Polarization of Transverse Seismic Waves

Markus Båth

"And perpendicular now and now transverse,
Pierce the dark soil and as they pierce and pass
Make bare the secrets of the Earth's deep heart."

P. B. Shelley, Prometheus Unbound.

Summary

A theoretical investigation is made of the changes of the polarization of transverse seismic waves during their propagation through the Earth. The polarizations have been computed theoretically and numerically for reflexion at the core boundary and at the Earth's surface, for refraction and reflexion at the base of the crust and for passages through continuously varying media. It is demonstrated that great changes of the vibration properties (vibration angle and particle orbit) may occur in all cases except for continuously varying media, through which transverse waves propagate with practically unchanged vibration properties. The consequences of these results for earthquake mechanism studies, based on transverse waves, are discussed.

1. Introduction

Transverse seismic waves or S waves are composed of two components, $SH$ and $SV$. In studies of S-wave propagation and especially of the behaviour at discontinuity surfaces it is most convenient to investigate $SH$ and $SV$ separately. The equations governing reflexion and refraction of $S$ waves are of such a nature that a separate treatment of $SH$ and $SV$ is possible. But the actual particle motion is the result of $SH$ and $SV$ motions, and a combination of $SH$ and $SV$ is necessary in studies of the $S$-wave polarization.

In this paper we shall present a theoretical investigation of $S$-wave polarization and especially its changes upon reflexion or refraction at discontinuity surfaces in the Earth as well as during passages through continuously varying media. The results may be of interest in elucidating some wave propagation phenomena and particularly with regard to earthquake mechanism studies, using $S$ waves.

2. Notation

Reflexion and refraction of seismic waves are dealt with in most textbooks in seismology, and therefore the fundamental ideas need only be briefly mentioned here. In our investigation of reflexion and refraction of $S$ waves we shall apply
the method used by Jeffreys (1926), and our notation will be almost the same as used by Jeffreys:

$x, y, z =$ rectangular coordinates; the waves propagate in the $xz$ plane and $z = 0$ coincides with a discontinuity surface;

$u, v, w =$ displacements along $x, y, z$ respectively;

$\Phi, \Psi =$ potentials, such that

$$u = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial z} \quad \text{and} \quad w = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi}{\partial x};$$

(1)

$\lambda, \mu =$ Lamé's parameters;

$\rho =$ density;

$c_P, c_S =$ wave velocities for $P, S$ respectively;

$t =$ time;

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2};$$

$p_{zx}, p_{xz}, p_{zy} =$ normal and tangential stresses acting on a discontinuity surface ($z = 0$);

$A, B, C =$ amplitude functions for $P, SV, SH$ respectively;

$A_{SH}, A_{SV} =$ amplitudes of $SH, SV$ respectively;

$e =$ angle of emergence

$\delta =$ vibration angle (see Section 3);

$\Delta =$ epicentral distance.

Quantities related to incident, reflected and transmitted (refracted) waves are denoted as follows:

incident waves: no accent or suffix ($B, C$, etc.),
reflected waves: suffix unity ($B_1, C_1$, etc.),
transmitted waves: accent ($B', C'$, etc.).

Similarly, quantities related to the incident side of a discontinuity are given with no accent ($c_P, c_S, \rho$, etc.), those belonging to the other side have an accent ($c_P', c_S', \rho'$, etc.).

Any part of $\Phi, \Psi, v$ can be written as follows with usual notation:

$$\begin{align*}
\Phi & \sim A \exp[i(k(az + x - \omega t))] \\
\Psi & \sim B \exp[i(k(\beta z + x - \omega t))] \\
v & \sim C \exp[i(k(\gamma z + x - \omega t))].
\end{align*}$$

(2)

$\Phi, \Psi, v$ satisfy the equations of motion:

$$\begin{align*}
\rho \frac{\partial^2 \Phi}{\partial t^2} & = (\lambda + 2\mu) \nabla^2 \Phi \\
\rho \frac{\partial^2 \Psi}{\partial t^2} & = \mu \nabla^2 \Psi \\
\rho \frac{\partial^2 v}{\partial t^2} & = \mu \nabla^2 v.
\end{align*}$$

(3)
Inserting the expressions (2) of $\Phi$, $\Psi$, $v$ into (3) we find

$$\omega^2 = cp^2(1 + a^2) = cs^2(1 + b^2) = cs^2(1 + \gamma^2)$$

$$= cp^2(1 + a'^2) = cs^2(1 + b'^2) = cs^2(1 + \gamma'^2).$$

Furthermore, $\alpha = \tan e (P \text{ wave})$, $\beta = \tan e (SV)$, $\gamma = \tan e (SH)$. The equations of motion evidently express only Snell’s refraction law. In case $\lambda = \mu$ (Poisson’s relation) holds, i.e. $c_P = c_S \sqrt{3}$, we get $\beta^2 = 2 + 3a^2$.

