Wave Equation with Mass and Spin Spectrum Based on $O(3,3)$ Group for Relativistic Deformable Model

Takehiko Takabayasi

Department of Physics, Nagoya University, Nagoya

(Received May 24, 1967)

On the basis of the fact that the group $SL(4,R)$ of relativistic linear deformation and rotation is isomorphic to $SO(3,3)$, new transformation properties under the physical Lorentz transformation is assigned to the internal coordinate in the bilocal-type model, which is adopted as a simple relativistic example having internal movement. The new viewpoint leads to both integer and half-integer spin states and also to the first order wave equation, which contains a mass spectrum corresponding to infinite-dimensional representations of the inner Lorentz group.

On the other hand several versions of bilocal model which belong to the conventional identification of inner Lorentz group and represent oscillator or rotator type internal motion are successively discussed in the Appendix.

§ 1. Introduction

The relativistic description of a model with internal movement must correspond to representations of the direct product group

$$G^\text{orb} \times G^\text{in},$$

where $G^\text{orb}$ denotes the orbital Poincaré group acting on the “center-of-mass” coordinates $X^a$, alone, and $G^\text{in}$ is a non-compact group containing as subgroup the inner (homogeneous) Lorentz group $L^\text{in}$ whose generators $S_{\mu\nu}$ are responsible for the spin. The physical Lorentz group $L$

$$L \subset L^\text{orb} \times L^\text{in}$$

is generated by the angular momentum tensor

$$M_{\mu\nu} = X_{[\mu} P_{\nu]} + S_{\mu\nu},$$

where $P_a$ is the translation operator satisfying $[X_{\mu}, P_a] = ig_{\mu a}$. In particular $P_0$ represents the energy of the system and $X_0$ means the usual time.

The early Majorana model and the Yukawa bilocal model in its original version can be regarded as the simplest examples conforming to the above general framework. For the former the inner group is

---

*) This general conception has been naturally formed in the course of investigation of various relativistic models with internal motion for a unified description of hadrons by the author. The same statement was made by Budini and Fronsdal through more abstract analysis. See also references 3), 3a).
Wave Equation with Mass and Spin Spectrum

\[ G^{in} = Sp(2, R) \approx O(3, 2) \supseteq L^{in}, \]

and the model realizes its infinite-dimensional unitary representation, with descending mass spectrum \( m = \kappa/(J + \frac{1}{2}) \). This model is regarded as built on a two-component real internal coordinate \( q_{a}(\alpha=1,2) \). On the other hand the bilocal model has a real 4-vector \( x_{a} \) as internal coordinate, but in the original Yukawa version, because of the restrictions \((x_{a}x^{a} - r^{2})\phi = 0, (r=\text{const}), \) and \( P_{a}x_{a}\phi = 0, \) the system contains again two internal degrees of freedom alone. Here the inner group is \( G^{in} = O(3, 1) \) and the mass is completely degenerate. (cf. Appendix II-(vii)).

In each of the above models the existing two internal degrees of freedom are just responsible for the spin quantum numbers \( J \) and \( J_{a}. \) Starting with a larger group for \( G^{in} \) and its representations realized on a larger set of internal variables, one can accommodate various other quantum numbers. Such simple examples are the bilocal model without subsidiary condition discussed after Yukawa \( ^{7} \) and the spinor model introduced by the authors \( ^{5,6,8} \). Each of them has four internal degrees. The internal group \( G^{in} \) is either \( U(3, 1) \) or \( GL(4, R) \) for the former, and \( G^{in} = U(2, 2) = U(1) \times O(4, 2) \) for the latter.

The purpose of this paper is to present a new point of view for treating relativistic deformation-rotation within the above general conception, in taking a bilocal-type model without subsidiary condition as a simple example. The new method rests on the natural possibility of taking \( SL(4, R) = O(3, 3) \) as the inner group \( G^{in} \), which contains as subgroup unconventional inner Lorentz group to be identified as \( L^{in} \). This has the advantage that it leads to integer and half-integer spin states associated with first-order wave equation. This is a rather striking result since so far a first-order relativistic wave equation containing half-integer spin has been attained within our conception (1) only by the model with spinorial coordinates \( ^{8,9} \) (or the equivalent infinite-component wave function \( ^{10} \)) or by adjoining the Dirac \( \gamma \)-matrices corresponding to a finite-dimensional non-unitary representations of \( L^{in} \). \( ^{3,5,8} \)

§2. General outline of bilocal model

In this section we briefly describe some general properties of the bilocal theory to establish notations and to supply the common basis for the theory in the next section and for those in Appendix II.

Since the bilocal model has real 4-vector internal coordinate \( x_{a} \), the inner group \( G^{in} \), is taken to be either \( U(3, 1) \) or \( GL(4, R) \). Both of them are 16-parameter groups but are different. To see this explicitly we denote the canonical-

\( ^{*) \) The next simplest examples are the model with three internal degrees. This is realized by a bilocal model associated with a single subsidiary condition, which may be \( x_{a}x^{a} = r^{2} \), or \( P_{a}x_{a} = 0 \), or still another one (see (ii), (iii), (v), (vii), (viii), and (ix) of Appendix II). For a relativistic model with three internal degrees, see also the footnote on p. 971.
conjugate to $x_\mu$ by $p_\mu$ (which is also a real 4-vector):

$$[x_{\mu}, p_{\nu}] = i g_{\mu\nu}, [x_{\mu}, p^\nu] = i \delta_{\mu\nu},$$

and further introduce the non-hermitian 4-vector*)

$$a_{\mu} = \sqrt{\frac{1}{2}} (x_{\mu} + i p_{\mu}) = \sqrt{\frac{1}{2}} \left( x_{\mu} + g_{\mu\nu} \frac{\partial}{\partial x_\nu} \right),$$

and its hermitian conjugate**)

$$a^*_{\mu} = \sqrt{\frac{1}{2}} (x_{\mu} - i p_{\mu}) = \sqrt{\frac{1}{2}} \left( x_{\mu} - g_{\mu\nu} \frac{\partial}{\partial x_\nu} \right),$$
satisfying

$$[a_{\mu}, a^*_{\nu}] = g_{\mu\nu}. \quad (4)$$

Then the rotation-vibration tensor

$$K_{\mu\nu} = a^*_{\mu} a_{\nu} = \frac{1}{2} \left[ (x_{\mu} p_{\nu} + p_{\mu} x_{\nu}) + i (x_{\mu} p_{\nu} - p_{\mu} x_{\nu}) \right]$$

satisfies the $U(3,1)$ algebra:**)

$$K^*_{\mu\nu} = K_{\nu\mu},$$

$$[K_{\mu\nu}, K_{\rho\sigma}] = g_{\mu\nu} K_{\rho\sigma} - g_{\rho\sigma} K_{\mu\nu},$$

and realizes its symmetric representations, each of which is labeled by the eigenvalue of the $U(3,1)$ invariant

$$K^a_{\mu} = a^*_{\mu} a^a = \frac{1}{2} (x_{\mu} x^a + p_{\mu} p^a) - 2, \quad (7)$$
alone.***)

On the other hand the set of

$$B^a_{\mu} = \frac{1}{2} (x_{\mu} p^a - x^a p_{\mu} - i \delta_{\mu\nu})$$
represents the operator of linear deformation-rotation and they are generators of $GL(4, R)$:**)

$$(B^a_{\mu})^* = B^a_{\mu},$$

$$[B^a_{\mu}, B^b_{\nu}] = i \left( -\delta^a_{\nu} B^b_{\mu} + \delta^a_{\mu} B^b_{\nu} \right). \quad (9)$$

*) Here we are setting the scale factor of length as $l_0 = 1$.

