz method is most effective when dealing with complex domains, arbitrary initial and boundary values, and inhomogeneous material properties; direct comparison with the perturbation method of Soler and Brull[5] would not do justice to either approach.

The authors feel that higher-order element displacement functions would significantly improve the stress calculations and might reduce dispersion of the signal to some extent at the same time. The purpose in the paper was, however, to discuss the lowest order of displacement and temperature expansions which would satisfy spatial continuity requirements of the variational principle, realizing that such expansions are not the end result of the investigation.

[5] See footnote 3 to Dillon discussion.

Unsteady Laminar Motion of a Newtonian Fluid Contained Between Concentric Rotating Cylinders[1]

C. F. LO.[2] The paper presents the solution of the problem solved by the method of separation of variables and by use of Duhamel's theorem. However, the solution can be easily obtained by applying the generalized finite Hankel transform[3] without excessively formal calculation. The special case of the integral transform for the problem is quoted here. The transform is defined by

$$\bar{f}(\lambda_i) = \int_{R_1}^{R_2} f(r) r U_1(\lambda_i r) dr$$

where

$$U_1(\lambda_i r) = [-\beta_1\lambda_i Y_1'(\lambda_i R_1) + \alpha_1 Y_1(\lambda_i R_1)]J_1(\lambda_i r) - [-\beta_1\lambda_i J_1'(\lambda_i R_1) + \alpha_1 J_1(\lambda_i R_1)]Y_1(\lambda_i r)$$

and λ_i the ith positive root of the equation

$$\beta_2\lambda_i U_1'(\lambda_i R_2) + \alpha U_1(\lambda_i R_2) = 0, \qquad U_1'(\lambda_i R_2) = \left.\frac{dU_1(\lambda_i r)}{d\lambda_i r}\right|_{r=R_2}$$

The inversion formula is

$$f(r) = \sum_{i=1}^{\infty} \left\{ \bar{f}(\lambda_i) U_1(\lambda_i r) \bigg/ \int_{R_1}^{R_2} r U_1^2(\lambda_i r) dr \right\}$$

Some other properties can be found; see Lo.[3]

Authors' Closure

The paper referenced by the discusser will be read with interest by the authors, when it is available.

[1] By A. V. Farnsworth, Jr., and W. Rice, published in the June 1968 issue of the JOURNAL OF APPLIED MECHANICS, Vol. 35, TRANS. ASME, Vol. 90, Series E, pp. 419–420.

[2] Research Engineer, ARO, Inc., Arnold Air Force Station, Tenn.

[3] Lo, C. F., "A Generalization of the Finite Hankel Transform and Applications," submitted to *Quarterly Journal of Mechanics and Applied Mathematics*.

Potential Vortex Flow Adjacent to a Stationary Surface[1]

W. S. LEWELLEN.[2] Kidd and Farris have obtained a numerical solution of Gol'dshtik's problem. They have confirmed his analytic proof that a unique and continuous solution exists for a tangential Reynolds number less than 4.8. Perhaps the most interesting feature of their velocity profiles is the fact that there is no net radial flow in the interaction layer. On the other hand, it is not surprising that the resulting profiles do not obey all of the classical boundary-layer assumptions at these low Reynolds numbers.

It is difficult to accept the authors' implication that their results are relevant to the high Reynolds numbers experiments carried out by Kendall [1][3] and by Savino and Keshock [2]. It is true that radial flow reversals were observed in their experiments, but this is entirely consistent with the high Reynolds number solutions of King and Lewellen [3]. The vortex distribution is not precisely potential in their experiments; i.e., there is a decay in the circulation ($v_\theta r$) as r decreases. In this case a terminal similarity solution to the boundary-layer equations is available and it exhibits the flow reversal.

There should be no great mystery surrounding the boundary layer produced by a potential vortex flow adjacent to a stationary surface. It is not the boundary-layer assumption that breaks down, but the requirement of a similar flow that is not satisfied at high Reynolds numbers.

Equation (20) of the paper shows that the tangential velocity profile must monotonically increase from zero at the wall to its potential vortex value away from the wall. Thus the centrifugal force in the interaction layer is everywhere less than or equal to the free-stream value. If the pressure is constant across the boundary layer, as it must be in Kidd and Farris's problem for high Reynolds numbers, then the radial velocity will be directed radially inward with no reversal. The angular momentum defect of the fluid close to the wall must increase with decreasing radius due to the shear losses at the surface. However, the similarity transformation of equations (10)–(14) would require any angular momentum defect to increase with increasing radius. This inconsistency leads one to the conclusion that this similarity transformation must be abandoned at high Reynolds numbers.

In the more general case, when the vortex is nonpotential and the circulation decreases with decreasing radius in the outer flow, the tangential velocity can actually exceed the outer value at points within the boundary layer. In these cases the boundary layer can carry an excess angular momentum which decreases with decreasing radius. A similarity solution is then possible [3] which describes how the boundary layer terminates near the axis.

If the potential vortex problem is completed by specifying conditions at some radius, rather than using the similarity constraint, then the boundary layer will grow radially inward from this point. This problem was first solved approximately by Taylor [4]. Anderson [5] has integrated the exact boundary-layer equations numerically for the problem of a finite stationary disk perpendicular to a potential vortex. The boundary layer grows from the outer edge of the disk and the radial mass flow in the layer increases with decreasing radius. A more complete discussion of this and other rotating boundary layers is available in a review article by Rott and Lewellen [6].

The actual solution produced by Kidd and Farris appears to require more than just a potential vortex perpendicular to a stationary surface. It also involves an axisymmetric radial jet

[1] By G. J. Kidd, Jr., and G. J. Farris, published in the June, 1968, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 35, TRANS. ASME, Vol. 90, Series E, pp. 209–215.

[2] Massachusetts Institute of Technology, Cambridge, Mass.

[3] Numbers in brackets designate References at end of Discussion.