Vector and Axial Vector Mesons and Chiral Current Algebra

Takanori SHIOZAKI
Department of Physics, Nagoya University, Nagoya

(Received August 21, 1967)

As the chiral partners of vector gauge mesons the axial vector mesons are introduced by the generalization of the Yang-Mills theory. The connection to the chiral current algebra with these vector and axial vector fields is discussed.

Many important properties of the strong and weak interactions have been successfully predicted in terms of the algebra of vector and axial vector currents in $U(3) \times U(3)$ and $SU(2) \times SU(2)$ in conjunction with the Partial Conservation of Axial Vector Currents (PCAC). This suggests that the chiral group may provide a good symmetry of elementary particle dynamics. In this note we show that as the chiral partners of the vector mesons (nonet), the axial vector mesons (nonet) are naturally introduced in Lagrangian field theory. If the Lagrangian of matter fields is invariant (or partially invariant) under a constant parameter gauge transformation of a general semi-simple Lie group, the gauge transformation may depend on space time coordinates by introducing the vector gauge particles coupled universally with the matter fields as shown by Yang and Mills, by Utiyama, and by Gell-Mann and Glashow. Generalizing the gauge transformation to chiral semi-simple Lie group, vector and axial vector gauge mesons transforming as a regular representation of the group are naturally introduced. Although the Lagrangian is generally not strictly invariant under the chiral gauge transformation, the non-invariant part of the Lagrangian may be assumed to have a simple transformation property. The concept of chiral gauge invariance (or partial invariance) leads to the conservation (or partial conservation) of vector and axial vector currents. The space integrals of vector and axial vector currents form a chiral algebra of semi-simple Lie group.

Let us consider the matter field Lagrangian $L_m (F_a, \partial_\mu F_a)$ which has simple transformation properties (invariant or partially invariant) under an infinitesimal constant parameter chiral gauge transformation. Denoting the matter field $F_a$, it

The non-invariant part of the Lagrangian is a meson mass term which leads to the partial conservation of axial vector current (PCAC). In the succeeding note we consider a phenomenological theory in which the constant parameter chiral gauge invariance is broken in a pseudoscalar meson mass term, and the space time depending gauge invariance is broken by vector and axial vector meson mass terms.
and infinitesimal gauge parameters as $\alpha_i^+$, and $\alpha_i^-$, the $\gamma_a$ transforms as:

$$\gamma_a \rightarrow \gamma_a + i \alpha_i^+ (T_i^+ \gamma_a),$$  
(1)

$$\gamma_a \rightarrow \gamma_a + i \alpha_i^- (T_i^- \gamma_a),$$

where $T_i^+$ are matrix representations of the semi-simple Lie algebras on $\gamma_a$. They are assumed to satisfy the following commutation relations:

$$[T_i^+, T_j^-] = 0,$$
$$[T_i^-, T_j^+] = i f_{ijk} T_k^-,\tag{2}$$

where $f_{ijk}$ are totally antisymmetric structure constants of Lie algebra. It is convenient to work with vector and axial vector gauge transformations defined by simultaneous transformations of $\gamma_a$ with $\alpha_i^+ = \alpha_i^- = \alpha_i^Y$ and $\alpha_i^+ = -\alpha_i^- = \alpha_i^A$. Defining

$$(T_i^+ \gamma_a) = (T_i^+ \gamma_a) + (T_i^- \gamma_a),$$
$$(T_i^- \gamma_a) = (T_i^- \gamma_a) - (T_i^+ \gamma_a),$$

$\gamma_a$ transforms under chiral transformations as

$$\gamma_a \rightarrow \gamma_a + i \alpha_i^Y (T_i^+ \gamma_a),$$
$$\gamma_a \rightarrow \gamma_a + i \alpha_i^A (T_i^- \gamma_a),\tag{3}$$

where $T_i^Y$ and $T_i^A$ satisfy the following commutation relations:

$$[T_i^Y, T_j^Y] = i f_{ijk} T_k^Y,$$
$$[T_i^Y, T_j^A] = i f_{ijk} T_k^A,$$
$$[T_i^A, T_j^A] = i f_{ijk} T_k^Y.$$

Let us consider the gauge transformation for which $\alpha_i^Y$ and $\alpha_i^A$ depend on the space time coordinates.

