American Journal of Physics

NOTES AND DISCUSSIONS | APRIL 01 2015

High-field level crossing in atomic hydrogen FREE

Kara M. Lynch; Kieran Flanagan; H. Henry Stroke

Am. J. Phys. 83, 375–376 (2015)
https://doi.org/10.1119/1.4901810

NOTES AND DISCUSSIONS

High-field level crossing in atomic hydrogen

Kara M. Lynch
Instituut voor Kern en Stralingsfysica, KU Leuven, B-3001 Leuven, Belgium

Kieran Flanagan
Nuclear Physics Group, University of Manchester, Manchester M13 9PL, United Kingdom

H. Henry Stroke[a)]
Department of Physics, New York University, New York, New York 10003 and CERN, 1211 Geneva 23, Switzerland

(Received 8 July 2014; accepted 3 November 2014)

In hydrogen, an external magnetic field, which we calculate to be ≈16.65 T, cancels the internal field caused by the electron motion in the magnetic sublevels with $m_J = +1/2$. This results in an energy-level degeneracy between states with nuclear magnetic sublevels m_I of opposite signs. The evaluation of this field has been calculated previously with the use of the low-field quantum numbers F, m_F. We show that this calculation is considerably simpler in the high-field m_J, m_I representation. A comparison is given with the earlier work.

[http://dx.doi.org/10.1119/1.4901810]

I. INTRODUCTION

Level crossings in atoms have been the subject of study for nearly one century. The polarization behavior of radiating atoms near zero magnetic field, in a region where level separation into distinct magnetic quantum substates is being established, known as the Hanle effect,[1] is a textbook subject.[2] This study of the Hanle effect allows the determination of spectral linewidths and shapes, and hence parameters such as lifetimes and collision cross sections. Atomic level crossings at higher (laboratory) magnetic fields have been exploited in hyperfine structure (hfs) studies.[3] Hyperfine structure denotes the small energy splittings in an electron energy level caused by the interaction of the atomic electron with nuclear magnetic (and possibly electric) multipole moments. Another type of hfs level crossing, e.g., in atoms with electron (spin plus orbital) angular momentum $J = 1/2$, occurs when the internal magnetic field B_0 created by the atomic electron is just equal and opposite to an externally applied field B. This situation has been studied formally by Dickson and Weil.[4] When the applied field is below B_0, the field seen by the nucleus is essentially that produced by the electron; it is in opposite directions for the two sets of electron magnetic substates $m_J = \pm 1/2$. Above B_0, the magnetic field direction is determined by B. At B_0 for one of the electron substates, the nuclear magnetic moment μ_I is in zero magnetic field and the energies of the nuclear magnetic sublevels become degenerate. We show that this "crossing field" is simple to calculate in the high magnetic field representation.

II. J = 1/2 ATOM IN HIGH *B*-FIELD: m_I, m_J REPRESENTATION

The energy E of the $J = 1/2$ atom with nuclear spin I is given by the Breit-Rabi formula[5] in terms of I, J, the total angular momentum $\mathbf{F} = \mathbf{I} + \mathbf{J}$, and the magnetic quantum number m_F. It is simpler for our purpose to express the Hamiltonian $\mathcal{H}$, given by

$$\frac{\mathcal{H}}{h} = a\mathbf{I}\cdot\mathbf{J} + \mu_B(g_J\mathbf{J}\cdot\mathbf{B} + g_I\mathbf{I}\cdot\mathbf{B}), \qquad (1)$$

in terms of the I and J decoupled high-field quantum numbers m_I and m_J instead of F and m_F:

$$E = am_Im_J + \mu_B(g_Jm_JB + g_Im_IB), \qquad (2)$$

where $g_I = \mu_I/I$, $g_J = \mu_j/J$, and $a = \Delta\nu/(I + 1/2)$ with $\Delta\nu$ the zero-field energy separation of the two hfs levels, F (for hydrogen, $F = 1$ and $F = 0$). In this expression, both g_I and g_J are expressed in units of the Bohr magneton $\mu_B = e\hbar/(2mc)$. From Eq. (2), we can see that for states with $m_J = 1/2$, there is a magnetic field B_0 for which $E_{m_I} = E_{-m_I}$ (these levels cross); the external field just cancels that produced on the nuclear site by the atomic electron, and these two m_I states become degenerate, so that

$$am_Im_J + \mu_Bg_Im_IB_0 = a(-m_Im_J) + \mu_Bg_I(-m_I)B_0. \qquad (3)$$

From this we obtain for the crossing field

$$B_0 = \frac{am_J}{\mu_Bg_I}. \qquad (4)$$

This result is valid not only for hydrogen, but also for other $J = 1/2$ atoms. Putting in the hydrogen values for $a = \Delta\nu \approx 1420.4$ MHz, $m_J = 1/2$, the Bohr magneton in frequency units $\mu_B \approx 1.404\,\text{MHz/G}$ ($1\,\text{G} = 10^{-4}\,\text{T}$), $g_I = \mu_I/I$, and recalling that the value of g_I has to be expressed in Bohr magnetons, we obtain $B_0 = 166{,}481\,\text{G} \approx 16.65\,\text{T}$, in accord with the value obtained in Ref. 4 (and presented in Sec. III). It is suggested in Ref. 4 that with present-day technology one may reach this field in the laboratory. We should note, however, that the approach to the crossing magnetic field has such a weak dependence on B, that B_0 would be difficult to determine. Otherwise one would have a new tool to determine the spatial dependence of the hfs interaction on the nuclear magnetic structure.

