Two-Dimensional Euclidean Group and the Partial-Wave Expansion. I

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(Received December 25, 1967)

When we decompose the invariant amplitude \( f(s, t) \) into the partial-wave scattering amplitudes for fixed \( t \), we must consider four different cases according to the values of \( t \). The amplitude \( f(s, t) \) is considered as a function over a little group. For \( t=0 \), two cases occur; we treat the case where the little group is two-dimensional Euclidean group \( E_2 \). Taking \( f(s, t=0) \) as a function over \( E_2 \), we get the partial-wave expansion according to the unitary irreducible representations of \( E_2 \) and can pick out the Regge behavior from this expansion. We find that this group-theoretical method is essentially the "eikonal" approximation representation.

§ 1. Introduction

It is said that particles should be classified according to the irreducible unitary representations of the Poincaré group. These representations are distinguished according to whether the square of the momentum transfer, \( \sum_{\mu=0}^{3} Q_{\mu} Q_{\mu} = Q_0^2 - Q^2 = M^2 \), is positive, negative or zero. When \( M^2 > 0 \), the little group is \( SO(3) \); when \( M^2 < 0 \), it is \( SO(2, 1) \). On the other hand, when \( M=0 \), there are two cases: one is specified by the little group of \( SO(3, 1) \) (the Lorentz group) and the other by two-dimensional Euclidean group \( E_2 \) (i.e. semi-direct product of 2-dimensional rotations and translations). Photons and neutrinos are classified by the latter.

Turning to the Regge pole theory where Regge poles pass in \( t \)-channel, we have to classify Regge particles (Regge trajectories) according to the values of \( t \). When \( t > 0 \), this classification is based on \( SO(3) \); \( t < 0 \), on \( SO(2, 1) \). The former is the Regge theory and the latter the Toller\(^3\) theory. For \( t = 0 \), two cases are to be considered as mentioned above. When the masses of the particle \( A \) and \( C \) are equal, we can make \( Q_\mu = p_\mu^B - p_\mu^A \) into \((0, 0, 0, 0)\) by some
Lorentz transformation. Hence the little group is $SO(3, 1)$ and this case is treated in detail by M. Toller, although the difficulty arises, namely that he cannot pick out the Regge trajectories above $J = -1$.

In the case of unequal masses $(m_A \neq m_B)$, e.g., when Regge poles are the intermediate states in $\pi N$ channel, the situation is quite distinct. We cannot take $(Q_n)$ into $(0, 0, 0, 0)$ by any Lorentz transformations and the little group is the so-called two-dimensional Euclidean group $E_2$. This case is treated in this paper. In §2 we describe briefly the irreducible unitary representations of $E_2$, and in §3, using the result of §2, we obtain partial-wave expansion of the scattering amplitude regarding it as a function over this group. In §4 we examine some properties of this expansion. Some comments and discussions are given in §5. A more mathematically strict and different way of treating $E_2$ in the Regge theory, in order to discuss the daughter trajectories, is under preparation and will appear in II of this series.

§2. Mathematical preliminaries for two-dimensional Euclidean group

Unitary irreducible representations of 2-dimensional Euclidean group $E_2$ are completely determined by S. Ito. Moreover those of $n$-dimensional Euclidean group $(n = 1, 2, 3, \ldots)$ are determined by S. Ito, Gelfand and Neumark. We mention briefly the main results of the representations of $E_2$.

1°. The group $E_2$ is defined as the set of all two-dimensional transformations (i.e. rotations and translations) in the two-dimensional Euclidean space $\mathbb{R}^2$. Let us identify an element of $E_2$ with

$$g(\theta, v) = \begin{pmatrix}
\cos \theta & \sin \theta & x_1 \\
-\sin \theta & \cos \theta & x_2 \\
0 & 0 & 1
\end{pmatrix},$$

where $v = (x_1, x_2) \in \mathbb{R}^2$. Let $T$ be a set of all rotations and $V$ all translations, i.e.

$$T = \{g(\theta, 0)\}, \quad V = \{g(0, v)\}.$$ Then $E_2$ is the semi-direct product of $T$ and $V$. Now we denote the unitary irreducible representations of $E_2$ by $(E_2, \pi, \mathcal{S})$ which are classified as follows. The first is the one-dimensional irreducible representation $(E_2, \pi_{tr}, \mathcal{S})$, specified by

$$(A) \quad \pi(g(\theta, v)) = \exp(in\theta), \quad n \in \mathbb{Z} (\text{whole integers}).$$