The generalization of the Yang-Mills trick\(^{1,9,20}\) to chiral gauge transformations allows us to introduce the vector and axial vector gauge mesons, $V_{\mu\nu}$ and $A_{\mu\nu}$, which transform as regular representations of the chiral group:

$$V_{\mu\nu} \rightarrow V_{\mu\nu} - f_{ij\lambda} \epsilon^j_{\mu\nu} V_{\lambda\rho} + \frac{1}{g} \partial_\mu \alpha_i^Y,$$
$$A_{\mu\nu} \rightarrow A_{\mu\nu} - f_{ij\lambda} \epsilon^j_{\mu\nu} A_{\lambda\rho}.$$

\(^{10}\) If $q_a$ field is considered to be represented by the quarks $q$ of the group of $U(3)_+ \times U(3)_-$, $q$ transforms under $T_i^Y = (\lambda_i/2) \gamma_5 q + \lambda_i q^T T_i^Y q$, where $\gamma_5 q^T = \pm q^T$ and $\lambda_i$ are Gell-Mann's nine $3 \times 3$ matrices.

\(^{11}\) For $q_a$ not transforming as Eqs. (3) and (4) such as the pseudoscalar meson nonet $\Phi_{ab}^a$ and the baryon nonet $B_a^b$ in a succeeding note\(^{13}\), an analogous treatment is possible. Under the axial vector gauge transformation, $\Phi_{ab}^a$ transforms non-linearly. In such a case we may consider the unitary function of $\Phi_{ab}^a$ which transforms linearly under gauge transformations.
Vector and Axial Vector Mesons and Chiral Current Algebra

\[ V_{\mu \nu} \rightarrow V_{\mu \nu} - f_{ijk} \partial_j A_{\mu k}, \]
\[ A_{\mu \nu} \rightarrow A_{\mu \nu} - f_{ijk} \partial_j A_{\mu k}, \]

where \( g \) denotes universal coupling constant of vector and axial vector mesons with matter fields. Vector and axial vector field strengths \( F_{\mu \nu} \) and \( G_{\mu \nu} \) are defined by the following expressions:

\[ F_{\mu \nu} = \partial_\mu V_{\nu} - \partial_\nu V_{\mu} - g f_{ijk} V_{\mu j} V_{\nu k}, \]
\[ G_{\mu \nu} = D_\mu A_{\nu} - D_\nu A_{\mu}, \]

where we denote

\[ D_\mu A_{\nu} = \partial_\mu A_{\nu} + g f_{ijk} V_{\nu j} A_{\mu k}. \]

The field strength transforms as follows under the chiral gauge transformation:

\[ F_{\mu \nu} \rightarrow F_{\mu \nu} - f_{ijk} \partial_j F_{\nu k}, \]
\[ G_{\mu \nu} \rightarrow G_{\mu \nu} - f_{ijk} \partial_j G_{\nu k}, \]
\[ F_{\mu \nu} \rightarrow F_{\mu \nu} - f_{ijk} \partial_j F_{\nu k}, \]
\[ G_{\mu \nu} \rightarrow G_{\mu \nu} - f_{ijk} \partial_j G_{\nu k}. \]

From these transformation properties we can construct the partially gauge invariant Lagrangian \( L^a \), the invariance is broken by the mass terms of gauge mesons.\(^\dagger\)

\[ L^a = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} m^2 V_{\mu \nu} V^{\mu \nu} - \frac{1}{4} G_{\mu \nu} G^{\mu \nu} - \frac{1}{4} m^2 A_{\mu \nu}^2. \]

The response of \( L^a \) under gauge transformation is

\[ \delta L^a = \frac{m^2}{g} V_{\mu \nu} \partial_\mu \partial_\nu V^{\mu \nu} + \frac{m^2}{g} A_{\mu \nu} \partial_\mu \partial_\nu A^{\mu \nu}. \]

The chiral gauge invariant coupling with matter fields is attained by replacing \( \partial_\mu \mathcal{F}_a \) in \( L^a(\mathcal{F}_a, \partial_\mu \mathcal{F}_a) \) by \( (\tilde{D}_\mu \mathcal{F})_a \) defined by

\[ (\tilde{D}_\mu \mathcal{F})_a = \left[ (\partial_\mu i g V_{\mu \nu} T^\nu a - i g A_{\mu \nu} T^\nu a) \right]_a. \]

