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A note on Newton's shell-point equivalency theorem

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An isotropic sphere or shell of matter can be regarded as acting as a point mass at its center so far as its gravitational force on an external test mass is concerned. This is well-known, but what is often not appreciated is that this behavior also holds for a central potential of the harmonic form $V(r) \sim r^2$. This paper proves this assertion and shows that *only* the gravitational and harmonic potentials possess this property.
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I. INTRODUCTION

Newton's theorem that an isotropic shell or sphere can be regarded as acting as a point mass located at its center when computing the force it exerts on an external test mass is a cornerstone of his gravitational theory and its application to celestial mechanics. This beautiful result can be regarded as a manifestation of the inverse-square nature of the gravitational force or, equivalently, of its corresponding inverse-first-power potential. Richard Feynman describes this theorem in almost supernatural terms: "When we add the effects all together, it seems a miracle that the net force is exactly the same as we would get if we put all the mass in the middle!"[1]

Proofs of this theorem and its corollary that a test mass trapped within a uniform spherical shell experiences no net force can be found in Feynman and many mechanics and dynamics texts in common use; these usually proceed by integrating the potential and then extending the argument to a solid sphere.[2–7] An exception is Goldstein's venerable *Classical Mechanics*, where he probably considered the proof too trivial for this more advanced text.[8] Griffiths analyzes the equivalent electrostatic case for a uniform shell of electrical charge via both Gauss's law and direct integration of the potential, leaving to an exercise a direct integration of the electric field.[9] Another approach to the interior of a uniform shell beng force-free is to imagine an off-center test mass equipped with an observer who looks at patches of the shell in opposite directions which subtend the same solid angle. Since the areas of the patches will be proportional to the squares of the observer's distances to them but the forces they exert depend on the inverse squares of these distances, the forces exerted by the two patches are equal and cancel; the argument is then extended to cover the entire sphere. If the force were not inverse-square, this argument would obviously break down and the test mass would experience a net force; this point will be revisited later. This force-free behavior inside the shell is, however, entirely irrelevant to Newton's issue of wanting to prove that the force exerted by the shell on an *external* test mass acts as if the entire mass of the shell were concentrated at its center; *a priori*, there is no reason that gravity *has* to be inverse-square.

More advanced students may be aware that if gravity were instead a *linear* attractive force, planetary orbits would still be closed ellipses, albeit with the attracting mass being located at the geometric center of the ellipse as opposed to being at one of the foci; this motion also arises with a mass attached to springs that extend off to the x and y directions. While these are obviously quite different scenarios, it is striking that both inverse-square and linear forces give rise to closed elliptical orbits. It can be shown that the gravitational and harmonic potentials are, in fact, the *only* integer-power ones that can support closed orbits. This behavior is now regarded as a manifestation of Bertrand's theorem, which received its own section in the second edition of Goldstein's book. A paper recently published in this journal offers a more advanced look at the dynamical symmetries underlying this powerful theorem.[10] However, none of these sources point out that the harmonic potential also satisfies shell-point equivalency just like its gravitational cousin, and that it is the only other power-law potential which does so. A proof of this assertion is a straightforward extension of the usual gravitational case and is developed here. This is not a new result, but it seems not to be well known in the physics teaching community. That this is the case first came to my attention when it appeared in a paper on cosmology, although this is likely not the first source to point this out.[11]

The purpose of this paper is to bring this behavior to wider attention. To give this development a motivation for classroom use, pose the following situation to your students: "Imagine that you are Newton, but you can use the full machinery of calculus and potential theory now taught to physics students. You hypothesize that gravity is a central power-law force such that the force exerted by an element of mass dM of a uniform shell on an external test mass m is $\boldsymbol{F} = +Knm(dM)r^{n-1}\hat{\boldsymbol{r}}$, where r is the element-to-test mass distance and $\hat{\boldsymbol{r}}$ is a unit vector from the element to the test mass. The potential energy is then $U(r) = -Km(dM)r^n$, and the potential function is $V(r) = U(r)/m = -K(dM)r^n$. Now consider these two questions. First, for what value(s) of the power n does point-shell equivalency hold? Second, of these possible values, which gives a period-distance behavior consistent with Kepler's Third Law, $T^2 \propto r^3$? For this latter question, assume a circular orbit and invoke a simple centripetal force argument." Note that if $K < 0$ and $n < 0$ ($n > 0$) the force will be repulsive (attractive) and the reverse for $K > 0$, but these considerations do not affect the conclusions.

