\[ U(s) - U_0(s) = \sum_{i=1}^{N} H_i(-s) + \int_{0}^{\infty} d\tau \left[ g_i(\tau) \right] \]

\[ + \int_{0}^{\infty} \sum_{i=1}^{N} c_i e^{-b_i(\tau - s)} \alpha_i(\sigma - \omega) d\sigma - x_d(\tau) \]

\[ k_i \sum_{n=1}^{N} e^{-\alpha_n(\sigma - \omega)} + \sum_{i=1}^{N} \int_{0}^{\infty} d\tau \left[ g_i(\tau) \right] \]

\[ + \int_{0}^{\infty} \sum_{i=1}^{N} c_i(-b_i) \alpha_i(\sigma - \omega) \alpha_i(\sigma - \omega) \alpha_i(\sigma - \omega) \alpha_i(\sigma - \omega) \]

\[ k_i \sum_{n=1}^{N} e^{-\alpha_n(\sigma - \omega)} e^{-b_i(\tau - s)} \alpha_i(\sigma - \omega) \alpha_i(\sigma - \omega) \alpha_i(\sigma - \omega) \alpha_i(\sigma - \omega) \]  

where

\[ H_i(-s) = \sum_{n=1}^{N} k_i \int_{0}^{\infty} d\tau \left[ g_i(\tau) - x_d(\tau) \right] \]

and for \( i = 2, \ldots, N \)

\[ H_i(-s) = \sum_{n=1}^{N} k_i \int_{0}^{\infty} d\tau \left[ g_i(\tau) \right] \]

Since the values of the integrals in equations (68) and (69) are constants, the right-hand sides in these equations are functions of \( -s \) only.

In equation (67), substituting for \( g_i \) and interchanging the order of integration and carrying out the integration gives

\[ U(s) = U_0(s) + k_i W_i(-s) X_i^*(s) \]

\[ - \sum_{i=1}^{N} k_i W_i(-s) G_i^*(s) + k_i W_i(s) W_i(-s) U(s) + H(-s) \]

where the quantities \( W_i(s) \), \( G_i^*(s) \), and \( X_i^*(s) \) are defined in equations (10), (20), and (21) and

\[ H(-s) = \sum_{i=1}^{N} H_i(-s) \]

Acknowledgment

The author thanks Dr. Raymond E. Goodson of the School of Mechanical Engineering, Purdue University, for his valuable discussions.

References


Discussion

R. Oldenburger

Considering the frequent occurrence of transportation lags in engineering applications, the treatment by H. C. Khatri of such systems is a much needed addition to the theory of optimal control. In Khatri's paper no bounds on the controlling variables are assumed. The problem remains of treating systems where the controlling variables are bounded. The case where the pure delay is in a feedback path has already been treated by G. L. Kharatishvili [13] and N. N. Krasovskii [14]. Here the delay is in the controlled variable or variables, and not the controlling variables.

In equations (2) and (9) Khatri uses as an index of performance an integral of a quadratic form that is a linear combination of squares of the errors in the controlling and controlled variables and derivatives of the controlled variable. Although this is a satisfactory index of performance, chosen for mathematical convenience, the practical control engineer is primarily concerned with minimizing the maximum error in the controlled variable and simultaneously optimizing other quantities such as over-swings and under-swings [15]. A solution of this problem would be most valuable. The two solutions may prove to be almost identical.

It will not be clear to the average reader how Dr. Khatri obtained equation (4). It is hoped that he will give a thorough derivation of this equation in his closure.

Additional References


H. R. Sebesta

The author provides an interesting and different approach to the problem of determining an optimal controller for systems with input delays. However, it is pointed out that the same problem is solved by Merriam [17, 18] in a manner which gives a feedback form for the optimal controller. The advantages of a closed-loop system over the open-loop control implied by equations (32) of the paper under discussion may be argued with some justification.

Actually, it is possible to show that a linear feedback system may be used to optimally control processes that are much more general than either those cited by Merriam, or those described

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H. R. Sebesta

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in the paper. In particular, we can consider (see references [19], [20]) systems described by the linear vector equation.