The gauge invariance of the matter Lagrangian \( \tilde{L}_\nu = \tilde{L}_\nu(\mathcal{F}_a, \tilde{D}_\mu \mathcal{F})_a \) is ensured by observing that \( (\tilde{D}_\mu \mathcal{F})_a \) transform under chiral gauge transformation as follows:

\[ (\tilde{D}_\mu \mathcal{F})_a \rightarrow (\tilde{D}_\mu \mathcal{F})_a + i \xi_\mu T^\mu (\tilde{D}_\nu \mathcal{F})_a, \]
\[ (\tilde{D}_\mu \mathcal{F})_a \rightarrow (\tilde{D}_\mu \mathcal{F})_a + i \xi_\mu a (T^\mu a (\tilde{D}_\nu \mathcal{F})_a). \]

\(^\dagger\) Inclusion of the mass terms in the Lagrangians \( L^a \) of gauge mesons destroys the gauge invariance and also the possibility of renormalization.\( ^7 \)
The minimally interacting Lagrangian for gauge mesons and matter fields can be introduced as

$$L = L^0 + \bar{\Psi}_\nu (\gamma^\mu \bar{D}_\mu \Psi)_a.$$  \hspace{1cm} (11)

The equations of motion for vector (and axial vector mesons) are derived therefrom as

$$\partial_\mu F_{\mu\nu} - g f_{ijk} F_{\rho\sigma j} V_{i\sigma} - g f_{ijh} G_{\rho \sigma j} A_{i\sigma} - m^2 V_{i\nu} + g \tilde{j}^V_{\mu i} = 0,$$

$$\partial_\mu G_{\rho \sigma i} - g f_{ijk} F_{\rho \sigma j} A_{ik} - g f_{ijh} G_{\rho \sigma j} V_{ik} - m^2 A_{i\nu} + g \tilde{j}^A_{\mu i} = 0,$$  \hspace{1cm} (12)

where \(\tilde{j}^V_{\mu i}\) and \(\tilde{j}^A_{\mu i}\) are respectively the vector and axial vector current densities of the matter fields \(\Psi_a\) and they are derived from the matter Lagrangian \(\bar{\Psi}_\nu (\gamma^\mu \bar{D}_\mu \Psi)_a\) by the following equations:

$$\tilde{j}^V_{\mu i} = -i \frac{\partial \bar{\Psi}_\nu (\gamma^\mu \bar{D}_\mu \Psi)_a (T_i^V \Psi^*)_a}{\partial (\bar{D}_\mu \Psi)_a},$$

$$\tilde{j}^A_{\mu i} = -i \frac{\partial \bar{\Psi}_\nu (\gamma^\mu \bar{D}_\mu \Psi)_a (T_i^A \Psi^*)_a}{\partial (\bar{D}_\mu \Psi)_a}.$$  \hspace{1cm} (13)

The current densities of the system including gauge mesons are expressed as follows:

$$j^V_{\mu i} = -f_{ijk} F_{\rho\sigma j} V_{i\sigma} - f_{ijh} G_{\rho \sigma j} A_{ik} + \tilde{j}^V_{\mu i},$$

$$j^A_{\mu i} = -f_{ijk} F_{\rho\sigma j} A_{ik} - f_{ijh} G_{\rho \sigma j} V_{ik} + \tilde{j}^A_{\mu i}.$$  \hspace{1cm} (14)

The equations of motion in Eq. (12) are reduced with the aid of Eq. (14) to

$$\partial_\mu F_{\mu\nu} + g j^V_{\mu\nu} = m^2 V_{\mu i},$$

$$\partial_\mu G_{\rho \sigma i} + g j^A_{\rho \sigma i} = m^2 A_{i\nu}.$$  \hspace{1cm} (15)

The conservations or partial conservations of the currents are expressed as follows:

$$\partial_\mu j^V_{\mu i} = m^2 \partial_\mu V_{\mu i},$$

$$\partial_\mu j^A_{\rho \sigma i} = m^2 \partial_\mu A_{\rho \sigma i}.$$  \hspace{1cm} (16)