Those stress components, which will be used, have the following expressions:

$$p_{zz} = \lambda \nabla^2 \Phi + 2\mu \left( \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial z \partial x} \right)$$

$$p_{xx} = \mu \left( 2 \frac{\partial^2 \Phi}{\partial x \partial z} + \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial z^2} \right)$$

$$p_{xy} = \frac{\partial v}{\partial z}.$$  

(5)

Our calculations are valid only for plane wave fronts and for plane discontinuity surfaces. The results are therefore not valid near an earthquake focus, where the front cannot be assumed plane.

3. Determination of the vibration angle

The vibration plane is defined as a plane along the direction of wave propagation and oriented such that the vibration takes place in this plane. The vibration angle $\delta$ is the angle between the vibration plane and the vertical plane of propagation. Therefore, looking in the direction of propagation, we have for linearly polarized $S$ waves

$$\tan \delta = \frac{A_{SH}}{A_{SV}}.$$  

(6)

$\delta$ is counted clockwise from the vertical direction from $0^\circ$ to $180^\circ$ ($0^\circ \leq \delta \leq 180^\circ$).

In all our computations we assume the incident $S$ wave to be linearly polarized, whereas the reflected or refracted $S$ wave may be linearly or elliptically polarized, depending upon the angle of emergence. An elliptically polarized $S$ wave corresponds to a complex expression for $B_1/B$ or $B'/B$.

For an incident $S$ wave we have

$$SH : v = C \exp[i(x+\gamma z - \omega t)]$$

$$SV : \Psi = B \exp[i(x+\beta z - \omega t)].$$

(7)

The expressions for $v$ and $\Psi$ give us the amplitudes in the incident wave:

$$A_{SH} = C$$

$$A_{SV} = iKB(1 + \beta^2)^i$$

(8)

remembering that $A_{SV}$ is the maximum amplitude of $(\omega + \omega^2)^i$. We thus find

$$\tan \delta = \frac{C}{iKB(1 + \beta^2)^i}.$$  

(9)
Similarly, we find for a reflected $S$ wave:

$$\begin{align*}
v_1 &= C_1 \exp[i\kappa(x-\gamma z-\omega t)] \\
\Psi_1 &= B_1 \exp[i\kappa(x-\beta z-\omega t)]
\end{align*}$$

which give

$$\begin{align*}
\tan \delta_1 &= \frac{A_{SH1}}{A_{SV1}} = \frac{C_1}{i\kappa B_1(1+\beta^2)^{\frac{1}{2}}}
\end{align*}$$

and

$$\begin{align*}
\tan \delta_1 &= \frac{C_1/C}{B_1/B}.
\end{align*}$$

For a transmitted (refracted) $S$ wave we get in the same way:

$$\begin{align*}
v' &= C' \exp[i\kappa(x+\gamma' z-\omega t)] \\
\Psi' &= B' \exp[i\kappa(x+\beta' z-\omega t)]
\end{align*}$$

and

$$\begin{align*}
\tan \delta' &= \frac{A_{SH'}}{A_{SV'}} = \frac{C'}{i\kappa B'(1+\beta'^2)^{\frac{1}{2}}}
\end{align*}$$

and

$$\begin{align*}
\tan \delta' &= \frac{(1+\beta'^2)^{\frac{1}{2}}}{B'/B} = \frac{c'S'}{c'S B'/B'}.
\end{align*}$$

These formulae can be used for computation of the vibration angle only for linearly polarized waves. In case of elliptical polarization, i.e. $B_1/B$ or $B'/B'$ being complex quantities, we let the vibration plane coincide with the major axis of the ellipse. New formulae are then required for the vibration angle.