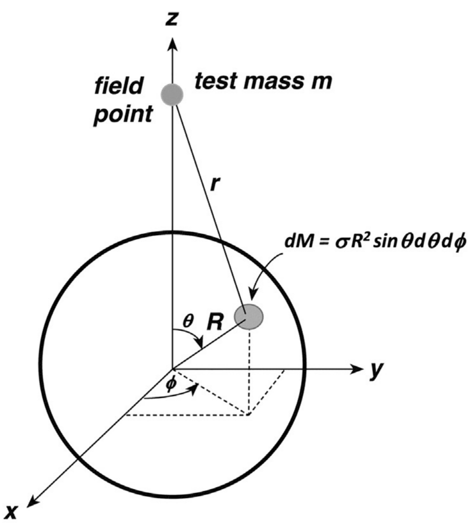

Fig. 1. Isotropic shell of radius R and areal mass density σ centered on the origin. A field point is located at distance z along the z-axis.

II. SHELL-POINT EQUIVALENCY

See Fig. 1. A spherical shell of radius R and areal mass density σ is located with its center at the origin. As usual, a field point is located at distance z from the center along the z-axis, and a patch of surface of area of the shell $dA = R^2 \sin\theta\, d\theta\, d\phi$ is located at spherical coordinates (θ, ϕ). The patch will have mass $dM = \sigma R^2 \sin\theta\, d\theta\, d\phi$. Let the patch-to-field-point distance be r; from the law of cosines

$$r^2 = z^2 + R^2 - 2zR\cos\theta. \tag{1}$$

As described above, the potential is written in the generalized power-law form $V = -K(dM)r^n = -K\sigma R^2 r^n \sin\theta\, d\theta\, d\phi$, where n is intended to be an integer. The units of K will depend on the choice of n; when $n = -1$, set $K \to G$ to recover the Newtonian case. The total potential at the field point is, after accounting for a factor of 2π when integrating over ϕ,

$$V = -2\pi K\sigma R^2 \int_0^\pi r^n \sin\theta\, d\theta. \tag{2}$$

Setting $a = z^2 + R^2$, $b = 2zR$ and making the substitution $w = b\cos\theta$ simplifies this to

$$V = \frac{\pi K\sigma R}{z}\int_b^{-b} (a - w)^{n/2} dw. \tag{3}$$

For $n \neq -2$ (which is discussed below), this integral is straightforward and reduces to

$$V = -\frac{KM}{2(n+2)zR}\left[(z+R)^{n+2} - (z-R)^{n+2}\right], \tag{4}$$

where M is the total mass of the shell. This form assumes $z > R$; if $z < R$, replace $(z - R)$ within the square brackets by $(R - z)$. For $K = G$ and $n = -1$, this reduces to the standard form $-GM/z$.

Assume that $n > -2$. The two terms within the square bracket can then be expanded using the binomial theorem

$$(a + x)^N = \sum_{j=0}^{N}\binom{N}{j} x^j a^{N-j}$$

to give

$$V = -\frac{KM}{2(n+2)}\sum_{j=0}^{n+2}\binom{n+2}{j}\left[1 - (-1)^j\right] z^{n+1-j} R^{j-1}. \tag{5}$$

The force on the test mass is then

$$\begin{aligned} \boldsymbol{F} &= -m\left(\frac{\partial V}{\partial z}\right)\hat{z} \\ &= \frac{KMm}{2(n+2)}\sum_{j=0}^{n+2}\binom{n+2}{j}\left[1 - (-1)^j\right] \\ &\quad \times (n+1-j) z^{n-j} R^{j-1}\,\hat{z}. \end{aligned} \tag{6}$$

Clearly, no even-j terms will survive because of the term in square brackets.

Shell-point equivalency means that all terms in $\boldsymbol{F}$ must be independent of R. Cases of interest can be analyzed very directly:

(a) If $n = -1$, the only surviving term in the sum will be that for $j = 1$, giving $\boldsymbol{F} = -(KMm/z^2)\hat{z}$, the Newtonian result.

(b) For $n = 0$, again only $j = 1$ contributes, but the factor of $(n + 1 - j)$ within the sum vanishes. There is no net force, exactly as one would expect for $V = $ constant.

(c) For $n = +1$, there are non-vanishing terms in the sum for $j = 1$ and $j = 3$. The first of these is independent of R as in case (a) above, but the second gives a term of the form R^2/z, destroying point-shell equivalency.

(d) For $n = +2$, the harmonic potential, the non-vanishing terms in the sum are again $j = 1$ and $j = 3$. In this case, however, the contribution from the $j = 3$ term vanishes because $(n + 1 - j)$ reduces to zero. The $j = 1$ term yields $\boldsymbol{F} = (2KMmz)\hat{z}$, which possesses point-shell equivalency. If $K > 0$ as with the Newtonian case, this will be a *repulsive* force. This might seem strange but can be understood via energy conservation. The potential in this case is $V = -KMz^2$. A repulsive force will endow a test mass with increasing kinetic energy; as z increases, the potential energy must correspondingly become more negative.

(e) For $n > 2$, there will be terms arising from $j = 1, 3, 5, \ldots$, but terms from the factor of R^{j-1} in Eq. (6) will survive, destroying point-shell equivalency.