**) The indices $\mu, \nu, \cdot \cdot \cdot$ run over 1, 2, 3, 0; $x_1$ is the relative time coordinate. Our metric is $g_{\mu\nu} = \text{diag}(1,1,1,-1)$; $x_{\mu}, p_{\mu} = -\frac{1}{i} g_{\mu\nu} \frac{\partial}{\partial x_\nu}, a_{\mu}$, and $a^a_{\mu}$ are contravariant vectors. The asterisk signifies the hermitian conjugate. The latin indices $i, j, k, \cdot \cdot \cdot$ run over 1, 2, 3.

***) The SU(3,1) subalgebra is formed by $\bar{K}_{\mu\nu} = K_{\mu\nu} - (1/4) g_{\mu\nu} K^a_{\mu}$ with $K^a_{\mu} = 0$. This $\bar{K}_{\mu\nu}$ is further separated into the symmetric part $H_{\mu\nu} = \bar{K}_{\mu\nu} + \bar{K}_{\nu\mu} = K_{\mu\nu} + K_{\nu\mu} - (1/2) g_{\mu\nu} K^a_{\mu}$, and the antisymmetric part $s_{\mu\nu} = -i(\bar{K}_{\mu\nu} - \bar{K}_{\nu\mu})$. 
Wave Equation with Mass and Spin Spectrum

If we use
\[ B_{\mu \nu} = g_{\mu \lambda} \hat{B}^\lambda = \frac{1}{2} (x_{\mu \lambda} p_\nu - x_{\nu \lambda} p_\mu - i g_{\mu \nu}) \]
(9) is rewritten as
\[ B_{\mu \nu}^* = B_{\nu \mu}, \]
\[ [B_{\mu \nu}, B_{\rho \sigma}] = i (- g_{\mu \rho} B_{\nu \sigma} + g_{\mu \sigma} B_{\nu \rho}). \]

Both \( U(3,1) \) and \( GL(4, R) \) have a Lorentz group \( O_s(3,1) \) as common subgroup, with the generators
\[ s_{\mu \nu} = -i (K_{\mu \nu} - K_{\nu \mu}) = -i (a^*_\mu a_\nu - a^*_\nu a_\mu) \]
(11)
which satisfy
\[ [s_{\mu \nu}, s_{\rho \sigma}] = i (- g_{\mu \rho} s_{\nu \sigma} + g_{\mu \sigma} s_{\nu \rho} + g_{\mu \nu} s_{\rho \sigma} - g_{\rho \sigma} s_{\mu \nu}), \]
and also
\[ s_{\mu \nu} s_{\rho \sigma}^* = 0. \]

Here \( s_{\mu \nu} \) denotes the dual defined by
\[ s_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} s^\rho \sigma. \]
(14)

Our Levi-Civita tensor has \( \varepsilon_{1234} = 1 \); clearly \( \varepsilon^{\mu \nu \rho \sigma} = - \varepsilon_{\mu \nu \rho \sigma}. \) (14) contains
\[ \tilde{s}_{ij} = s^{ij} = \varepsilon_{ijkl} s_{kl}, \]
\[ -\tilde{s}_{jk} = s_{jk} = \frac{1}{2} \varepsilon_{kl} s_{ij} s_{kl}. \]

The \( s_{\mu \nu} \) components are all hermitian and correspond to special infinite-dimensional unitary representations\(^*\) of \( O_s(3,1) \).

Now (4) means that \( a_i \) and \( a_0^* \) are annihilation operators while \( a_i^* \) and \( a_0 \) are creation operators. Therefore we also write \( b_0 = a_0^* \), which gives \([b_0, b_0^*] = 1\). Then any of \( n_i = a_i^* a_i, \ldots, n_0 = b_0^* b_0 = a_0^* a_0 = 1\), has the eigenvalue spectrum \( 0, 1, 2, \ldots \). Accordingly the quantity
\[ \mathcal{N} = \sum_{\mu} K_{\mu \mu} = \sum_{\mu} n_{\mu} + 1 \]
(15)
has the spectrum of all positive integers. (Note that \( \mathcal{N} \) is different from (7) which is \( K_{0 \mu} = \sum_{\mu} n_{\mu} - n_0 = -1 \).)

The space components of \( s_{\mu \nu} \) are

\(^*\) The relation (13) holds for a bilocal model and for the Majorana model, and limits realizable representations of \( O_s(3,1) \) for them. In other cases, e.g. the spinor model\(^1,3,4\) or a trilocal model (see Appendix IV), (13) do not generally apply.
which are clearly $\mathcal{A}N = 0$ operators and have the eigenvalue spectrum

$$s_\lambda = 0, \pm 1, \pm 2, \ldots,$$

while the space-time components

$$s_{\mu\nu} = i(a_\mu b_\nu - a_\nu b_\mu)$$

are $\mathcal{A}N = \pm 2$ operators.

With the usual identification

$$L_{\text{in}} = O(3, 1),$$

$s_{\mu\nu}$ represents the spin-tensor of the system. Thus the covariant spin pseudovector is given by

$$W_\mu = -\bar{s}_{\mu\nu} P^\nu = i\varepsilon_{\mu\nu\rho\sigma} K^{\rho\sigma} P^\nu,$$

that is,

$$W_\mu = P_\mu - \varepsilon_{i\mu} P_{\mu i}, \quad W_\rho = P_\rho s_1,$$

and the spin $J$ is determined from

$$-P_\mu P^\mu J(J + 1) = W_\mu W^\mu.$$

The model then gives integer spin states alone. The wave equation based on (19) is of oscillator or rotator type. Bilocal models in this usual category are collectively discussed in Appendix II.

§3. Model based on new inner Lorentz group, the wave equation

From the original standpoint of the usual bilocal theory, $x_\mu$ must transform under physical homogeneous Lorentz transformation in exactly the same way as $X_\mu$, so that (19) is required. However, if we take into account that the internal coordinates $x_\mu$ are not directly observable ones,\(^a\) the identification (19) is not the unique possibility for the structure. We start from the inner group

$$G_{\text{in}} = GL(4, R),$$

rather than $U(3, 1)$. Then (21) contains several $O(3, 1)$ subgroups other than $O(3, 1)$, and one of those $O(3, 1)$ can be chosen as $L_{\text{in}}$. As is seen presently we adopt such a new choice. This implies that we regard $L_{\text{in}}$ as transformations mixed up of rotation and deformation, and thus we depart from the usual bi-

\(^a\) cf. the comment of L. Rosenfeld in 	extit{Proceedings of the International Conference on Elementary Particles, Kyoto 1965}, p. 152.
Wave Equation with Mass and Spin Spectrum

ocal model. Clearly, this new possibility is due to the fact that the model contains the relative-time degree\(^*\) (connected to relativity) to carry \(GL(4,R)\). We shall see that the new scheme leads to integer as well as half-integer spin states and also to the first order wave equation.