For a reflected, elliptically polarized $S$ wave we may put

$$\begin{align*}
\frac{A_{SH1}}{A_{SH}} &= \frac{C_1}{c} = C; \\
\frac{A_{SV1}}{A_{SV}} &= \frac{B_1}{c} = a+ib.
\end{align*}$$

The complex expression for the reflected $SV$ wave corresponds to an amplitude $=(a^2+b^2)^{\frac{1}{2}}$ and a phase shift $=\tan^{-1}(b/a)$ compared to the incident $SV$. The ellipse, representing the particle motion, has the following equation, $\xi$ and $\eta$ being rectangular coordinates in the plane of the ellipse with $\xi$ horizontal:

$$\frac{(a^2+b^2)A_{SV}^2}{c^2 A_{SH}^2} \xi^2 + \eta^2 - \frac{aA_{SV}}{c A_{SH}} \xi \eta = b^2 A_{SV}^2.$$  

We rotate the $\xi\eta$ axes so as to coincide with the axes of the ellipse, which leads to the following formula for $\delta_1$:

$$\tan 2\delta_1 = \frac{2ac \tan \delta}{a^2+b^2-c^2 \tan^2 \delta}.\quad (18)$$

Similarly, in case of an elliptically polarized, transmitted wave:

$$\begin{align*}
\frac{A_{SH'}}{A_{SH}} &= \frac{C'}{c} = C; \\
\frac{A_{SV'}}{A_{SV}} &= \frac{B'}{c} = a+ib.
\end{align*}$$

and

$$\begin{align*}
\tan 2\delta' = \frac{2ac \tan \delta}{a^2+b^2-c^2 \tan^2 \delta}.
\end{align*}$$
Note the meaning of $a$, $b$ in this case as compared to the case of reflected waves. The special case of circular polarization will arise if the conditions $a = 0$ and $\tan \delta = b/c$ are both fulfilled; the vibration angle ($\delta_1$ or $\delta'$) is then indefinite.

The formulae (18) and (20) reduce to the corresponding equations (12) and (15) for linearly polarized waves, if we put $b = 0$. The equations (18) and (20) are more general, and they can be used in all cases, when the incident vibration is linear.

4. Reflexion at the Earth’s core (ScS)

For an $SV$ wave incident upon the Earth’s core we have the following expressions for the potentials, considering the fact that there is no transmitted $S$ wave:

$$\begin{align*}
\Psi' &= B \exp[i\kappa(x - \beta z - \omega t)] + B_1 \exp[i\kappa(x + \beta z - \omega t)] \\
\Psi'' &= 0 \\
\Phi &= A_1 \exp[i\kappa(x + \alpha z - \omega t)] \\
\Phi' &= A' \exp[i\kappa(x - \alpha' z - \omega t)].
\end{align*}$$

(21)

We have assumed $\lambda \neq \mu$ in this case, which is also in agreement with our present knowledge of $\lambda$ and $\mu$ at the core boundary, and we use the complete expressions (4), obtained from the equations of motion. The boundary conditions require continuity of $\omega$, $p_{zz}$ and $p_{xz}$ at $z = 0$, i.e. using equations (1) and (5):

$$\begin{align*}
\frac{B_1}{B} + \frac{A_1}{B} + A' &= -i \\
\frac{B_1}{B} + \frac{A_1}{B} -\frac{2\alpha}{1 - \beta^2} &= -i \\
\frac{B_1}{B} + \frac{A_1}{B} \frac{2\beta}{\beta c_s^2} &= 1.
\end{align*}$$

(22)

The unknowns are $B_1/B$, $A_1/B$, $A'/B$. Solving for $B_1/B$ we find

$$\frac{B_1}{B} = -1 + \frac{2}{1 + \frac{\rho'(1 + \beta^2)\beta}{4\rho'\beta} + \frac{(1 - \beta^2)^2}{4\alpha\beta}}.$$  

(23)

For an $SH$ wave incident at the core boundary the corresponding expressions are

$$\begin{align*}
v &= C \exp[i\kappa(x - \gamma z - \omega t)] + C_1 \exp[i\kappa(x + \gamma z - \omega t)] \\
v' &= 0.
\end{align*}$$

(24)

The boundary condition is that $p_{xy}$ is continuous at $z = 0$, which gives

$$\mu(-\gamma C + \gamma C_1) = 0$$

or $C_1/C = +1$ for all angles of emergence.
Polarization of transverse seismic waves

