An Exact Form of First-Order Self-Energy in Random Lattice Problems

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It has been pointed out that, in the problems of random lattices, the restricted summations over impurity sites must be treated exactly without neglecting contributions from the higher-order terms of the impurity concentration \( c \).1) With consideration for this point, the expression for the first-order self-energy \( \Sigma(1) \) has been given and the dual symmetry of said expression has been proved by the present author and Matsubara.2) That is, if we avoid the multiple existence of more than one impurity at the same site and evaluating proper diagrams of first order in the sense that the repeated scattering by a single impurity alone is considered as shown in Fig. 1, \( \Sigma(1) \) is written in the form

\[
\Sigma(1) = NV \sum_{i=1}^{\infty} P_i(c) (V \mathcal{D})^{i-1},
\]

where \( P_i(c) \) is determined by a generating function \( g(x; c) = \ln[1 - c + c \exp(x)] \) (see Eq. (3.8) of reference 2), \( V \) being impurity potential and \( \mathcal{D} = \sum_k G(k) \). They have also shown that the above-mentioned first-order self-energy is expressed in the form of a continued fraction,3)

\[
\Sigma(1) = NVI(V \mathcal{D}; c) = NV[1 + V \mathcal{D} f_1(V \mathcal{D}; c)],
\]

where

\[
f_1(x; c) = \frac{c(1-c)}{1-(1-2c)x+x^2f_2(x; c)},
\]

and

\[
f_n(x; c) = \frac{n(n-1)c(1-c)}{1-n(1-2c)x+x^2f_{n+1}(x; c)}, \quad (n \geq 2)
\]

(2c)

The first approximant of \( f_1(x; c) \) obtained by putting \( f_2(x; c) = f_1(x; c) \) yields the same result as that by Taylor.4) The purpose of the present short note is to give a physical meaning of said first approximant of the continued fraction originally derived from purely mathematical point of view.

In evaluating every term of Eq. (1), the correction factors or renormalization from all types of higher-order diagrams, proper or improper, which contain more than one impurity site but with the same number of interaction lines, are taken into account. If we take diagrams of fourth-order in \( V \) as an example, the contribution from the proper diagram in Fig. 2b is considered as well as the contributions from the improper fourth-order diagrams in Fig. 2c to renormalize the factor which is to be assigned to the diagram 2a. There are two types of diagrams in Fig. 2c; improper diagrams and noncrossed proper diagrams. It must be noted, however, that the substantial cluster diagram 2b, or the crossed diagram

\[
\Sigma(1) = \text{etc.}
\]

(2a)

Fig. 1.

Fig. 2.
as generally called, is not a first-order diagram, i.e. not included in Fig. 1 while the other fourth-order diagrams are included in the first approximation. The same situation occurs for every term after the fourth order in $V$. Thus, it is concluded that $\Sigma(1)$ in Eq. (1) is rigorous when all of the higher-order self-energy parts are taken at once. On the other hand, it is necessary to eliminate the correction factors arising from the substantial cluster diagrams in order to make the first approximation self-contained.

This goal is achieved by the diagrammatic consideration. With reference to Figs. 3 and 4, the consistent first-order proper diagrams are expressed by diagrams in Fig. 3, while $\zeta$ is defined in terms of $I_1$ in a self-consistent manner as shown in Fig. 4. A straight full-line connecting two different impurity sites as shown in Fig. 4b designates an accidental coincidence of two independent impurity sites. This kind of diagram in which more than one impurity site are connected by a full line arises from the process of freeing the restricted summation. A wavy line represents a contribution from a diagram attained by the replacement of said wavy line by a single interaction line minus contributions from diagrams yielded by the separation of said wavy line from the original vertex, said wavy line itself being the sum of all kinds of first-order diagrams and newly-produced vertices being necessarily connected to the original vertex.

In view of the above prescriptions for counting diagrams, it is readily seen from Figs. 3 and 4 that contributions from all required diagrams and no redundant ones are included in $\Sigma(1, 1)$. Expression for $\Sigma(1, 1)$ is given from Fig. 3 in the form:

\[
\Sigma(1, 1) = \frac{c}{1 - V V_\zeta} = N V I_1(V \zeta; c).
\]

As a result, $\Sigma(1, 1)$ is written as

\[
\Sigma(1, 1) = N V I_1(V \zeta; c),
\]

\[
I_1(x; c) = \frac{c}{1 - x(1 - I_1(x; c))}.
\]

It is easily verified that $\Sigma(1, 1)$ in Eq. (4b) is the same as in Eq. (2a) when $f_2(x; c)$ is rendered equal to $f_1(x; c)$.

Thus, the meaning of taking the first approximant of the continued fraction determined by the Eqs. (2a) and (2b) has been explained hereinabove. By treating the correction factors strictly, spurious or unphysical poles are subdued.

Before concluding the present short article, it is worth while to note that, when the next order approximation to the self-energy is required, two procedures must be followed at the same time; considering the consistent second-order self energy $\Sigma(2, 2)$ and including the correction factors $\Sigma(1, 2)$ to the first-order self-energy arising from the cluster diagrams with two impurities. As the third-order approximation $\Sigma(3, 3) + \Sigma(2, 3) + \Sigma(1, 3)$ is to be evaluated where $\Sigma(n, n)$ indicates a consistent $n$-th order self-energy and $\Sigma(m, n)$ ($m < n$) correction factors to the $m$-th order self-energy coming from the $n$ cluster diagrams. Any higher-order approximation is obtained in the same way as mentioned and only in this way,
the approximation at every stage assured to be self-contained. The evaluation of actual forms for $\Sigma(1, 2)$ and $\Sigma(2, 2)$ is now in progress. A detailed derivation of $\Sigma(1, 2)$ as well as $\Sigma(1, 1)$ will appear in a forthcoming paper.

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