First we decompose\(^{11}\) the generators \(B_\mu^n\) of \(GL(4,R)\) into \(s_\mu\) of Eq.(11), the operator of pure dilatation

\[
A = B_\mu^n = \frac{1}{2} g_{\mu\nu} \{x_\mu, p_\nu\} = x_\mu p_\mu - 2i \frac{i}{2} g_{\mu\nu} (a_\mu^* a_\nu^* - a_\mu a_\nu),
\]

which is \(GL(4,R)\) invariant, and the deformation operator

\[
\Gamma_\mu = \frac{1}{2} (B_\mu + B_\nu) - \frac{1}{4} g_{\mu\nu} B_\rho^\sigma \\
= \frac{1}{2} (x_\mu p_\nu + x_\nu p_\mu) - \frac{1}{4} g_{\mu\nu} x_\rho p^\rho \\
= \frac{i}{2} \left( a_\mu^* a_\nu^* - a_\mu a_\nu \right) - \frac{1}{4} g_{\mu\nu} A.
\]

They are all hermitian, and \(\Gamma_\mu = \Gamma_\nu, \Gamma_\mu^\nu = 0\). The commutation relations are

\[
[s_\mu, \Gamma_\nu] = i \left( -g_{\mu\nu} \Gamma_\sigma + g_{\mu\sigma} \Gamma_\nu - g_{\nu\sigma} \Gamma_\mu + g_{\nu\mu} \Gamma_\sigma \right),
\]

\[
[\Gamma_\mu, \Gamma_\nu] = \frac{i}{4} \left( -g_{\mu\nu} s_\sigma + g_{\mu\sigma} s_\nu - g_{\nu\sigma} s_\mu + g_{\nu\mu} s_\sigma \right),
\]

in addition to (12). Equation (23) is also written as

\[
\Gamma_{ij} = \frac{i}{2} (a_i^* a_j^* - a_i a_j) - \frac{1}{4} \delta_{ij} A, \quad \Gamma_{\alpha\beta} = \frac{i}{2} (a_\alpha^* b_\beta^* - a_\beta^* b_\alpha).
\]

This shows that the space components \(\Gamma_{ij}\) are, like \(s_\mu\), \(\Delta \mathcal{N} = \pm 2\) operators, while the space-time components \(\Gamma_{\alpha\beta}\) are, like \(s_\mu\), \(\Delta \mathcal{N} = 0\) operators. Thus each \(\Gamma_{\alpha\beta}\) takes the eigenvalue spectrum

\[
\Gamma_{\alpha\beta} = 0, \pm \frac{1}{2}, \pm 1, \ldots.
\]

To see this explicitly with respect to \(\Gamma_{\alpha\beta}\) for example, we make the canonical transformation from the quantities \((b_\alpha, a_\beta)\) to \((c_1, c_2)\), by

\[
c_1 = (b_\alpha - id_\alpha) / \sqrt{2}, \quad c_2 = (b_\alpha + id_\alpha) / \sqrt{2}
\]

\(^*\) In fact even a 3-dimensional oscillator model can be made to incorporate an inner Lorentz algebra.\(^3\),\(^12\) (See also Appendix III.) In this model the internal coordinates \(x_i\) together with the conjugate momenta \(p_i\) originally give either \(GL(3,R)\) or \(U(3)\), but the latter is enlarged into \(U(3,1)\), which contains \(O(3,1)\).
and obtain $\Gamma_{\alpha}\equiv (c_1^* c_1 - c_2^* c_2)/2$. Also clearly $[s_{\alpha}, \Gamma_{\alpha}] = 0$, $(k$ not summed).

It is now important to note the isomorphism\(^3\)

$$SL(4, R) \approx SO(3, 3).$$

We can therefore recombine $s_{\mu}$ and $\Gamma_{\alpha}$ into the fifteen generators of $O(3, 3)$ which is now our group $G^{in}$. They are given by

$$S_i^\pm = \frac{1}{2} s_i \pm \Gamma_{\alpha},$$

$$R_{ij} = -\frac{1}{2} \varepsilon_{ijk}s_{kl} + \Gamma_{ij}^{\alpha},$$

where $\Gamma_{ij}^{\alpha} = \Gamma_{ij}^i$ is defined by

$$\Gamma_{ij}^{\alpha} = \Gamma_{ij}^j - \delta_{ji} \Gamma_{ji}^i$$

$$= \frac{1}{2} \left[ x_i p_j + x_j p_i - \delta_{ij} (x_i p_j + x_j p_i - i) \right]$$

$$= \frac{i}{2} \left[ (a_i^* a_j^* - a_j a_i) + \frac{1}{2} \delta_{ij} (a_i a_j - a_j^* a_i^* - b_i^* b_j + b_j^* b_i) \right].$$

Indeed $s_i^\pm$ are generators of the maximum compact subgroup $O'(3) \times O'(3) \approx SU^+(2) \times SU^-(2) \approx O(4)$ of the $O(3, 3)$ group, while $R_{ij}$ represent noncompact generators of $O(3, 3)$. They satisfy the following commutation relations of $O(3, 3)$ (see also Appendix I):

$$[S_i^+, S_j^-] = 0,$$

$$[S_i^+, S_j^+] = i\varepsilon_{ijk}S_k^+, \quad [S_i^-, S_j^-] = i\varepsilon_{ijk}S_k^-, \quad (33)$$

$$[S_i^+, R_{ij}] = i\varepsilon_{ijk}R_{kj}, \quad [S_i^-, R_{ij}] = i\varepsilon_{ijk}R_{ki}, \quad (34)$$

$$[R_{ij}, R_{kl}] = -i (\delta_{ik} \varepsilon_{jlm}S_m^- + \delta_{jl} \varepsilon_{kim}S_m^+).$$

We note that Eq.\,(31) is written conversely as

$$s_{\alpha} = -\varepsilon_{\alpha i j} R_{ij}, \quad \Gamma_{ij} = \frac{1}{2} (R_{ij} + R_{ji} - \delta_{ij} R_{ii}).$$

Evidently $O(3, 3)$ contains $O(3, 1)$ subgroups other than $O(3, 1)$, and we adopt for $L^{in}$

$$L^{in} = O(3, 1)'$$

whose generators $S^\prime_{\alpha}$ are\(^*)

$$S_i^\prime = \frac{1}{2} \varepsilon_{\alpha i j} S_{ij} = S_i^-, \quad S_{\alpha}^\prime = R_{\alpha},$$

\(^*) There are many other $O(3, 1)$ subgroups of $O(3, 3)$, and one of them can equally be chosen as $L^{in}$, but physically this does not make difference from the present choice.
Wave Equation with Mass and Spin Spectrum

\[ S'_{ik} = \frac{1}{2} \left( \epsilon_{hkl} x_{lj} p_{j} - x_{k} p_{l} + x_{l} p_{k} \right), \]
\[ S'_{00} = -\frac{1}{2} \epsilon_{kl} (x_{k} p_{l} + x_{l} p_{k}) \]
\[ + \frac{1}{2} \left[ x_{k} p_{l} + x_{l} p_{k} - \delta_{kl} (x_{l} p_{l} - x_{k} p_{k}) \right]. \]  

(38)

This means that we redefine the physical Lorentz generators by
\[ M_{\mu\nu} = X_{\mu} P_{\nu} + S'_{\mu\nu}. \]

(39)

Note that \( S'_{ik} \) combines rotation and “space-time deformation” while \( S'_{00} \) combines boost and spatial deformation. We see from (33) and (34) that under our \( L^a \) the 4-component quantity \( V_{\lambda} \) defined by
\[ V_{\lambda} = R_{\lambda\nu} V_\nu, \quad V_{\theta} = S'_{\theta} \]
behaves as a 4-vector, namely
\[ [S'_{\mu\nu}, V_{\lambda}] = i (g_{\mu\lambda} V_\nu - g_{\nu\lambda} V_\mu), \]