Numerical values are given in Table 1, computed for \( \rho = 5.4 \, \text{g/cm}^3, \rho' = 10.1 \, \text{g/cm}^3, c_s = 7.25 \, \text{km/s}, c_p = 13.7 \, \text{km/s}, c_p' = 8.0 \, \text{km/s} \). In addition to \( B_1/B \) we give numerical values of \( A_1/B \) and \( A'/B \) in order to be complete, even if the latter values are of no further use in this investigation. All numerical results have been checked by the energy equation, expressing the fact that the incident energy equals the sum of the energies of the reflected and transmitted waves:

\[
\left( \frac{B_1}{B} \right)^2 + \frac{\alpha_1}{\beta} \left( \frac{A_1}{B} \right)^2 + \frac{\alpha'_{p'}}{\beta_p} \left( \frac{A'}{B} \right)^2 = 1. \tag{26}
\]

Using the values of Table 1 it is possible to calculate \( \delta_1 \) for any given values of \( \delta \) and \( e \) by means of equation (18). The results are shown in Figure 1 for a representative selection of \( e \)-values. The curves for \( e = 10^\circ, 20^\circ \) and \( 80^\circ \) (not shown in Figure 1) are close to the straight line for \( e = 0^\circ, 90^\circ \), and \( e = 30^\circ \) is close to the curve for \( e = 60^\circ \). The inset figure shows the relation between the angle of emergence (\( e \)) at the core and the epicentral distance (\( \Delta \)) for an \( ScS \) wave from a surface focus, computed by means of Jeffreys-Bullen travel times (1940).

Figure 1 demonstrates that for all angles of emergence at the core, the vibration angles \( \delta \) and \( \delta_1 \) lie in different quadrants. This naturally means a considerable change of the particle motion due to the reflexion. But the reason is easy to visualize physically, as it is sufficient with a phase reversal of one of the \( S \) components to bring this about.

Table 1

Reflexion of \( SV \) at the Earth's outer core

<table>
<thead>
<tr>
<th>( e ) deg</th>
<th>( B_1/B )</th>
<th>( A_1/B )</th>
<th>( A'/B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-1)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>10</td>
<td>(-0.9744 + 0.2248i)</td>
<td>(-0.1292 + 0.0147i)</td>
<td>(-0.2962 + 0.0337i)</td>
</tr>
<tr>
<td>20</td>
<td>(-0.9583 + 0.2857i)</td>
<td>(-0.1485 + 0.0217i)</td>
<td>(-0.6118 + 0.0892i)</td>
</tr>
<tr>
<td>30</td>
<td>(-0.6478 + 0.0151i)</td>
<td>(-0.0684 + 0.1483i)</td>
<td>(-0.7614 + 0.0327i)</td>
</tr>
<tr>
<td>40</td>
<td>(-0.4447 + 0.0056i)</td>
<td>(-0.0111 + 0.1136i)</td>
<td>(-0.7270 - 0.0076i)</td>
</tr>
<tr>
<td>50</td>
<td>(-0.3965 + 0.0191i)</td>
<td>(+0.0044 - 0.2235i)</td>
<td>(-0.7439 + 0.0145i)</td>
</tr>
<tr>
<td>60</td>
<td>(-0.6378)</td>
<td>(+1.0437)</td>
<td>(-0.4792)</td>
</tr>
<tr>
<td>70</td>
<td>(-0.7866)</td>
<td>(+0.5917)</td>
<td>(-0.3717)</td>
</tr>
<tr>
<td>80</td>
<td>(-0.9410)</td>
<td>(+0.3191)</td>
<td>(-0.1947)</td>
</tr>
<tr>
<td>90</td>
<td>(-1)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

5. Reflexion at the Earth's surface (\( sS, SS, ... \))

This case was investigated by Jeffreys (1926, pp. 328–329 for \( SV \) and p. 324 for \( SH \)). The results are as follows:

\[
\frac{B_1}{B} = \frac{4x\beta - (1 + 3\alpha^2)^2}{4x\beta + (1 + 3\alpha^2)^2} \tag{27}
\]

and

\[
\frac{C_1}{C} = +1. \tag{28}
\]
In this solution we have assumed $\lambda = \mu$ or $\beta^2 = 2 + 3\alpha^2$, which is fairly true near the surface. The energy equation for the $SV$ wave reads as follows:

$$\left( \frac{B_1}{B} \right)^2 + \frac{\alpha}{\beta} \left( \frac{A_1}{B} \right)^2 = 1. \quad (29)$$

