If \( r \) is considerably larger than \( y(x) \) for all values of \( \omega \) then these forces are equal to \( \omega^2m_\phi \).

If we let

\[ c_i = \sum_0^1 m_\phi \phi(x_i) \quad (14) \]

and let \( y(x) \) be the deflection shape due to both the original unbalance and the added balance weights \( m_\phi \) then we can write

\[ y(x) = \sum_i d_i \phi(x_i) \quad (15) \]

where

\[ d_i = \frac{\beta_i^2}{1 - \beta_i^2} (a_i + c_i) \quad (16) \]

corresponding to \( b_i \) in equation (13).

The problem now is to find optimum values for the unbalance \( m_\phi \). The optimum we have chosen is to make \( y \) zero at several discrete positions and speeds. Let the measured values of \( y \) (without balance weights) at these several conditions be \( y \) where \( y \) is a vector of \( k \) elements. Then, from equations (10) and (12) we may write

\[ y = Aa \quad (17) \]

where \( a \) is a vector of the \( a_i \) and \( A \) is the square matrix of the \( k \) elements \( a_i \). Since we wish to solve for \( A \), \( A \) must be inverted. Thus there can be no more than \( k \) \( a_i \) and if there are fewer than \( k \) \( a_i \) or, if the coefficients of some of the \( a_i \) in \( A \) are zero, \( A \) will be singular. Now the relative values of \( a_i \) are given by the relative values of the modes shown dashed in Fig. 3 and similar figures. It is obvious that for all measurements taken at \( \beta \) less than 5, \( A \) will be almost singular unless \( n \) is 1 or 2; i.e., only the first two modes are excited. From equation (15) it is clear that our condition requires \( c_i = -a_i \). Since there are \( k \) and only \( k \) \( a_i \) there are \( k \) and only \( k \) \( (m_\phi) \).

Therefore we must measure at exactly as many points as we have balance weights. If we measure more points and have more balance weights than there are excited modes in the unbalanced rotor, the solution will be almost singular and the results are apt to be such as to cause large unbalance in the higher modes. Actually, it is possible to use more than \( k \) measurement points, but then the equations must be reduced to \( k \) by least squares methods. If some other balance criterion were used, it would be reduced to the form of equation (15) by suitable transformation and the same conclusions would hold. One possible minimal criterion would be

\[ \int_0^L y^2(x)dx = \text{min} \]

**DISCUSSION**

Josef K. Sevcik

The aim of this paper is to demonstrate the feasibility of high-speed balancing of flexible rotors and that it can be achieved with a minimum number of measurements. The analysis and experimental investigation deal with simply supported uniform shaft, but from the Appendix it can be deduced that the same reasoning may be applied to any rotating body with general mass and flexibility distribution along its axis, and confined by any linear boundary conditions.

A very limited amount of experimental data is presented, so that no comparison is possible between unbalanced and balanced shaft. The authors explain more at length the discrepancies encountered in checking the calculated response coefficients \( \alpha \), and the application of a shake test analysis which helped to define these coefficients experimentally. In addition to the response coefficients \( \alpha \), the natural frequencies and flexural modes have also changed. To the factors affecting these changes should be added, that it is the over-all system, shaft and balancing apparatus included, which also contributed to these deviations.

In the actual balancing, where the ratio between the weights of balanced piece and of balancing machine will be higher than in the described laboratory test, the difficulty in calculating the response coefficients will be greater, and the calculation will have to be replaced by the shake test analysis, which will only add to the number of measurements.

The elimination of initial runout at measuring planes is explained very well in the outline of the balancing procedure. It would have been of great interest to show that the effect of bearing clearance can also be accounted for. Since the system damping changes the phase angle of the measured deflections in relation to the unbalance in the vicinity of natural frequency, making the angular location of these readings difficult, it would be interesting to know at what speeds the described shaft was actually balanced.

With reference to equations (7), it should be said that two such sets were apparently written for proper resolution of unbalance components: one in the plane of timer mark, and one in the plane perpendicular to the former.

The balancing conditions were defined in connection with these equations, i.e., zero deflections at the measuring planes and speeds. No wonder then that conclusion No. 4 has been reached, i.e., that “the number of balance weights and measurement points should be equal to the number of critical speeds.”

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But it should not be forgotten that also other balancing conditions, genuinely different, can be set up, such as zero reactions for instance or minimum kinetic energy. The number of balancing equations (7) would then be equal to the number of balancing conditions, and would require an equal number of balancing weights and points of measurement. The minimum number of balancing conditions is given by the number of flexural modes for which the balancing is required, being the last one, eventually, above the maximum operating speed. On the other hand, it should be remembered that the first two modes in balancing precede the bow shaped mode normally considered as the first one of a simply supported shaft. Low-speed balancing takes care of those two modes at which the rotor behaves like a rigid body.

It seems that conclusion No. 2 partially contradicts the No. 4 conclusion, and partially is identical with it. It could be deleted.

The first sentence of conclusion No. 3 emphatically condemns all but high-speed balancing, but the present paper does not offer sufficient proof for such a strong stand. The second sentence of the conclusion is more likely to be agreed upon, with some qualification of course, such as: “It is not necessary to balance a flexible shaft much above its maximum operating speed, provided that the supports of the high-speed balancing machine are able to impose on the shaft the same flexural modes as the engine will.”

I can readily agree with the first conclusion. It has yet to be seen how practical the procedure will be. The balancing results, as presented in Figs. 6 and 7, do not show too great accuracy, which indicates the need for further development of the method. The practicality would also be judged by the time spent in balancing of a typical piece. It is believed to be premature to reach any conclusion on this subject at this time.

The authors should be commended for the work presented in this paper, and for the idea of response coefficients which they call acceleration mobility. It certainly makes the high-speed balancing understandable, and puts in proper perspective the various parameters important in this type of balancing.
It is my opinion that perhaps some references of similar type of work could have been mentioned, such as: “Fundamentals of Systematic Vibration Elimination From Rotors With Elastic Shafts” (Grundlagen einer systematischen Schwingungsent-Störung wellenelastischer Rotoren) by Prof. Klaus Federn of Darmstadt, Germany (VDI—Berichte Bd. 24/1957).

Authors' Closure

The authors appreciate comments from one with as much experience as Mr. Sevcik has in the problems of balancing flexible rotors and are looking forward to the publication of his method of distributing balance weights from low-speed measurements.

Most of his comments appear to stem from a misunderstanding of the nature of our experimental apparatus. As mentioned in the body of the paper, it was built for experimental purposes only and was deliberately made to be a very flexible low-speed shaft with stiff ball bearings. Mr. Sevcik is quite correct when he says the rigid body modes must be included in any practical case. Conclusion 4 is incorrectly stated, as he points out, and should refer only to balance weights and not measurement points.

Only actual experience can show whether calculations alone would give sufficiently accurate values for response coefficients and to what extent supplementary measurements would be necessary. One would hope that, as usual, the accuracy of computed results would improve with experience.

The authors thank Mr. Sevcik for mentioning one of Dr. Federn's many important contributions to the over-all problem. This oversight was because his work is in a different direction. More directly pertinent are three papers appearing in vol. 1, no. 1, of the Journal of Mechanical Engineering Science, June, 1959. These are: “The Vibration of Rotating Shafts,” by Dr. R. E. D. Bishop, pp. 50-65; “The Vibration and Balancing of an Unbalanced Flexible Rotor,” by R. E. D. Bishop and G. M. L. Gladwell, pp. 66-77; and “The Receptances of Uniform and Non-Uniform Rotating Shafts,” by G. M. L. Gladwell and R. E. D. Bishop, pp. 78-91.