Discussion

Critical Shear Stress of Plates Above the Proportional Limit[1]

E. Z. Stowell.[2] In this paper, among the conclusions the author states the following:

"Confirmation of the secant modulus in shear instability as well as that previously obtained in compression suggests that a re-examination of fundamental plate-instability theory for extension into the inelastic range is desirable."

Such a re-examination of fundamental plate-instability theory beyond the elastic range has been undertaken by the NACA on the basis of modern plasticity considerations. By employing the stress-strain relations of Ilyushin a basic differential equation of equilibrium for a buckled plate in the plastic range has been derived. Solutions of this equation for the case of pure axial compression show that long columns, which bend without appreciable twisting, require the tangent modulus, and that long flanges, which twist without appreciable bending, require the secant modulus. Structures which both bend and twist when they buckle require a modulus which is a combination of the secant modulus and the tangent modulus. A paper is in preparation giving the details of this investigation.

The fundamental differential equation of equilibrium for a buckled plate in the plastic range has also been solved for the case of a long plate with elastically restrained edges under pure shear. The solution shows that long plates under shear require a modulus which is a combination of the secant modulus and the tangent modulus, obtained from the axial stress-strain curve. When the elastic restraint along the edges is made infinite, the condition of a long plate with clamped edges, such as was studied by the author, is obtained. His reduction factor η_s has been computed from the theory using his axial stress-strain curve for 24S-O alloy. Fair agreement with the experimental data was obtained.

The theory shows that use of the shear-secant modulus as proposed by the author is satisfactory, provided the stress-strain curve is capable of representation by a simple power law in the plastic region. The stress-strain curve for 24S-O alloy can be so represented. There are materials, however, for which the stress-strain curve cannot be described accurately by a simple power law, and for these materials the use of the shear-secant modulus probably would not be sufficiently accurate.

Walter Ramberg.[3] The author has performed a service for structural engineers in suggesting a method for computing buckling stresses in shear in the plastic range, and in supporting this method by test results. The method is so simple that it will be widely used by designers. It is desirable therefore that it should be checked by further tests on plates of other thicknesses and materials than the one thickness and one material used by the author.

If such additional tests are made it is desirable also that enough gages be placed upon at least some of the specimens to confirm that the strain distribution across the width is uniform in accordance with the theory. The resulting body of test data, together with the data already presented in the paper, would serve not only to establish the author's formula, but also to indicate the adequacy of the shear test of a pair of long strips for obtaining the stress-strain curve in shear and to provide experimental material which must be explained by any theories of plastic instability, such as the theory recently developed by Ilyushin.[4]

Consideration of the theory of plastic instability suggests replacing the purely empirical relation $\tau = 0.5\sigma$, $\gamma = 1.3\epsilon$, used by the author, either by the relation $\tau = 0.5\sigma$, $\gamma = 1.5\epsilon$, which corresponds to the maximum-shear theory of yielding, or by the relation $\tau = \sigma/\sqrt{3}$, $\gamma = \sqrt{3}\epsilon$, which corresponds to the octahedral shear theory.

The relation $\tau = 0.5\sigma$, $\gamma = 1.3\epsilon$ was derived by the discusser to fit to experimental data on tension and torsion tests of 17S-T aluminum-alloy tubes. For chromium-molybdenum steel tubes, the best fit was obtained with $\tau = \sigma/\sqrt{3}$, $\gamma = 1.4\epsilon$. Osgood's work[5] on 24S-T aluminum-alloy tubes under combinations of internal pressure and axial load shows a slightly better fit for the maximum-shear theory than for the octahedral shear theory.

The maximum shear relation $\tau = 0.5\sigma$, $\gamma = 1.5\epsilon$ would also give a better fit to the strain data shown in Fig. 2 of the paper. The curve in Fig. 9 of the paper would remain unaffected provided we take $\nu = 0.5$.