III. $J = 1/2$ ATOM IN HIGH B FIELD: F, m_F REPRESENTATION

For completeness, in the following we derive the result given in Eq. (4) in the more cumbersome low-field F, m_F representation of the Breit-Rabi equation. We have $m_F = m_I + m_J$ and we start with the Breit-Rabi equations[5] for the hfs energies of the $J = 1/2$ atom in an external magnetic field B:

$$\frac{E(F, m_F)}{h} = -\frac{\Delta\nu}{2(2I+1)} + m_F g_I \mu_B B \pm \frac{\Delta\nu}{2}\left(1 + \frac{4m_F x}{2I+1} + x^2\right)^{1/2}. \qquad (5)$$

Here, the magnetic field parameter x for $J = 1/2$ is

$$x = (g_J - g_I)\frac{\mu_B B}{\Delta\nu} = (g_J - g_I)\frac{\mu_B B}{a}, \qquad (6)$$

and the $\pm$ signs denote the states $F = I \pm 1/2$. The crossing m_I levels of interest occur only within one of the F levels, characterized here at high field by $m_J = +1/2$. Since $m_{F_\pm} = m_J \pm m_I$, and correspondingly we have E_+ and E_-, these two levels become degenerate at a crossing field given by

$$\frac{1}{h}(E_+ - E_-) = 0 = 2g_I \mu_B B m_I + \frac{a}{2}\left[\left(1 + \frac{4(m_J + m_I)x}{2I+1} + x^2\right)^{1/2} - \left(1 + \frac{4(m_J - m_I)x}{2I+1} + x^2\right)^{1/2}\right]. \qquad (7)$$

The high field is characterized by $x \gg 1$, so we can replace $1 + x^2 \approx x^2$ in this expression (which may affect the result by less than one part in 10^4), and then expand the square roots to obtain

$$0 \approx 2g_I \mu_B B m_I + \frac{a}{2}\frac{4m_I}{(2I+1)}, \qquad (8)$$

or

$$B = \frac{a}{(2I+1)g_I \mu_B}. \qquad (9)$$

Inserting $I = 1/2$ and $m_J = 1/2$, we obtain

$$B_0 = \frac{a m_J}{\mu_B g_I}, \qquad (10)$$

identical to the result obtained in the "high-field" representation given by Eq. (4).

IV. CONCLUSION

The F, m_F representation used to obtain Eq. (10) is obviously more laborious than the simple calculation in the m_J, m_I scheme. We also did not have to make the approximation $x \gg 1$ as done for Eq. (8). The advantage of using the more appropriate high-field Hamiltonian representation at the start is clear.

ACKNOWLEDGMENTS

K.M.L was supported by the FWO Pegasus Marie Curie Fellowship No. 267216; K.T.F. had support from the STFC Continuation Grant No. ST/J000159/1 and Advanced Fellowship Scheme Grant No. ST/G006415/1. H.H.S. gratefully acknowledges support of our work by Edward Schneiderman.

[a]Electronic mail: henry.stroke@nyu.edu

[1]W. Hanle, "Über magnetische Beeinflüssung der Polarisation der Resonanzfluoreszens," Z. Phys. **30**, 93–105 (1924).

[2]A. C. G. Mitchell and M. Zemansky, *Resonance Radiation and Excited Atoms* (Cambridge U.P., London, 1934), pp. 262–270.

[3]See, for example, O. Redi and H. H. Stroke, "Level-crossing determination of the $6s6p\,^3P_1$ hfs of ^{193m}Hg," Phys. Rev. A **9**, 1776–1782 (1974).

[4]R. S. Dickson and J. A. Weil, "Breit-Rabi Zeeman states of atomic hydrogen," Am. J. Phys. **59**, 125–129 (1991). The existence of this energy level crossing in hydrogen is also noted by D. Budker, D. F. Kimball, and D. P. DeMille, *Atomic Physics* (Oxford U.P., New York, 2004), p. 18.

[5]G. Breit and I. I. Rabi, "Measurement of Nuclear Spin," Phys. Rev. **38**, 2082–2083 (1931).