The vector and axial vector fields themselves must satisfy the same subsidiary conditions such as current conservation or partial conservation. The space integrals of the fourth components of these currents, or the vector and axial vector charges satisfy the commutation relations of the chiral semi-simple Lie algebra.\(^6\) The charges \(T_i^V\) and \(T_i^A\)

\(^6\) Owing to the form of the Lagrangian \(L^0\) chosen for vector and axial vector mesons, the canonical conjugate momenta for the fourth components of the vector and axial vector fields vanish identically. This difficulty may be avoided by considering that the four components of the gauge mesons are not independent of each other, and by making use of equations of motion the fourth components are expressed as functions of other field components and their canonical momenta. The subsidiary conditions for gauge mesons are \(\partial_\mu V_{\mu i} = 0\) and \(\partial_\mu A_{\rho \sigma i} = 0\) only in the case of current conservation.
The commutation relations among the space components and the time components of the current densities depend on the model for matter field Lagrangian $\mathcal{L}_\pi$ chosen, and more singular Goto-Imamura, Schwinger terms \( \delta^{(4)} \) may appear, depending on $\mathcal{L}_\pi$. Instead of considering the commutator among current densities which have unspecified Goto-Imamura, Schwinger terms, we consider the field commutation relations of vector and axial vector gauge fields.\(^{10}\)

By making use of equations of motions Eq. (12), a straightforward calculation leads to the commutation relations for chiral gauge fields. If we define

\[
V_{\alpha i} = (m^2/g) V_{\alpha i}, \quad A_{\alpha i} = (m^2/g) A_{\alpha i},
\]

the commutation relations take the following form:

\[
[V_{\alpha i}(x), \ V_{\beta j}(x')]_{L \cdot t'} = i f_{ijk} V_{\beta k}(x) \delta^3(x - x'),
\]
\[
[V_{\alpha i}(x), \ A_{\beta j}(x')]_{L \cdot t'} = i f_{ijk} A_{\beta k}(x) \delta^3(x - x'),
\]
\[
[A_{\alpha i}(x), \ A_{\beta j}(x')]_{L \cdot t'} = i f_{ijk} V_{\beta k}(x) \delta^3(x - x').
\]

(17)

\[
[V_{\alpha i}(x), \ V_{\beta j}(x')]_{L \cdot t'} = i f_{ijk} V_{\beta k}(x) \delta^3(x - x') - i \frac{m^2}{g^2} \delta_{ij} \partial_i \delta^3(x - x'),
\]
\[
[V_{\alpha i}(x), \ A_{\beta j}(x')]_{L \cdot t'} = [V_{\alpha i}(x), \ A_{\beta j}(x')]_{L \cdot t'} = i f_{ijk} A_{\beta k}(x) \delta^3(x - x'),
\]
\[
[A_{\alpha i}(x), \ A_{\beta j}(x')]_{L \cdot t'} = i f_{ijk} V_{\beta k}(x) \delta^3(x - x') - i \frac{m^2}{g^2} \delta_{ij} \partial_i \delta^3(x - x')
\]

\[(a = 1, 2, 3)\]

(18)

and

\[
[V_{\alpha i}(x), \ V_{\beta j}(x')]_{L \cdot t'} = [V_{\alpha i}(x), \ A_{\beta j}(x')]_{L \cdot t'} = [A_{\alpha i}(x), \ A_{\beta j}(x')]_{L \cdot t'} = 0.
\]

(19)

(\( a, b = 1, 2, 3 \))

The fourth components of these fields themselves form the chiral Lie algebra. Owing to the facts that the commutation relations among the space components of
these fields vanish, the field algebras themselves do not form, for example, the chiral algebras such as $SU(6) \times SU(6)$.

Acknowledgements

The author is grateful to the members of the Elementary Particle Physics Laboratory of Physics Department of Nagoya University for valuable discussions, especially to Professor T. Takabayasi, Professor Y. Ohnuki, Dr. A. Toyota and Dr. S. Kitakado. He would also like to thank Professor S. Hayakawa for careful reading of the manuscript.

References

4) M. Gell-Mann and M. Lévy, Nuovo Cim. 16 (1960), 705.
6) M. Gell-Mann, Physics 1 (1964), 63.