The octahedral shear relation $\tau = \sigma/\sqrt{3}$, $\gamma = \sqrt{3}\,\epsilon$, gives a worse fit to the strain data in Fig. 2: making $\nu = 0.5$ would lead to replacing Equation [12] by

$$\frac{E}{\sigma_{0.7}}\sqrt{3}\,K_s\left(\frac{t}{b}\right)^2 = \frac{\sqrt{3}\,\tau_{cr}}{\sigma_{0.7}} + \frac{3}{7}\left(\frac{\sqrt{3}\,\tau_{cr}}{\sigma_{0.7}}\right)^n$$

Solutions of this equation would be given by multiplying both ordinates and abscissas of the curves in Fig. 9 by $2/\sqrt{3}$. The result would be to make the computed critical stresses in the plastic range slightly greater than those given by the curve in Fig. 8 of the paper. The difference is small for buckling problems, and it is in the same sense as for other comparisons of measured and computed buckling stresses.

Author's Closure

The theoretical development of effective moduli for use in buckling of thin plates above the proportional limit, as first accomplished by Ilyushin and then by Stowell of NACA, is an outstanding contribution and one which is a logical development of the research work pursued on the subject.

There is a fundamental difference between the work of Ilyushin and Stowell, however. Ilyushin assumes that a strain reversal occurs at the critical stress whereas Stowell assumes that no strain reversal occurs. Consequently, Stowell's results are lower than those of Ilyushin. In both cases, Poisson's ratio is assumed to have a value of $\nu = 0.5$, although the critical stress is in the region of the knee of the stress-strain curve.

It is necessary to determine which process of strain reversal is correct in buckling of plates. Work on columns by Shanley[6] and

[1] By George Gerard, published in the March, 1948, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 70, pp. 7–12.

[2] NACA, Langley Field, Va.

[3] Physicist, National Bureau of Standards, Washington, D. C.

[4] "Stability of Plates and Shells Beyond the Proportional Limit," by A. A. Ilyushin (translated from Russian), NACA Tech. memo. 1116, October, 1947.

[5] "Combined-Stress Tests on 24 S-T Aluminum Alloy Tubes," by W. R. Osgood, Journal of Applied Mechanics, Trans. ASME, vol. 69, 1947, p. A-147.

[6] "Inelastic Column Theory," by F. R. Shanley, *Journal of the Aeronautical Sciences*, vol. 14, May, 1947, pp. 261–267.

Wang[7] has shown that the assumption of no strain reversal is correct when applied to columns. On plates, however, the assumption of no strain reversal is not an obvious physical assumption. For example, if the top-of-the-knee method is used experimentally for determining buckling stresses on plates, a strain reversal has occurred at the critical stress.

Also, if experimental work on plates is to be compared with theory, it is necessary to develop a method for plates comparable with the Southwell method for columns, so that experimental results on specimens subject to initial imperfections can be compared with theoretical results for theoretically perfect plates.

The author agrees with Dr. Ramberg that a highly desirable field for further research work is the development of a method for determining experimentally shear stress - strain curves. For a particular material, the ultimate choice of affinity constants given by either the maximum-shear theory of yielding or the octahedral shear stress theory depends upon such a stress-strain curve.

For the particular data presented in the paper, the affinity constant for stresses corresponds to that of the maximum shear theory. The affinity constant for the strain of 1.3 used in the paper, as compared to 1.5 as obtained from maximum shear theory, appears to depend upon the choice of the value assumed for Poisson's ratio. In the elastic range $\nu = 0.3$, in the plastic range $\nu = 0.5$, while in the region of the yield stress, the value of Poisson's ratio probably varies between the two limits and, consequently, no set value can be assumed.

The Flutter of a Uniform Wing With Tip Weights[1]

R. H. Scanlan.[2] If we understand the authors' point correctly, we infer that they definitely do not suggest the methods of this paper as a means of flutter analysis for use in the engineering office, but rather they attempt to assess the value of a certain so-called Rayleigh-type approach commonly used in field calculations. They are to be congratulated upon the attempt. We think, however, and perhaps they will agree, that a conclusive answer to the question of the validity of such an approach would lie only in a statistical set of results comparing, case by case, many airplane-flutter calculations performed both by what they call the Rayleigh-type analysis and by some alternative method of better accuracy and universal applicability.