The pendulum swings back in Einstein's favor; a comment on "The equivalence principle as a stepping stone from special to general relativity: A Socratic dialog" [Am. J. Phys. 74, 22–25 (2006)]

Jeremy Bernstein

Stevens Institute of Technology, Hoboken, New Jersey 07030

(Received 22 November 2014; accepted 5 December 2014)

[http://dx.doi.org/10.1119/1.4904424]

In a 2006 article in this journal,[1] Drake discusses the effect of the Earth's gravitational field on the dilation of time as measured by clocks located on the surface of the Earth. In particular, he compares the time as measured by a clock at the equator to the time measured by an identical clock at the pole. This was a comparison already made by Einstein in his1905 relativity paper where he wrote, "Thence we conclude that a balance clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions."[2] It is interesting that Einstein specified the type of clock. If by a balance clock he meant a pendulum then his statement is correct, though not for the reasons he thought (he did not yet know about the effect of gravity on time). On the other hand, using an atomic clock and including the effect of gravity the statement is false. In his 2006 article, Drake does not specify the clocks to which his arguments apply, and that is the purpose of this comment.

At any point on the Earth's surface, there is both a gravitational and a centrifugal force, the latter being zero at the poles and maximal at the equator. Hence, there is an effective potential that takes into account both forces, though it is quite complex because the Earth is not a sphere. In the paper cited this potential is given by[3]

$$\Phi_{\rm eff} = -\frac{GM_e}{r} - \frac{J_2 GM_e a^2(1-3\cos^2\theta)}{2r^3} - \frac{1}{2}\omega^2 r^2 \sin^2\theta, \tag{1}$$

where $GM_e = 3.986\,004\,42 \times 10^{14}\,\mathrm{m^3/s^2}$ is the product of the gravitational constant and Earth's mass, $J_2 = 1.082\,636 \times 10^{-3}$ is a measure of Earth's equatorial bulge, $a = 6\,378\,137\,\mathrm{m}$ is Earth's equatorial radius, and $\omega = 7.292\,116 \times 10^{-5}\,\mathrm{rad/s}$ is Earth's rotation rate. As complicated as this potential may look, it has the same value at all points on the Earth's surface as the reader can verify by studying the value at the pole and the equator where the polar radius is $6\,356\,760\,\mathrm{m}$ with a small error. This is as it must be because otherwise Earth's crust would move.

The time dilation generated by the potential, if it is weak, is given by[4]

$$\frac{dt}{d\tau} = 1 - \frac{\Phi_{\rm eff}}{c^2}, \tag{2}$$

with τ being the proper time. This is the formula that applies to atomic clocks where the time is defined in terms of the frequency of an atomic transition; it tells us that the time measured this way would be the same at the pole and the equator. But what about the pendulum clock?

The key point about the pendulum is that when it is in free fall it no longer oscillates—there is no force of gravity. Thus, the period of the pendulum must vary inversely as the gravitational acceleration g to some power. But this power must be 1/2 because g is the only parameter in the problem that has a time dimension (g has dimensions $\mathrm{m/s^2}$). If the length of the pendulum is L then the period must be proportional to $\sqrt{L/g}$. A standard calculation shows that the constant of proportionality is 2π, so the period T is given approximately by

$$T = 2\pi\sqrt{\frac{L}{g}}. \tag{3}$$

We have argued that the gravitational potential is constant on Earth's surface, which means the *total* derivative with respect to r must vanish, but not the partial derivative. Indeed, the acceleration in the radial direction is given by

$$g = \frac{\partial}{\partial r}\Phi_{\rm eff}\,. \tag{4}$$

Taking the derivative of Eq. (1), we are led to

$$g = \frac{GM_e}{r^2} + \frac{3J_2 GM_e a^2(1-3\cos^2\theta)}{2r^4} - \omega^2 r \sin^2\theta\,, \tag{5}$$

and plugging in the numbers, we find that

$$g_{\rm pole} = 9.896\,551\,34\,\mathrm{m/s} \tag{6}$$

while

$$g_{\rm eq} = 9.839\,650\,901\,\mathrm{m/s}. \tag{7}$$

So Einstein was right after all, if by a balance clock he meant a pendulum clock. A pendulum balance clock *does* go slower at the equator than the pole!

ACKNOWLEDGMENTS

The author is grateful to Jim Peebles, Wolfgang Rindler, Luis Alvarez-Gaume, and an anonymous referee for very helpful comments.

[1] S. M. Drake, "The equivalence principle as a stepping stone from special to general relativity: A Socratic dialog," Am. J. Phys, **74**, 22–25 (2006).

[2] A. Einstein, *The Principle of Relativity with notes by A. Sommerfeld* (Dover Publications, New York, 1952), pp. 49–50. The German term he uses isunruhuhr, which is better translated as "spring wound clock." I have chosen the pendulum because of the effects of gravity. The spring wound clock would read the same at the pole as the equator. As the rotational speed of the Earth at the equator is only about 478 m/s, I have neglected the effects of special relativity in this note, which are of order 10−12 and thus much smaller than the general relativistic effects I am considering.

[3] Implicit in this discussion is the underlying metric $-c^2\,d\tau^2 = -(1+\frac{2\Phi}{c^2})c^2\,dt^2 + (1-\frac{2\Phi}{c^2})dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$. In a private communication, Luis Alvarez-Gaume has remarked that this metric is not technically correct because the Earth is rotating so that θ is a function of time. Thus, this diagonal form of the metric is only an approximation (although in this case a very good one).

[4] See appendix of Ref. 1.