As for other possibilities, setting $n = -2$ in Eq. (3) leads to a logarithmic form for V where terms in R remain; similarly, for $n < -2$, the terms in the potential cannot be brought to a common denominator without the appearance of offending R-terms. For non-integer values of n, the finite binomial expansions will become infinite binomial series, not all of the terms of which can be forced to vanish through the factors of $(n + 1 - j)$, so point-shell equivalency will be impossible. Only $n = -1$ and $n = +2$ can give equivalency.

The question regarding Kepler's third law is easier to deal with. For a satellite of mass m in a circular orbit of speed v and radius r around an attractor mass, the magnitude of the centripetal force must be mv^2/r. Set $v = 2\pi r/T$ to get this in terms of the period T. For our general power law, the magnitude of the force is $|Kn|mMr^{n-1}$, which gives

$$T^2 = \frac{4\pi^2}{|Kn|M} r^{2-n}. \tag{7}$$

Only $n = -1$ satisfies the empirical form of Kepler's third law. For the harmonic case where $n = 2$, the period is *independent* of the orbital radius, analogous to how the period of a mass-spring system is independent of its initial displacement.

As implied in the Introduction, another point of difference between the harmonic and Newtonian cases is that the net force is *not* zero within the shell for the harmonic case: $\boldsymbol{F} = -(2KMmz)\hat{z}$. If $K > 0$ (< 0), the force will be attractive toward (repulsive from) the center with the result that the center will be a point of stable (unstable) equilibrium. Worthwhile student exercises might be to check this by direct integration and also to use the solid-angle reasoning to confirm the stability/instability conclusion.

That the harmonic and gravitational potentials both give closed orbits and shell-point equivalency is presumably not a coincidence; I leave it to readers with more advanced command of dynamics to explore what underlying similarities make this so. Sometimes, a fresh look at an old problem reveals new insights.

I am grateful for comments and suggestions from reviewers who considered an earlier version of this paper. In particular, one reviewer encouraged me to consider the development of Eq. (6) more closely.

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[1]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. I, Sec. 13.
[2]J. B. Marion, *Classical Dynamics of Particles and Systems*, 2nd ed. (Academic Press, New York, 1970), Chap. 8.
[3]A. P. French, *Newtonian Mechanics* (W. W. Norton, New York, 1971), Chaps. 11 and 13.
[4]G. R. Fowles, *Analytical Mechanics*, 3rd ed. (Holt, Rinehart and Winston, New York, 1977), Chaps. 3 and 5.
[5]A. P. Arya, *Introduction to Classical Mechanics* (Prentice Hall, Englewood Cliffs, NJ, 1990), Chaps. 7 and 10.
[6]S. T. Thornton and J. B. Marion, *Classical Dynamics of Particles and Systems*, 5th ed. (Brooks/Cole-Thomson Learning, Belmomt, CA, 2004), Chap. 8.
[7]R. D. Gregory, *Classical Mechanics* (Cambridge U. P., Cambridge, 2006), Chap. 3.
[8]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950), Chap. 3.
[9]D. J. Griffiths, *Introduction to Electrodynamics*, 2nd ed. (Prentice Hall, Englewood Cliffs, NJ, 1989), Chap. 2.
[10]C. Carimalo, "Dynamical symmetries behind Bertrand's theorem," Am. J. Phys. **89**(11), 1012–1017 (2021).
[11]V. G. Gurzadyan, "The cosmological constant in the McCrea-Milne cosmological scheme," Observatory **105**, 42–43 (1985).

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Reference 31 of the original paper[1] contains two arithmetical errors and one bibliographical error. The conclusion, however, remains correct. The footnote should read:

The book *The Second Law* by P. W. Atkins (Scientific American Books, New York, 1984) promotes the idea that entropy is a measure of homogeneity. (This despite the everyday observation of two-phase coexistence.) To buttress this claim, the book presents five illustrations (on pages 72, 74, 75, and 77) of "equilibrium lattice gas configurations." Each configuration has 100 occupied sites on a 40×40 grid. If the occupied sites had been selected at random, then the probability of any site being occupied would be 100/1600, and the probability of any given pair of sites both being occupied would be $(1/16)^2$. The array contains $2 \times 40 \times 39$ adjacent site pairs, so the mean number of occupied adjacent pairs would be $2 \times 40 \times 39 \times (1/16)^2 = 12.19$. The actual numbers of occupied adjacent pairs in the five illustrations are 7, 3, 7, 4, and 3. A similar calculation shows that the mean number of empty rows or columns in a randomly occupied array is $2 \times 40 \times (15/16)^{40} = 6.05$. The actual numbers for the five illustrations are 5, 5, 4, 4, and 0. I am confident that the sites in these illustrations were occupied not at random, but rather to give the impression of uniformity.

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[1]D. F. Styer, "Insight into entropy," Am. J. Phys. **68**, 1090–1096 (2000).