Table 2 gives numerical values for a series of $e$-values, and Figure 2 shows the relation between $\delta_1$ and $\delta$. The inset figure $e(\Delta)$ refers to $S$ waves from a surface focus, assuming $c_S = 3.2\text{ km/s}$. Unless the wavelength is very short, this may be too low a value for $c_S$. The value immediately below Moho ($c_S = 4.4$ km/s, see Figure 4) would be more appropriate in most cases. The $\delta_1-\delta$ curves in Figure 2 are not influenced by the value of $c_S$. If an $SS$ wave is observed at a station at distance $\Delta_1$, the $e$-value at the reflexion point corresponds to the distance $\Delta_1/2$ in these graphs.

The $\delta_1-\delta$ relation is more complicated for reflexions at the surface of the Earth than for reflexions at the core boundary. Especially within certain ranges of the
Polarization of transverse seismic waves

Table 2

Reflexion of SV at the Earth's surface

<table>
<thead>
<tr>
<th>$e$ (deg)</th>
<th>$B_i/B$</th>
<th>$A_i/B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>10</td>
<td>$-0.4596 + 0.8881i$</td>
<td>$+0.5312 - 0.3232i$</td>
</tr>
<tr>
<td>20</td>
<td>$+0.3994 + 0.9168i$</td>
<td>$+0.5040 - 0.7693i$</td>
</tr>
<tr>
<td>30</td>
<td>$+0.8750 + 0.4844i$</td>
<td>$+0.2165 - 0.8386i$</td>
</tr>
<tr>
<td>40</td>
<td>$+0.9968 + 0.0793i$</td>
<td>$+0.0178 - 0.4495i$</td>
</tr>
<tr>
<td>45</td>
<td>$+i$</td>
<td>$0$</td>
</tr>
<tr>
<td>50</td>
<td>$+0.9859 + 0.1675i$</td>
<td>$-0.0801 + 0.9498i$</td>
</tr>
<tr>
<td>55.7</td>
<td>$0$</td>
<td>$-2.5373$</td>
</tr>
<tr>
<td>57.2</td>
<td>$+0.0715$</td>
<td>$-2.0468$</td>
</tr>
<tr>
<td>60</td>
<td>$0$</td>
<td>$-1.7320$</td>
</tr>
<tr>
<td>62.5</td>
<td>$-0.1160$</td>
<td>$-1.5938$</td>
</tr>
<tr>
<td>70</td>
<td>$-0.4829$</td>
<td>$-1.2446$</td>
</tr>
<tr>
<td>80</td>
<td>$-0.8621$</td>
<td>$-0.6778$</td>
</tr>
<tr>
<td>90</td>
<td>$-i$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Fig. 2.—Reflexion at the Earth's surface (sS, SS,...).
Markus Båth

e-values, there are rapid variations of the $\delta_1-\delta$ relation for surface reflexions (Figure 2). Thus $\delta_1$ and $\delta$ are in the same quadrant for $e = 15^\circ-0^\circ-54^\circ-2$ and for $e = 55^\circ-7^\circ-60^\circ-0$, but in different quadrants for other $e$-values. This is naturally only another way of expressing the conditions under which $B_1$ has the same or the opposite phase relative to $B$.

Figure 3 illustrates a particular case of reflexion at the Earth's surface, in which an incident linearly polarized wave gives rise to a reflected wave of elliptic polarization and with a different vibration angle.

![Figure 3](https://example.com/figure3.png)

**Fig. 3.**—Particle orbits before and after reflexion at the Earth's surface in a particular case ($e = 10^\circ; \delta = 45^\circ$).

6. Refraction and reflexion at the base of the crust ($S$, $Ss$, $SS$, ...)

An $SV$ wave incident from below to the base of the crust represents a much more general case than the preceding ones, as both reflected and refracted $P$ and $S$ waves will arise. The potentials are as follows:

\[
\begin{align*}
\Psi &= B \exp[i\kappa(x+\beta z - \omega t)] + B_1 \exp[i\kappa(x-\beta z - \omega t)] \\
\Psi' &= B' \exp[i\kappa(x+\beta' z - \omega t)] \\
\Phi &= A_1 \exp[i\kappa(x-\alpha z - \omega t)] \\
\Phi' &= A' \exp[i\kappa(x+\alpha' z - \omega t)].
\end{align*}
\]