(41)

and also that \( S'_{\theta} \) is invariant under \( L^a \).*) Under the physical Lorentz transformation generated by (39), the internal coordinates \( x_\mu \) transform according to the following unconventional linear transformation. First under space rotation,
\[ [S'_{ik}, x_j] = -\frac{i}{2} (\epsilon_{hkl} x_j + \delta_{ik} x_l), \]
\[ [S'_{ik}, x_k] = -\frac{i}{2} x_j, \]

(42)

so that \( x_k \) and \( x_\theta \) are mixed. Under a boost,
\[ [S'_{ik}, x_j] = \frac{i}{2} (\epsilon_{hkl} x_j + \delta_{ik} x_l - \delta_{ij} x_k - \delta_{lk} x_i), \]
\[ [S'_{ik}, x_k] = \frac{i}{2} (\epsilon_{hkl} x_i - \delta_{ik} x_l). \]

(43)

Now \( S'_{\theta} \) consists of two independent terms \( s/2 \) and \( \Gamma_{ia} \), each of which has the eigenvalue spectrum \( 0, \pm \frac{1}{2}, \pm 1, \ldots \), so that \( S'_{\theta} \) also takes the same spectrum:
\[ S'_{\theta} = 0, \pm \frac{1}{2}, \pm 1, \ldots, \]

(44)

and the same is true for \( S'_{\theta} \):

*) The 4-component quantity \( V'_{\lambda} = (R_{\lambda\nu} - S'_{\lambda\nu}) \) gives another vector, obeying \([V'_{\lambda}, V'_{\nu}] = -i S'_{\lambda\nu}, [V_{\nu}, V'_{\lambda}] = -i g_{\nu\lambda} S'_{\theta} \). Thus the sixteen \( G(4, R) \) quantities are grouped into a skew-tensor \( S'_{\mu\nu} \), two polar vectors \( V_{\lambda} \) and \( V_{\theta} \), a scalar \( S'_{\theta} \), and a pseudoscalar \( A \), with respect to the subgroup \( O(3,1)' \) and the parity defined by (49) below.
\[ S'_s = S^*_s = 0, \pm \frac{1}{2}, \pm 1, \ldots \]  

(45)

From this it is clear that the spin \( J \) in this model takes integer and half-integer eigenvalues \( 0, \frac{1}{2}, 1, \ldots \).

From (33) and (34) we also see that \( V_\lambda \) of (40) satisfies

\[ [V_\lambda, V_\mu] = -i S'_\mu, \]  

(46)

whence \( S'_\mu \) and \( V_\lambda = S^*_\lambda \) form together the algebra of an \( O(3, 2) \) subgroup of \( O(3, 3) \), and accordingly we can assume the first order wave equation

\[ (V_\lambda P^\lambda \pm i) \psi = 0. \]  

(47)

This is quite analogous to the case of Majorana wave equation. In (47) \( \psi \) is a single-component function \( \psi(X_\mu, x_\mu) \), \( P^\lambda = -i \frac{\partial}{\partial x^\lambda} \), and each \( V_\mu \) is a linear combination of the \( SL(4, R) \) generators \( x_\mu p^\lambda = -i \frac{\partial}{\partial x^\mu} \), as seen from (40):

\[ V_\lambda = -\frac{1}{2} \epsilon_{\mu_1 \mu_2} (x_\mu p_{\mu_1} + x_{\mu_2} p^\mu) + \frac{1}{2} (x_\mu p_{\mu_1} + x_{\mu_2} p_\mu) \]

\[ \qquad - \frac{1}{2} \delta_{\mu_1 \mu_2} (x_{\mu_1} p_{\mu_2} - x_{\mu_2} p_{\mu_1}) \]  

(48)

Thus our wave equation (47) is of first order in \( \frac{\partial}{\partial x^\mu} \) as well as in \( \frac{\partial}{\partial x^\mu} \).

Here we consider space inversion, under which \( X_\lambda \rightarrow -X_\lambda, X_\mu \rightarrow X_\mu \). With respect to inversion we have to assign new transformation properties to our internal variables so as to be consistent with their unconventional Lorentz transformation properties given above. We consider the following peculiar reciprocity transformation:

\[ x_\lambda \rightarrow p_\lambda, \quad x_0 \rightarrow -p_0, \]

\[ p_\lambda \rightarrow -x_\lambda, \quad p_0 \rightarrow x_0, \]  

(49)

whence \( a_\lambda \rightarrow -ia_\lambda, a_0 \rightarrow ia_0 \). This is essentially the product of the normal internal reciprocity \( x_\mu \rightarrow p_\mu, p_\mu \rightarrow -x_\mu \) and the conventional space inversion. Clearly this is a canonical transformation leaving the commutation relation (3) invariant.*

It is easily seen that with (49) the \( U(3, 1) \) and \( GL(4, R) \) quantities transform as

\[ K_{ij} \rightarrow K_{ij}, \quad K_{0\mu} \rightarrow K_{0\mu}, \]

\[ K_{\mu0} \rightarrow -K_{\mu0}, \quad K_{ij} \rightarrow -K_{ij}, \]

* Evidently (49) is outside the \( GL(4, R) \) group, which consists of linear mixings among \( x_\mu \) alone. Thus \( A \) is \( GL(4, R) \) invariant but changes sign under (49), as indicated below.
Wave Equation with Mass and Spin Spectrum

\[ B_{ij} \rightarrow -B_{ji}, \quad B_{0\alpha} \rightarrow -B_{\alpha 0}, \quad B_{i0} \rightarrow B_{0i}, \quad B_{ij} \rightarrow B_{ji}, \]

whence

\[ s_i \rightarrow s_i, \quad s_{i0} \rightarrow -s_{0i}, \quad A \rightarrow -A, \quad \Gamma_{ij} \rightarrow -\Gamma_{ji}, \quad \Gamma_{i0} \rightarrow \Gamma_{0i}, \quad S_i^k \rightarrow S_i^k, \quad R_{ij} \rightarrow -R_{ij}, \]

containing

\[ S'_i \rightarrow S'_i, \quad S'_{i0} \rightarrow -S'_{0i}, \quad V_i \rightarrow -V_i, \quad V_{0} \rightarrow V_0. \quad \text{(50)} \]

Therefore, if we assume that the internal variables transform as (49) under the physical space-inversion, this is precisely consistent with their proper Lorentz transformation properties, and also the wave equation (47) is invariant under space inversion, owing to (50). Note that with the prescription (49) we are again departing from the conventional bilocal model.