The other class of unitary irreducible representations is realized over the space $\mathcal{S}$ which is defined as

$$\mathcal{S} = L^1(C), \quad \text{where } C = \{(x_1, x_2); x_1^2 + x_2^2 = 1, x_1, x_2 \in \mathbb{R}\}$$

$$= \{ (\cos \varphi, \sin \varphi); 0 \leq \varphi < 2\pi \}.$$
This representation is denoted by \((E_2, \pi, \mathbb{S})\).

\[
\begin{align*}
& \text{For any } f \in L^2(C), \\
& (\pi(g(0, 0))f)(\varphi) = f(\varphi - \rho), \\
& (\pi(g(0, \varphi))f)(\varphi) = \exp[i\rho(x_1 \cos \varphi + x_2 \sin \varphi)] f(\varphi),
\end{align*}
\]

where \(\rho\) is real and positive.

All the unitary irreducible representations of \(E_2\) are exhausted by (A) and (B), that is, they are unitarily equivalent to either \((E^z, \pi_m, \mathbb{S}_j)\) or \((E^z, \pi_p, \mathbb{S}_j)\).

2°. Plancherel's formula and expansion theorem

Let \(f(g) = f(0, x)\) (where \(x \in \mathbb{R}^2\)) be a function over \(E_2\), and let

\[
F_n(x) = \frac{1}{(2\pi)^{1/2}} \int_0^{2\pi} f(\theta, x) \exp(-i\theta) d\theta.
\]

Then

\[
\| f \|_2^2 = \sum_{n=-\infty}^{\infty} \| f_n \|_2^2,
\]

\[
\| f_n \|_2^2 = \int_{\mathbb{R}^2} |F_n(x)|^2 dx.
\]

Moreover, define

\[
\tilde{F}_n(\nu) = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_n(x) \exp[i \langle x, \nu \rangle] dx,
\]

where \(\langle x, \nu \rangle\) means a Euclidean inner product. Introducing a polar coordinate \((\rho, \phi)\) of \(\nu\), i.e. \(\nu = (\rho \cos \phi, \rho \sin \phi)\), we have

\[
\| f \|_2^2 = \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \rho d\rho \int_0^{2\pi} d\phi |\tilde{F}_n(\rho, \phi)|^2.
\]

This is the so-called Plancherel's formula. Further, we can use the following expansion theorem according to the unitary irreducible representations of \(E_2\):

\[
f(\theta, x) = \sum_{n=-\infty}^{\infty} \exp(i\theta) F_n(x)
\]

\[
= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp(i\theta) \int_0^{2\pi} d\rho \int_0^{2\pi} d\phi \exp[-i\rho(x_1 \cos \phi + x_2 \sin \phi)] \tilde{F}_n(\rho, \phi).
\]

Note that this theorem is applicable to those functions \(f(\theta, x)\) which are square integrable in \(x\) and differentiable with respect to \(\theta\). The structures of \(E_n(n = 1, 2, 3, \ldots)\) are quite simple because they are "solvable" Lie groups, whereas \(SO(3, 1), SO(2, 1), \ldots\) are the usual groups which we encounter in particle physics—are "semi-simple". For the proof of the theorems mentioned above, see references 3) and 4).
§ 3. Little groups and partial-wave expansion

1°. Consider the configuration in Fig. 1, where the masses of A and C are unequal, whereas the masses of A and C are equal to those of B and D, respectively:

\[ m_A = m_B, \quad m_C = m_D, \quad m_A \neq m_C. \]  

Mandelstam variables, \( s \) and \( t \) are defined as follows:

\[ t = (p^C - p^A)^2 = \sum_{\rho=0}^{3} Q_{\rho} Q_{\rho} = Q_s^2 - Q^2, \]

\[ s = (p^A + p^B)^2. \]  

When \( t = 0 \) and \( m_A \neq m_C \), we cannot bring \( (Q_{\rho}) = (Q_0, Q_1, Q_2, Q_3) \) into \( (0, 0, 0, 0) \) by any Lorentz transformations, and we take \( (0, 0, q, q) \) as a representative of \( (Q_{\rho}) \). All the inhomogeneous Lorentz transformations which leave the above representative vector invariant form a group—two-dimensional Euclidean group \( E_2 \). In particular we may take generally \( p^A = (0, 0, m_A) \). Hence \( p^C = (0, 0, q, q + m_A) \), and using \( (p^C)^2 = m_C^2 \), \( q \) is given as \( q = (m_C^2 - m_A^2)/2m_A \).