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t - h(t)) \]  

where \( A(t) \) and \( B(t) \) are continuous \( n \times n \) and \( n \times m \) matrices, respectively. The time delay \( h(t) \) is a continuous function satisfying the conditions

\[ h(t) \geq 0 \text{ and } \frac{dh(t)}{dt} < 1. \]  

The optimal control law which results in a minimum for the quadratic criteria

\[ J = \frac{1}{2} \int_{0}^{T} [x(T')Q(T)x(T') + u'(T')W(T)u(T')]dT \]  

is

\[ u(t) = -W^{-1}(t)B(t)'\psi(t)\xi(t) \]

where \( x(t) \) is the solution of the equation

\[ \dot{\psi}(t) = A(t)\psi(t) + B(t)x(t), \quad 0 \leq t \leq T - h(T), \]

\[ u(t) = -\psi(t)B(t)'x(t), \quad 0 < t < T, \]

\[ u(t) = 0, \quad T - h(T) < t \leq T, \]

The n \( \times \) n matrix \( R \) is the solution of the nonlinear Riccati equation

\[ \dot{R} - FR + RA + Q = 0; \quad \xi(t_0) \leq t \leq T, \]

with the boundary condition

\[ R(T) = S \]  

and

\[ F \triangleq \frac{B(t)W^{-1}(t - h(t))R(t)'}{1 - h(t)} \]

The function \( \psi(t, t) \) is the transition matrix and may be expressed as

\[ \psi(t, t) = \psi(t, t_0)\varphi(t, t_0), \quad t_0 \leq t \leq T, \]

\[ \psi(t, t_0) = \psi(t, t_0), \quad t_0 \leq t \leq T, \]

where \( \varphi(t, t) \) is the solution of the equation

\[ \frac{d\varphi(t, t)}{dt} = A(t)\varphi(t, t) \]

and \( \bar{\psi}(t, t) \) is that of

\[ \frac{d\bar{\psi}(t, t)}{dt} = A(t)\bar{\psi}(t, t) \]

for the respective boundary conditions

\[ \varphi(t, t) = I \quad \text{and} \quad \bar{\psi}(t, t) = I. \]

The Riccati equation for the sample problem of the paper is

\[ \dot{R} - R^2 - 2bR + 1; \quad a \leq t \leq T. \]

The solution for the boundary condition,

\[ R(T) = 0; \]

as given by Athans and Falb [16], is

\[ R(t) = \frac{\sqrt{b^2 + 1} - b + (\sqrt{b^2 + 1} + b)S(t-T)1 - p_0^2\sqrt{b^2 + 1}1 - \psi(t, t)}{1 - p_0^2\sqrt{b^2 + 1}(t - T)}. \]

This will reduce to

\[ R = \sqrt{b^2 + 1} - b \]

when \( T \to \infty \). Corresponding to equations (81) and (82), we have, for example of the paper,

\[ \psi(t + a, t) = e^{-\sqrt{b^2 + 1}a}, \quad 0 \leq t \leq a, \]

\[ \psi(t + a, t) = e^{-a\sqrt{b^2 + 1}}, \quad t > a. \]

Thus, the optimal-control law takes the form

\[ u(t) = -K(t)x(t) \]

where

\[ K(t) = (-b + \sqrt{b^2 + 1})e^{-a\sqrt{b^2 + 1}}, \quad t > a. \]

Note that for the time lag \( a \to 0 \) the feedback gain is in-variant throughout the entire process. This is in agreement with results given elsewhere [16, 17].

In general, it has been established that for linear systems, with or without time lags in the control vector, the optimal control for a quadratic performance index is a linear combination of the state components. For systems with lags, it is necessary to qualify this statement for the case when

\[ u(t) = g(t), \quad g(t) \neq 0, \quad t < t_0. \]

In this instance, the optimal control over the interval \( t_0 \leq t \leq \xi(t_0), \] i.e., may be expressed as

\[ u(t) = u_0(t) + u_0(t) \]

where \( u_0(t) \) is defined by equation (75) and \( u_0(t) \) is a function which depends on \( g(t) \). Over the remaining time of operation i.e., \( t \geq \xi(t_0) \), the optimal control law expressed in equation (75) remains valid.