Now the wave function \( \psi \) is equivalently regarded as an infinite-component wave function \( \psi(n_1, n_2, n_3, n_4, X_\mu) \), where each \( n_i \) and \( n_4 \) run over all non-negative integers, and for them \( a_i^* \) and \( b_0^* \) are respective raising operators and \( a_i \) and \( b_0 \) the lowering operators, e.g.\(^*\)

\[ a_i^\psi(n_1, n_2, n_3, n_4, X_\mu) = \sqrt{n_1} \psi(n_1 - 1, n_2, n_3, n_4, X_\mu), \]
\[ b_0^\psi(n_1, n_2, n_3, n_4, X_\mu) = \sqrt{n_4 + 1} \psi(n_1, n_2, n_3 + 1, X_\mu). \]

In this representation each \( V_\lambda \) is an infinite-dimensional hermitian matrix, to be read off explicitly from

\[ V_\lambda = \frac{i}{2} \left[ \varepsilon_{ikj} (b_0 a_i - b_i a_0^*) - (a_0 a_i - a_i a_0^*) \right] + \frac{i}{4} \delta_{ik} (a_0 a_i - a_i a_0^* - b_0^* + b_0^*), \]
\[ V_0 = \frac{i}{2} (-a_0^* a_i + a_i^* a_0 + a_0 b_0^* - a_i^* b_0). \]

The Lagrangian density is written as

\[ L(X) = i \sum_{n, \alpha} \psi^\ast(n, X) \langle n | V_\lambda | n' \rangle \frac{\partial \psi(n', X)}{\partial X_\lambda} \]
\[ - \kappa \sum_n \psi^\ast(n, X) \psi(n, X), \]

where \( n \) stands for \( n_1, n_2, n_3, n_4 \). For any two solutions \( \psi \) and \( \varphi \) of the wave equation (47) the invariant inner product can be defined by

\[ (\psi, \varphi) = \int d^4 X \sum_{n, \alpha} \psi^\ast(n, X) \langle n | V_\psi | n' \rangle \varphi(n', X), \]

\(^*\) More adequately this should be written like \( (a_i \psi)(n_1, n_2, n_3, n_4, X_\mu) = \sqrt{n_1 + 1} \psi(n_1 + 1, n_2, n_3, n_4, X_\mu), \) \( (b_0^* \psi)(n_1, n_2, n_3, n_4, X_\mu) = \sqrt{n_4} \psi(n_1, n_2, n_3 + 1, X_\mu). \)
under the assumption of convergence of the right-hand side expression. The current density

$$j_n(X) = \sum_{n'} \psi^*(n, X) \langle n | V_n | n' \rangle \psi(n', X)$$  \hspace{1cm} (51)$$
satisfies $\partial j_n(X)/\partial X_n = 0$. The norm $(\psi, \psi) = \int j_n(X) d^3X$ is not positive-definite. Thus the infinite-dimensional representation is in fact "pseudo-unitary". Still it is the advantage of the first order wave equation that the electromagnetic interaction is simply given by the coupling of $A_n(X)$ to $j_n(X)$ of (51).

If we consider in the rest frame, the wave equation (47) has the symmetry $O(3, 1)^{''}$, which is another Lorentz subgroup (of $G^{\alpha\beta}$) generated by $S_{\alpha\beta}$ with

$$S_i' = S_i = S_i^-, S_{ii} = R_{ii}.$$  \hspace{1cm} (52)$$
Then the complete set of internal quantum numbers is given by those of the maximum compact subgroup $O'(3) \times O'(3)$:

$$S_i^+ S_i^+, S_i^-, S_i S_{i-}, S_{i}^-,$$  \hspace{1cm} (53)$$
in which the latter two represent the spin quantum numbers (magnitude and third component of spin). The mass spectrum is given by

$$m = \pm \varepsilon / V_0 = \pm \varepsilon / S_i^+,$$  \hspace{1cm} (54)$$
where $S_i^+$ runs as

$$S_i^+ = -t, -t+1, \ldots, t$$
with

$$S_i^+ S_i^+ = t(t+1), \hspace{1cm} t=0, 1/2, 1, \ldots.$$  \hspace{1cm} (55)$$
Thus in general the wave equation (47) gives the mass spectrum

$$m = \pm \varepsilon / \nu, \hspace{1cm} \nu = 1, 2, 3, \ldots.$$  \hspace{1cm} (55)$$
The appearance of double sign corresponds to particle-antiparticles. Each mass level is infinitely degenerate with respect to different spin values. On the other hand the wave equation (47) also allows space-like solutions. Thus this wave equation is not immediately a physically reasonable one.

The wave equation (47) can be generalized, e.g. by adding a term $S_i^+$, such that

$$(V_\lambda P^\lambda + \varepsilon + \kappa S_i^+) \psi = 0,$$  \hspace{1cm} (56)$$
which is still invariant under our Lorentz transformation (39) and space inversion (49). The rest-symmetry is reduced from $O(3, 1)^{''}$ to its rotation subgroup $O(3)^-$ generated by $S_i^-$, and the mass formula is altered to $m^2 = 4\varepsilon^2 / \nu^2 - \kappa^2$.\vspace{1cm}
The purpose of this paper was rather to present a new viewpoint for treating the bilocal-type model and more generally a relativistic deformable model. Indeed the essential point in the present scheme is based on the existence of the linear group $SL(4, \mathbb{R}) = O(3, 3)$ of deformation-rotation, so that the method is readily adapted, e.g. to a multi-local model which has multiple relative vectors $x^\nu_i$ corresponding to a full 3-dimensional structure\(^{11,14,15}\) (cf. Appendix IV). Naturally for this case the mass formula is different from (55), since the constant $\kappa$ in the wave equation (47) is to be replaced there by an operator representing mutual movement among the relative vectors $x^\nu_i$. This will lead to a more realistic model for a unified description of hadrons. This point will be treated subsequently.

**Appendix**

I. *Unified expression of $O(3, 3)$ algebra*

The $O(3, 3)$ generators of (30) and (31) are incorporated into a 6-dimensional antisymmetric tensor $S_{AB}$, $(A, B = 1, 2, 3, 4, 5, 6)$, as follows:

\[
S_{ij} = \varepsilon_{ijb} S^+_b, \quad S_{i+3,j+3} = -\varepsilon_{ijb} S^-_b, \quad S_{i,j+3} = -S_{j+3,i} = R_{ij},
\]

namely

\[
S_{AB} = \begin{pmatrix}
0 & S^+_1 & S^+_2 & R_{11} & R_{12} & R_{13} \\
0 & S^+_2 & S^+_3 & R_{21} & R_{22} & R_{23} \\
0 & R_{31} & R_{32} & R_{33} \\
0 & -S^-_1 & S^-_2 \\
0 & S^-_2 & S^-_3 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then all relations of (33) and (34) are incorporated into the single equation

\[
[S_{AB}, S_{CD}] = i(-g_{CB} S_{AD} + g_{AD} S_{CB} + g_{DB} S_{AC} - g_{AC} S_{DB}),
\]

where the six-dimensional metric tensor is

\[
g_{AB} = \text{diag}(1, 1, 1, -1, -1, -1).
\]

II. *Bilocal wave equations based on the inner Lorentz group $O(3, 1)$*

Various bilocal wave equations with the conventional inner Lorentz group (19) have been treated in the literature.\(^{16}\) In the following, however, we shall consider some wave equations belonging to this usual category successively and quickly, since our argument and the wave equations themselves are not necessarily the same as those given by other authors and will be helpful to the clari-
fication of the problem,

It is intuitively expected that a simple wave equation based on the bilocal model will lead to a mass spectrum of a harmonic oscillator consistent with relativity, but in fact this is not realized with a simple wave equation alone, due to the general circumstance that a unitary irreducible representation of the homogeneous Lorentz group corresponds to a descending mass spectrum. This fact is not surprising because a unitary irreducible representation corresponds to the case where internal movement is contained just kinematically. To admit a "dynamical" internal movement we should need reducible representations and then we can obtain physically reasonable mass spectrum.

We consider several models in succession.