It is clear that the authors do not intentionally employ the terms "Rayleigh-type flutter analysis" loosely. They apply them specifically to a flutter analysis based upon the first bending and first torsion uncoupled modes, i.e., one in which the modal shapes of the flutter system are assumed to be a combination of these modes as individually unaffected in shape by the existence of the flutter air forces. The authors are then comparing two methods of flutter analysis: One (their method) wherein the air forces do affect the mode shapes at flutter, and another (which they call the Rayleigh type), in which these forces do not affect flutter-mode shape.

However, the use of the words "Rayleigh-type analysis" conceivably may include other meanings than the one the authors have assumed. Care should be taken not to apply their conclusions indiscriminately to all types of analysis which equally reasonably may be termed the Rayleigh type. Most accurately, a Rayleigh analysis applies only to a system in free vibration wherein a fair approximation of the mode shape is made (usually by satisfying boundary conditions), and the resulting energy calculations, being not too much in error, lead to good approximations of the natural frequency. Flutter analyses not much different from those involving only bending and torsion uncoupled modes can be made, introducing higher bending and higher torsion modes at will, raising the number of degrees of freedom by one for each uncoupled mode included. Such analyses, loosely speaking, are also of the "Rayleigh type."

Moreover, coupled or natural, rather than uncoupled, modes may be employed initially, effectively replacing each pair or group of uncoupled modes which normally couple by a single mode, thus reducing the needed number of degrees of freedom. This approach, by employing the fact that any motion (including the flutter motion) is the superposition of an infinite set of natural modes, can achieve a good approximation to the flutter mode by including sufficient modes of this infinite series. In general, this method is more economical of degrees of freedom than the use of uncoupled modes (which are not normal modes).

Thus in the reasonable realm of the term "Rayleigh-type analysis" flutter-mode accuracy can be increased indefinitely. This accuracy of course extends not only to modes but also to phase relationships of distinct particles, of which the authors also make considerable point for their method. These Rayleigh-type methods, having the desired features of accuracy just mentioned, have been in use throughout a large section of the aircraft industry for some years. It would be an appropriate study to compare the authors' method with them on various counts, such as accuracy, calculation speed, difficulty, and so forth.

The use of the word "exact" is unfortunate in connection with any flutter method; there are always open the questions of the degree, location, and type of approximation used. The approximation of the coupled-mode methods just alluded to lies in taking only a finite number of the infinite number of terms of the infinite series of normal modes. In the authors' method they have found it convenient to introduce an approximation into the calculation of the ξ_{sr} by means of the first few terms of a convergent power series. Which is the more permissible approximation will be determined by the specific case discussed. Profitably, then, many cases should be discussed.

In pursuing further our hypothetical set of statistics necessary for confirmation or refutation of a given method by comparative procedure it would be noticed that the authors' method would be inapplicable to those cases actually encountered in practice, at least without further approximation. This is not a fault of the paper, but it does inhibit extensive generalizations from its conclusions. We might suggest a paragraph to sum up the problem and the contributions of the paper to the immensely difficult task of weighing flutter methods:

For a wing having (*a*) a unique spanwise distributed torsional stiffness coefficient; (*b*) constant bending stiffness; and (*c*) constant torsional stiffness throughout, the following conclusion holds: A particularly simple type of flutter analysis of universal application to actual wings, which assumes that air forces do not affect the flutter mode, gives results quantitatively close to results of a method presented by the authors. This method is applicable only to this constant-section wing. There exist other methods of universal application which have been in use for some years which are analytically demonstrable as being at least as accurate potentially as either of the methods compared in the paper. These other methods, as well as the authors' method, take account of the effect of the air forces on the flutter mode. It should be noted that in general airplane wings having

[7] "Inelastic Column Theories and an Analysis of Experimental Observations," by C-T. Wang, *Journal of the Aeronautical Sciences*, vol. 15, May, 1948, pp. 283–292.

[1] By Martin Goland and Y. L. Luke, published in the March, 1948, issue of the Journal of Applied Mechanics, Trans. ASME vol. 70, pp. 13–20.

[2] Associate Professor of Aeronautical Engineering, Rensselaer Polytechnic Institute, Troy, N. Y.