Additional References


Author’s Closure

The author wishes to thank Professors Oldenburger and Sebesta for their interesting and enlightening comments.

Dr. Oldenburger raises an important question about optimal control of systems with transportation lag and bounded control variables. The results given in this paper are certainly not valid if the control variables are bounded. However, if the problem is formulated as in this paper it is possible, in principle, to solve for the optimal bounded control variable using the maximum principle given by Butkovskii [21]. It may be noted that in this case

\[ J = 1/2 \int_{0}^{T} [x(T')Q(T)x(T') + u'(T')W(T)u(T')]dT \]

Additionally, the author notes that the results given in this paper are certainly not valid if the control variables are bounded. However, if the problem is formulated as in this paper, it is possible, in principle, to solve for the optimal bounded control variable using the maximum principle given by Butkovskii [21]. It may be noted that in this case

\[ J = 1/2 \int_{0}^{T} [x(T')Q(T)x(T') + u'(T')W(T)u(T')]dT \]

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regimes of singular trajectories occur in a manner similar to that given in [22] for systems described by ordinary differential equations. These singular trajectories for systems described by integral equations are being investigated.

The author concurs with Dr. Oldenburger about the need for further investigation of selection of performance index for control systems.

Following Dr. Oldenburger's suggestion, details of the derivation of equation (4) are now given. In Appendix I it is stated that integration by parts of equation (52) yields equation (54) when the natural boundary condition, equation (4), is imposed. If the natural boundary condition is not imposed, equation (54) takes the form

\[
\int_{a}^{T} \left( \frac{\partial F}{\partial x}(x^*, u^*) \right) dt + \int_{a}^{T} W(t, r) n(t - r) dt = \int_{0}^{T} \left( \frac{\partial F}{\partial u}(x^*, u^*) \right) dt = 0
\]

Equation (54a) differs from equation (54) by the last term in \(x\):

\[
\text{Since } n(r) \text{ is arbitrary, equation (94) is satisfied if and only if [23]}
\]

\[
\left. \frac{\partial F}{\partial u} \right|_{t = T} = 0
\]

This is the natural boundary condition given in equation (4).

Dr. Sebesta has presented an excellent discussion for the solution of the problem considered in this paper by using differential equation model instead of the integral equation model of the system. Dr. Sebesta obtains the control law by first solving the nonlinear Ricatti equation, (78), and then substituting the results in equation (75) which is in terms of the transition matrix \(\Psi\).

The author feels that if the index of performance is an integral with a quadratic integrand it is easier to solve for the control law by first obtaining the control function using equation (32). Note that equation (32) is obtained directly from the system transfer function and the integrand of the index of performance. Dr. Sebesta has given the control law for the example considered in this paper. At first glance it might appear as though the two results are different. However, the results are identical because

\[
-b + (b^2 + 1)^{1/2} = \frac{1}{b + (b^2 + 1)^{1/2}}
\]

The system with time varying delay \(h(t)\) considered by Dr. Sebesta is certainly more general than the one treated in this paper. A necessary condition for optimal control of this more general system is now given in terms of the system transition matrix. The integral equation representation of equation (72) is given by [24]

\[
x(t) = \psi(t, u_0)(u_0) + \int_{0}^{t} \psi(t, r) B(r) u(r - h(r)) dr
\]

where the variables are as defined by Dr. Sebesta. It is assumed that

\[
u(r - h(r)) = 0 \quad \text{for} \quad r - h(r) < 0
\]

For minimization of index of performance given by equation (2) subject to the integral equation constraint (96) it is necessary that the controller satisfy the following necessary conditions

\[
\frac{\partial F(t)}{\partial x} = - \frac{d \xi(t)}{dt} B^T(\xi(t)) \int_{0}^{T} \psi'(r, \xi(t)) \frac{\partial F(r)}{\partial x} dr
\]

\[
\frac{\partial F(t)}{\partial u} = 0
\]

where the argument \(\xi(t)\) is obtained by inverting equation (77). The derivation of these necessary conditions is very similar to the derivation of necessary conditions for a system with time-invariant delay given in Appendix I.

Additional References