(i) The wave equation for the 2-point system bound by "relativistic Hooke potential" is transformed into the form

\[(P_\mu P^\mu + \kappa^2 \alpha_\mu \alpha^\mu + \kappa_0^2)\psi = 0.\]  \hspace{1cm} (A.1)

This is a simple wave equation of oscillator type and has the full non-compact \(U(3,1)\) symmetry whence it gives infinitely degenerate mass spectrum. Also (A.1) admits solutions with space-like \(P_\mu\). This is expected from the fact that it is derived from an action-at-a-distance Hooke potential working inside the particle.

(ii) If we require to obtain the covariant transcription of the nonrelativistic model, we should set the subsidiary condition

\[P_\mu x_\mu \psi = \frac{1}{\sqrt{2}} P_\mu (a_\mu + a_\mu^*) \psi = 0.\]  \hspace{1cm} (A.2)

One must then modify the wave equation (A.1) into

\[(P_\mu P^\mu + \kappa^2 O^{\mu\nu} a_\mu a_\nu + \kappa_0^2)\psi = 0,\]  \hspace{1cm} (A.3)

so as to be compatible with (A.2). Here \(O^{\mu\nu} = g^{\mu\nu} - P_\mu P_\nu / (P_\rho P^\rho)\) is the projection operator with the properties

\[O^{\mu\nu} P_\nu = 0, \quad O_\mu O_\nu = O_\mu.\]  \hspace{1cm} (A.4)

Thus (A.3) is really the fourth order wave equation

\[
\left[ (P_\mu P^\mu)^2 + \frac{\kappa^2}{2} \left( P_\mu P^\mu (x_\alpha \alpha^2 + p_\beta \beta^2) - (P_\rho P^\rho)^2 \right) - (P_\rho P^\rho)^2 \right] \psi = 0.
\]  \hspace{1cm} (A.5)

The internal symmetry is thus reduced from \(U(3,1)\) to \(U_\rho(3)\) generated by\(^8\)

\[K_\mu^\rho = O_\rho O^\nu K_{\rho\nu}.\]  \hspace{1cm} (A.6)

\(^8\) For simplicity we always assume a scalar wave function \(\psi(X_\mu, x_\mu)\). The case of spinor \(\psi_\mu(X_\mu, x_\mu)\) can be treated analogously.\(^5\),\(^6\),\(^7\)
This $U_p(3)$ characterizes an isotropic harmonic oscillator relativistically and contains the covariant-spin subgroup, since Eq. (20) is rewritten as

$$W_\mu = i\varepsilon_{\mu\nu\rho\sigma}K^{\nu\rho}P^\sigma.$$ 

The eigenvalues $N'$ of $K^{\mu\nu} = \omega a^\ast_\mu a_\mu$, which is $U_p(3)$ invariant, are $N' = 0, 1, 2, \cdots$, and then the $N'$-shell contains $(N' + 1)(N' + 2)/2$ states for which the spin $J$ varies within

$$J = N' - 2\omega \geq 0; \quad \omega = 0, 1, 2, \cdots.$$  

(A·7)

The mass spectrum is given by the formula

$$m^2 = \kappa^2 (J + 2\omega) + \kappa_0^2.$$  

(A·8)

In the case $\kappa^2 < \kappa_0^2$, (A·8) becomes the non-relativistic equidistant spectrum

$$m = \pm \left[ \kappa_0 + \frac{\kappa^2}{2\kappa_0} (J + 2\omega) \right].$$  

(A·9)

(The double sign corresponds to particle-antiparticle.)

Now the subsidiary condition (A·2) can be replaced by

$$P^\mu p_\mu \psi = \frac{i}{\sqrt{2}} P^\mu (a^\ast_\mu - a_\mu) \psi = 0,$$  

(A·10)

which is again compatible with the same wave equation (A·3). Thus (A·3) associated with (A·10) gives the same result as above. This fact was easily expected because our model is invariant under the “internal reciprocity”

namely $a_\mu \rightarrow -ia_\mu$. Indeed under (A·11), $K_{\mu\nu} = \text{invariant}$. [For $GL(4, \mathbb{R})$ quantities one has

$$B_{\mu\nu} \rightarrow -B_{\mu\nu}, \quad A \rightarrow -A, \quad r_{\mu\nu} \rightarrow r_{\mu\nu},$$

$$S_i \rightarrow -S_i, \quad R_{ij} \rightarrow -R_{ij},$$

under (A·11).]

Under the subsidiary condition (A·10), the plane-wave solution of (A·3) (with a definite 4-momentum $P_\mu$ with $-P_\mu P^\mu = \kappa_0^2$) in the internal ground state is

$$\psi_0 = \exp \left[ i P^\mu x_\mu - \frac{1}{2} \omega_{\mu\nu} x_\mu x_\nu \right].$$  

(A·12)

*) This is for the case of time-like $P_\mu$. For space-like $P_\mu$, $K_{\mu\nu}^\ast$ takes negative integer eigenvalues also.

**) Besides (A·8) the wave equation allows massless states satisfying $P^\mu x_\mu \psi = P^\mu p_\mu \psi = 0$. 

Downloaded from https://academic.oup.com/ptp/article-abstract/38/4/966/1879247 by guest on 11 April 2019
(iii) We now replace the subsidiary condition (A·2) or (A·10) by

\[ P^\mu a_\mu \psi = 0, \quad (A·13) \]

namely \((P_\alpha a_\alpha - P\beta a_\beta)\psi = 0\). This replacement is analogous to the case of Q.E.D. in which one goes over from the original Lorentz condition \(\partial^\mu A_\mu \psi = 0\) to the condition \(\partial^\mu A_\mu(+) \psi = 0\), picking the positive-frequency part \(A_\mu(+)\) of \(A_\mu\). An advantage of taking (A·13) is that this is compatible with the second order wave equation (A·1). For the system (A·1) with (A·13), the symmetry is still \(U_P(3)\) and the mass spectrum is the same as in (ii). The further advantage of this model is that it can accommodate the internal unitary symmetry \(SU(3)\) in a consistent way simply by going over to the model containing three relative vectors \(x_r^\mu (r=1, 2, 3)\).11,13

(iv) The infinite degeneracy of mass can be lifted, without recourse to subsidiary condition, by assuming a wave equation which reduces the \(U(3, 1)\) group to its maximum compact subgroup

\[ U_P(3) \times U_P(1), \]

where \(U_P(1)\) is generated by

\[ K = -P^\mu P_\mu K / (P P^\nu). \quad (A·14) \]

The simple wave equation of this kind is

\[ (\Theta_{\mu\nu} P^\mu P^\nu + \epsilon^2) \psi = 0, \quad (A·15) \]

where*)

\[ \Theta_{\mu\nu} = g_{\mu\nu} a_\mu^* a_\nu - (1+\alpha) a_\mu^* a_\nu, \quad (A·16) \]

and \(\alpha\) is a positive constant. This gives the mass spectrum

\[ m^2 = \epsilon^2 / (J+2\omega+\alpha k), \quad (A·17) \]

where \(k\) is the eigenvalue of \(K\):

\[ k = 1, 2, 3, \ldots. \quad (A·18)** \]

Eigenstates are labeled by the set of quantum numbers \((J, J_3, \omega, k)\). \((\omega\) is the quantum number of (A·7).) The spectrum is descending as in the case of the Majorana model.