An element of \( E_2 \) is written in general as

\[ S = \begin{pmatrix} e^{-i\beta} & (a + bi) e^{i\beta} \\ 0 & e^{i\beta} \end{pmatrix}, \quad a, b \in \mathbb{R}, \quad 0 \leq \beta < 2\pi. \]

From this, \( p^A \) is transformed into \( p^B = S p^A S^+ \) where \( p^A = -p^A \sigma^\rho \sigma_\rho^T \) and \( \sigma_\rho \) are the Pauli matrices (\( \sigma_0 \) is a unit matrix), etc., that is

\[ \begin{pmatrix} p_0^B + p_3^B \\ p_1^B - i p_2^B \\ p_2^B \\ p_3^B \end{pmatrix} = S \begin{pmatrix} p_0^A + p_3^A \\ p_1^A - i p_2^A \\ p_2^A \\ p_3^A \end{pmatrix} S^+. \]

In the \( 4 \times 4 \) matrix notation, we have

\[ \begin{pmatrix} p_1^B \\ p_2^B \\ p_3^B \\ p_4^B \end{pmatrix} = \begin{pmatrix} \cos 2\beta & \sin 2\beta & -a & a \\ -\sin 2\beta & \cos 2\beta & b & -b \\ a \cos 2\beta + b \sin 2\beta & b \cos 2\beta + a \sin 2\beta & \frac{1}{2} (2 - a^2 - b^2) & \frac{1}{2} (a^2 + b^2) \\ a \cos 2\beta + b \sin 2\beta & b \cos 2\beta + a \sin 2\beta & \frac{1}{2} (a^2 + b^2) & \frac{1}{2} (2 + a^2 + b^2) \end{pmatrix} \begin{pmatrix} p_1^A \\ p_2^A \\ p_3^A \\ p_4^A \end{pmatrix}. \]

In the case \( p^A = (0, 0, 0, m_A) \), \( p^B \) becomes

\[ \begin{pmatrix} p_1^B \\ p_2^B \\ p_3^B \\ p_4^B \end{pmatrix} = \begin{pmatrix} a m_A \\ b m_A \\ \frac{1}{2} m_A (a^2 + b^2) \\ \frac{1}{2} m_A (2 + a^2 + b^2) \end{pmatrix}. \]

By putting \( (a, b) = (r \cos \alpha, r \sin \alpha) \), we have \( s = m_A^2 (a^2 + b^2 + 4) = m_A^2 (r^2 + 4) \).
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Hence to take the asymptotic limit in $s \to \infty$ is equivalent to that in $r \to \infty$ unless $m_A$ vanishes.

2°. Let us now turn to the partial-wave expansion of the scattering amplitude $f(s, \ell)$. Take the Lorentz transformation belonging to a little group and transform $p^A$ into $p^B$. Then the partial-wave expansion is considered as a description of the behavior of the scattering amplitude for the transformation according to the unitary irreducible representations of that little group. For instance in the usual partial-wave expansion the little group is $SO(3)$, hence the amplitude is expanded in terms of the spherical harmonics $Y_{lm}(\theta, \phi)$—the basis of the $SO(3)$-representation. Here $\theta$ and $\phi$ are the parameters specifying the Lorentz transformation which brings $p^A$ into $p^B$. In our case the little group is $E_2$, therefore $f(s, \ell)$ is considered as a function over $E_2$ and the partial-wave expansion is defined according to the unitary irreducible representations of $E_2$ as follows:

$$ f(s, \ell = 0) = f(\theta, x) = f(\theta, r, \alpha) $$

$$ = \frac{1}{2\pi} \sum_{m = -\infty}^{\infty} \exp(in\theta) \int_{0}^{2\pi} \int_{0}^{\infty} d\rho \sin \phi \exp[-i\rho \cos(\phi - \alpha)] F_n(\rho, \phi). $$

In fact $f(s, \ell)$ does not depend on $\theta$ and $\alpha$, because by using the inverse formula, we find $F_n = 0$ unless $n = 0$;

$$ f(s) = f(s, \ell = 0) $$

$$ = \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} d\rho \sin \phi \exp[-i\rho \cos \phi] F_0(\rho, \phi). \quad \text{(3)} $$

§ 4. Some properties of partial-wave expansion

We reasonably assumed that $F_0(\rho, \phi)$ does not depend on $\phi$, because $f(s)$ is independent of $\alpha$. Then (3) is rewritten as

$$ f(s) = \frac{1}{2\pi} \int_{0}^{\infty} \int_{0}^{2\pi} d\rho \sin \phi \exp[-i\rho \cos \phi] F(\rho) $$

$$ = \frac{1}{2\pi} \int_{0}^{\infty} d\rho J_0(\rho \ell) F(\rho) $$

$$ = \frac{1}{2\pi} \int_{0}^{\infty} d\rho J_0(\rho \ell) \left[ F(\rho) \right]. \quad \text{(4)} $$

This is the required form of the partial-wave expansion according to $E_2$.