(v) The spectrum (A·17) is inverted by going over to the fourth order

*) \(\Theta_{\mu\nu}\) is essentially a symmetric tensor since only \(\frac{1}{2} (\Theta_{\mu\nu} + \Theta_{\nu\mu}) = g_{\mu\nu} K - \frac{1}{2}(1+\alpha)(K_{\mu\nu} + K_{\nu\mu})\) is relevant in (A·15). Recently wave equations of the type (A·15), by the use of a symmetric tensor operator \(\Theta_{\mu\nu}\), have been brought into discussions.13,19,17

**) (A·18) are eigenvalues of \(K\) for time-like \(P_\mu\). For space-like \(P_\mu\), \(K\) takes eigenvalues 0, -1, -2, ...
Wave Equation with Mass and Spin Spectrum

The plane-wave solution with the 4-momentum $P_\mu$ in the internal ground state ($J=\omega=0$, $k=1$) is given by

$$\psi_0 = \exp\left(iP_\mu x_\mu - \frac{1}{2} R_{\mu\nu} x_\mu x_\nu\right),$$

with $-P_\mu P^\mu = k_0^2 + \alpha k$, and $R_{\mu\nu} = g_{\mu\nu} - 2 P_\mu P^\nu/(P_\sigma P^\sigma)$. One can associate the subsidiary condition

$$(a_\mu^* a_\nu + \nu)\psi = 0,$$

where $\nu$ is a fixed integer $\nu \equiv 0$. This picks a definite irreducible representation of $U(3,1)$ according to $K^2 = -\nu$, i.e. $k = J + 2\omega + \nu$. Note that the subsidiary condition (A·21) is of a type different from (A·2) or (A·13) in that it is an equation concerning internal variables alone. It requires that the internal spatial vibration should balance with the vibration in relative time.

(vi) We now regard $\psi$ as a two-component quantity $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, where each $\psi_\alpha$ is Lorentz scalar, just as in the case of isospin doublet, and assume the second order wave equation:

$$\{P_\mu P^\mu + \kappa (\tau_3 x_\mu - \tau_2 P_\mu) P^\mu + \kappa_0^2\} \psi = 0,$$

where $\tau_i$ are Pauli matrices. This equation is invariant under $U_\gamma(3)$, and gives the mass formula

$$m^2 = k_0^2 + \kappa_0^2 \pm \kappa (\kappa_0^2 \xi^2 + 2k_0^2 \xi)^{1/2},$$

where $\xi = k - \frac{1}{2}(1 + \tau_3)$. [ $k$ is the quantum number of (A·18).] Each mass state is infinitely degenerate. If in particular $\kappa_0 = 0$, one has the equidistant mass formula

$$m^2_1 = 2\xi(k - 1), \quad m^2_0 = 2\xi k,$$

where the subscripts $p$ and $n$ refer to the states with $\tau_3 = 1$ and $-1$, respectively; and there are also massless states.

(vii) Thus far we have considered wave equations corresponding to oscillator type models. We can also consider bilocal wave equations of rotator type. A simple example is**

---

*) Similar equation is considered by Yukawa, Markov and by Nambu.

**) This equation was considered by Ginzburg.
This contains no spin-orbit coupling and has the symmetry group

\[ \mathcal{D}_{\text{orb}} \times O(3, 1) \times O(2, 1), \]

and the mass spectrum is again infinitely degenerate. In (A·25), \( O(2, 1) \approx SL(2, R) = SU(1, 1) \) is the dilatation-vibration group generated by

\[
L_1 = \frac{1}{2} A = \frac{1}{4} \{ x^\mu, p^\mu \}, \\
L_3 = \frac{1}{2} K^\mu_\mu + 1 = \frac{1}{4} (x^\mu x^\mu + p^\mu p^\mu), \\
L_2 = \frac{1}{4} (x^\mu p^\mu - p^\mu x^\mu),
\]
as pointed out by Yukawa. The \( O(2, 1) \) Casimir operator

\[ L_i L^i = -L_3^2 + L_3 + L_4 \]
is linked to that of \( O(3, 1) \) by

\[
\frac{1}{2} s_{\mu\nu} s^{\mu\nu} = s_s s_s - s_{ss} s_s = 4L_i L^i, \\
= (a^* a^n)^2 + 2a^* a^n - a^* a^n a_s a_s.
\]

Since also \( s_{\mu\nu} s^{\mu\nu} = 0 \) in the bilocal model, an irreducible representation of \( O(3, 1) \) is specified by the eigenvalue of the single quantity (A·26):

\[
\frac{1}{2} s_{\mu\nu} s^{\mu\nu} = j^2 - 1; \ j = 1, 2, 3, \ldots, \quad (A·27)
\]
j being the minimum spin. At the same time this corresponds to the irreducible representation \( D(j^{21}_{11}) \) of \( O(2, 1) \) according to Bargmann’s notation. Otherwise the equation allows continuous spectrum.

In the present case one can consistently impose the subsidiary condition, either

\[ x_\mu x^\mu = 2(L_2 + L_3) = r^2, \ r > 0 \quad (A·28) \]

meaning space-like \( x_\mu \), or (A·21), i.e. \( L_3 = -(\nu/2) + 1 \), and then the model has only three internal degrees of freedom. If in particular \( \kappa = 0 \) (whence \( m = \pm \kappa \)), one can impose (A·2), in addition to (A·28), and time-like \( P_\mu \) is confirmed. This is just the original bilocal model of Yukawa, and has only two internal degrees of freedom.

(viii) Another wave equation of the rotator type is the fourth-order one: The similar set of equations was considered by Wigner from a different physical viewpoint.

\[ (*) \quad (* * ) \]
\[ \{(P_\mu P^\nu)^2 + \kappa_0^2 P_\mu P^\nu - \kappa_1^2 W_\mu W^\nu\} \psi = 0. \]  
(A·29)

In contrast to (A·24), this equation has spin-orbit coupling contained in
\[ W_\mu W^\nu = -\frac{1}{2} \left( -P_\mu P_\nu \sigma_\alpha \sigma^\alpha + P^\rho P_\rho \{\sigma_\alpha, \sigma^\alpha\} \right), \]
and the \( L^{\text{orb}} \times L^\text{in} \) symmetry in the example (vii) is reduced here to the physical \( L \) itself, and one has the relativistic rotator mass formula
\[ m^2 = \kappa_0^2 + \kappa_1^2 J(J+1). \]  
(A·30)

The wave equation (A·29) is consistent with (A·28) and (A·2). By imposing these two subsidiary conditions the model has only two internal degrees, and the corresponding internal quantum numbers are again \( J \) and \( J_a \). Thus the set of equations (A·29), (A·28) and (A·2) represents the pure relativistic rotator. It has the spin spectrum\(^*) J=0, 1, 2, \cdots, \) and the mass spectrum \( m = \pm \sqrt{\kappa_0^2 + \kappa_1^2 J(J+1)} \). Otherwise the model allows no imaginary-mass solutions (i.e. space-like \( P_\mu \)) but allows massless solutions.

(ix) One can impose the subsidiary condition
\[ x_\mu x^\mu \psi = 0, \]  
(A·31)
in place of (A·28), on the bilocal wave function \( \psi \). (See reference 3.) Then the relative vector is restricted on its light cone and the system has again three internal degrees of freedom. This case is still regarded as a particular rotator-type model, which has some analogy to the spinor model.\(^**) The largest inner group consistent with (A·31) is the internal conformal group\(^***) \( C \), which is isomorphic to \( O(4,2) \):
\[ C = O(4,2) \supset SU(2,2) \supset O_4(3,1). \]

The generators of \( C \) consist of, besides \( \sigma_\mu \), the coordinate vector \( x_\mu \), the dilatation operator \( A = A - i = x_\mu p^\mu - 3i \), and the generator of non-linear transformations
\[ \beta^\prime_\mu = -x_\mu p_\rho p^\rho + 2x_\mu p_\rho p^\rho + 6i p_\mu. \]

III. Trick to obtain \( U(3,1) \) algebra by 3-dimensional oscillator variables

The bilocal model carries the \( U(3,1) \) algebra by \( K_\mu = a_\mu^* a_\mu \). We can suppress one degree of freedom by introducing the \( U(3,1) \) invariant condition (A·21). This condition is fulfilled by writing (we are now assuming \( \nu \geq 0 \)),
\[ a_0^* = (N+\nu)^{1/2} e^{-i\omega}, \quad a_\mu = e^{i\omega}(N+\nu)^{1/2}, \]
with \( N = a_\mu^* a_\mu \).

\(^*\) The realization of (A·29) which allows half-integer spin states is given in reference 23).

\(^**\) This is physically quite different from the usual conformal group.
where, however, \( N \) and \( w \) do not commute but obey\(^{30}\)

\[
[w, N] = i
\]

in agreement with \([(a^*_\alpha, a_\beta)] = 1\). Nevertheless, we disregard (A·32) and suppress \( w \), to write

\[
a^*_\alpha = a_\alpha = (N + \nu)^{1/2}.
\]

Inserting this into \( K_{\mu\nu} = a^*_\mu a_\nu \), one gets

\[
K_{ij} = a^*_i a_j, \quad K_{i\nu} = N + \nu, \quad K_{i\nu} = a^*_i (N + \nu)^{1/2}, \quad K_{ij} = (N + \nu)^{1/2} a_j,
\]

with complete elimination of \( a_\nu \) and \( a^*_\nu \). Indeed one can check that (A·33) preserve the reality property and commutation relations (6) of \( U(3, 1) \) generators, with the aid of the formula

\[
a_\nu f(N) = f(N+1) a_\nu, \quad f(N) a^*_\nu = a^*_\nu f(N+1).
\]

The result (A·33) agrees with that of Nambu\(^3\) and Fronsdal et al.\(^{12})^{*}\)

IV. Trilocal model and \( GL(4, \mathbb{C}) \)

The trilocal wave function \( \psi(X^1, X^2, X^3) \) is represented equivalently as \( \psi(X, x, y) \) with two relative coordinate vectors, \( x_\mu \) and \( y_\mu \). In this case one has two sets of vibrational modes and a full 3-dimensional rotation.\(^{25}\) The model is of interest also from its analogy to the quark (or Sakata) model. We only want to note here that this model is a natural one to carry the inner group

\[
G_{\text{in}} = GL(4, \mathbb{C}).
\]

This is clear because the internal coordinates are regarded as a complex vector

\[
z_\mu = \frac{1}{\sqrt{2}}(x_\mu + iy_\mu),
\]

and its linear transformation is exactly (A·34).

The momenta canonical-conjugate to \( z_\mu \) is given by

\[
\pi_\mu = \frac{1}{\sqrt{2}}(p_\mu - i q_\mu),
\]

where \( p_\mu \) and \( q_\mu \) are canonical conjugates to \( x_\mu \) and \( y_\mu \), respectively, with \([x_\mu, p_\nu] = [y_\mu, q_\nu] = i g_{\mu\nu}\). In fact one can check

\[
[z_\mu, \pi_\nu] = [z_\mu^*, \pi_\nu^*] = i g_{\mu\nu},
\]

\[
[z_\mu, \pi^*_\nu] = [z^*_\mu, \pi_\nu] = 0.
\]

Then \( GL(4, \mathbb{C}) \) generators are given by \( A_\mu^* = z_\mu \pi^*, \) satisfying

*) See also R.C. Hwa and J. Nuyts, Phys. Rev. 145 (1966), 1188.
Wave Equation with Mass and Spin Spectrum

\[ [A^u, A^v] = \imath(-\delta_{uv} A^u + \delta_{vw} A^v). \]  
(A·37)

Evidently \( A^u \) and \((A^u)^*\), or equivalently \( x_\mu p^\mu + y_\mu q^\mu \) and \( x_\mu q^\mu - y_\mu p^\mu \), are \( GL(4, C) \) invariants.

Now \( GL(4, C) \) contains both \( GL(4, R) \) and \( U(3, 1) \) as subgroups. The former is generated by

\[ B^u = A^u + (A^u)^* = z^u x^u + \pi^u z^u = x_\mu p^\mu + y_\mu q^\mu, \]  
(A·38)

which satisfy (9). From these \( B^u \) components one can derive \( O(3, 3) \) algebra as before. On the other hand the group of linear transformations which leave

\[ z^u z^u = \frac{1}{2} g_{u\nu} (x_\mu x_\nu + y_\mu y_\nu) \text{invariant} \]  
(A·39)

is clearly a \( U(3, 1) \) group\(^26\) and is shown to be generated by

\[ F^u_{\mu\nu} = i(A^u_{\mu\nu} - A^u_{\nu\mu}) = i(z^u_{\mu\nu} - \pi^u_{\mu\nu}). \]  
(A·40)

In fact from (A·37) we get

\[ F^u_{\mu\nu} = F_{\mu\nu}, \]

\[ [F^u_{\mu\nu}, F^v_{\rho\sigma}] = g_{u\rho} F^v_{\mu\sigma} - g_{u\sigma} F^v_{\mu\rho}. \]

This group, which we denote by \( U_F(3, 1) \), differs from the former \( U(3, 1) \) group generated by \( K_{\mu\nu} \), because \( U_F(3, 1) \) is a subgroup of \( GL(4, C) \) and is a completely geometrical transformation, being of first order in momenta. Besides (A·39), \( F^u_{\mu\nu} + 4 = x_\mu q^\mu - y_\mu p^\mu \), and \( x_\mu p^\mu + y_\mu q^\mu \) are also \( U_F(3, 1) \) invariants. The common subgroup of \( SL(4, R) \) and \( U_F(3, 1) \) is \( O(3, 1) \) whose generators are \( s_{\mu\nu} = B_{\mu\nu} - B_{\nu\mu} = -i(F_{\mu\nu} - F_{\nu\mu}). \)

References

3) Y. Nambu, Infinite Component Wave Equations, preprint.
3a) A. O. Barut and H. Kleinert, preprint.
4) E. Majorana, Nuovo Cim. 9 (1932), 335.
7) H. Yukawa, Phys. Rev. 91 (1953), 415, 416.
13) cf. S. Helgason, Differential Geometry and Symmetry Spaces.
16) We add the following papers:
   M. Markov, Nuovo Cim. Suppl. 3 (1956), 760;
17) I.M. Gel'fand, R.A. Minlos and Z.Y. Shapiro, *Representations of the rotation and Lorentz
groups and their applications* (English translation, 1963).
18) T. Takabayasi, Phys. Rev. 139 (1965), B1381.
21) V. Bargmann, Ann. of Math. 48 (1947), 568.
25) Trilocal model has been taken up occasionally: H.S. Green, *Proceedings of the International
32 (1964), 368.
26) The $U(3,1)$ group was considered from different points of view also in B. Kursunoglu, Phys. Rev.