We examine some properties of this expansion. For this purpose we have, using the inverse formula,
\[ \tilde{f}(\rho) = \frac{1}{2\pi} \int_{0}^{\infty} dr \, r J_{\rho}(\rho r) f(r). \]  

First we will see that the Regge behavior is really picked out by the expansion (4). By the term “Regge behavior” we mean that \( f(s) \) behaves asymptotically like some powers of \( s \), i.e. \( f(s) \sim s^{l} \) as \( s \to \infty \). If we put \( f(r) = r^{l} \), then owing to the integral formula of the Bessel functions, (5) becomes

\[
\tilde{f}(\rho) = \begin{cases} 
1/2\pi \rho & \text{for } l = -1, \\
\frac{1}{4\pi^{\rho^{2}}\rho^{l+1}} \Gamma(-l) \Gamma\left(\frac{1+l}{2}\right) \Gamma\left(-\frac{l}{2}\right) \sin(l+2)\pi & \text{for } -2 < l < -1, \\
\text{etc.}
\end{cases}
\]  

and vice versa; that is, the form of \( \tilde{f}(\rho) \sim 1/\rho^{l+1} \) gives that of \( f(r) \sim r^{l} \). As stated in §2, \( \tilde{f}(\rho) \) can be defined if \( f(r) \) is square integrable, i.e. \( l < -1 \). Moreover the expression (6) shows that \( l = -1 \) is included. Hence noting that \( r \sim \sqrt{s} \) for large \( s \), we can pick up all Regge trajectories for \( J \leq -1/2 \) by the expansion (4). However at the present time the present author cannot extract this behavior as “poles”.

Secondly we will show that the expansion (4) is essentially the impact parameter representation with \( \rho \) as the impact parameter. In fact for the \( t \)-channel \( (t > 0, s < 0) \), the impact parameter representation is

\[ f(s, t) = \sum_{n} \alpha(b, t) J_{n}(\sqrt{-s} \, b) b \, db, \]  

where \( b \) is an impact parameter and \( \alpha(b, t) \) is the scattering amplitude in the \( b \)-representation. From this we immediately conclude that (7) as \( t \to 0 \) is nothing but (4) except the difference of the minus sign before \( s \). This difference comes from the fact that we make \( t \) tend to zero from the positive value and it is not essential.\(^{8)}\) In the representation (7), we can observe the Regge behavior putting, à la R. Blankenbecler and M. L. Goldberger.\(^{9)}\)

\[ \alpha(b, t) = N(b, t) D^{-1}(b, t). \]

If \( N(b, t) \) approaches a constant and

\[ D(b, t) \sim b^{2 \alpha(t)} C(t), \]

as \( b \to 0 \), then

\[ f(s, t) \sim \tilde{\beta}(t) \left(-s\right)^{\alpha(t)} \left[\sin \pi \alpha(t)\right]^{-1}. \]

This shows the Regge behavior.

\(^{8)}\) The aspect that the \( E_{\rho} \)-representation assumes the similitude of the impact parameter representation is more apparent, if we see in the \( u \)-channel.
§ 5. Summary and comments

The structure of the solvable Lie group $E_2$ is rather different from and simpler than the usual semi-simple Lie groups. The partial-wave expansion based on the unitary irreducible representations of $E_2$ also picks out the Regge behavior, but unfortunately at the present time the author cannot exhibit the Regge behavior as "poles". An answer to the question whether this representation can be used in discussing the daughter trajectories and the consistent way of observing the connection between the Regge theory and this representation through "eikonal" method are under preparation along the line of $N$ over $D$ method and soon appear. Anyway, the range of the trajectories which we can pick out is restricted to and below $-1/2$ (cf. Fig. 2). This is the unfavorable feature of the partial-wave expansions based on the groups other than $SO(3)$.

Acknowledgements

The author would like to express his deep gratitude to Dr. T. Shintani for helpful mathematical discussions. He would also like to thank his colleagues of the laboratory of elementary particles for valuable discussions.

References

4) S. Ito, Nagoya Math. J., Nagoya University, 5 (1953), 79.
7) For the representations of the generalized Lorentz groups, see the next marvellous works.
He has obtained the unitary irreducible representations of the generalized Lorentz groups $SO(p, q)$ (to be published).