We have assumed $\lambda = \mu$, which is fairly true as in the preceding case. The boundary conditions imply continuity of $u, \psi, p_{xx}, p_{zz}$ at the discontinuity surface.
Polarization of transverse seismic waves

\[(z = 0),\] which gives the following equations:

\[
\frac{B'}{B} + \frac{B_1}{B} \frac{A'}{A_1} + \frac{A'}{A_1} = \beta
\]

\[
\frac{B'}{B} - \frac{B_1}{B} \frac{A'}{A_1} + \frac{A'}{A_1} = \beta
\]

\[
\frac{B'}{B} \frac{2\mu'\beta'}{\mu} + \frac{B_1}{B} \frac{2\beta}{\mu} + \frac{A'}{A_1} \frac{\mu'}{\mu} (1 + 3\alpha'^{2}) - \frac{A_1}{B} \frac{1 + 3\alpha^{2}}{1} = 2\beta
\]

\[
\frac{B'}{B} \frac{\mu'}{\mu} (1 - \beta'^{2}) - \frac{B_1}{B} \frac{1 - \beta^{2}}{1} + \frac{A'}{A_1} \frac{2\mu'\alpha'}{\mu} + \frac{A_1}{B} \frac{2\alpha}{1} = 1 - \beta^{2}
\]

with \(B'/B, B_1/B, A'/B, A_1/B\) as unknowns. The energy equation reads

\[
\left(\frac{B_1}{B}\right)^{2} + \frac{\beta'}{\beta} \frac{\rho'}{\rho} \left(\frac{B'}{B}\right)^{2} + \frac{\alpha}{A_1} \frac{\alpha'}{A_1} \frac{\rho'}{\rho} = 1.
\]

This is the general energy equation for \(SV\), of which the earlier ones, (26) and (29), are immediately obtained as special cases.

For an \(SH\) wave incident from below at the base of the crust, we have

\[
v = C \exp[i\kappa(x + \gamma z - \omega t)] + C_1 \exp[i\kappa(x - \gamma z - \omega t)]
\]

\[
v' = C' \exp[i\kappa(x + \gamma' z - \omega t)].
\]

The conditions for continuity in \(v\) and \(p_{xy}\) at \(z = 0\) lead to the relations

\[
C + C_1 = C' \quad \mu \gamma (C - C_1) = \mu' \gamma' C'
\]

which give the solutions

\[
\frac{C_1}{C} = \frac{\mu' \gamma' \gamma}{\mu' \gamma + \mu' \gamma'} \quad \frac{C'}{C} = \frac{2\mu' \gamma}{\mu' \gamma + \mu' \gamma'}
\]

The energy equation for \(SH\) is as follows:

\[
\left(\frac{C_1}{C}\right)^{2} + \frac{\mu' \gamma'}{\mu \gamma} \left(\frac{C'}{C}\right)^{2} = 1.
\]

In our numerical computations we followed Jeffreys (1926) and assumed \(c_s' = \frac{2}{3}c_s\) and \(\rho' = \frac{2}{3}\rho\). These values were taken by Jeffreys in 1926 as representative for the base of the granitic layer. Our present knowledge of the crust does not conform well with these numerical values, and they may correspond better, although not perfectly, with the conditions at the base of the crust (the Moho discontinuity). The numerical results of the computations of \(B'/B, B_1/B, A'/B\) and \(A_1/B\) from equations (31) and of \(C_1/C\) and \(C'/C\) from equations (35) for a series of \(\epsilon\)-values are compiled in Table 3. The \(\delta\)-relations are shown for representative \(\epsilon\)-values in Figure 4, and Figure 5 illustrates the particle motion of the refracted \(S\) wave for a given incident \(S\) wave in a particular case. The inset figure in Figure 4 gives the \(\epsilon(\Delta)\)-relation for an \(S\) wave from a surface focus and incident at Moho from below (\(c_s = 4.4\) km/s).
Table 3

Refraction and reflexion of SH and SV at Moho from below

<table>
<thead>
<tr>
<th>𝜖</th>
<th>C'/C</th>
<th>C_i/C</th>
<th>B'/B</th>
<th>B_i/B</th>
<th>A'/B</th>
<th>A_i/B</th>
</tr>
</thead>
<tbody>
<tr>
<td>deg</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>+0.607</td>
<td>-0.3993</td>
<td>+0.3350 +0.1871i</td>
<td>-0.3100 +0.5422i</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>+0.891</td>
<td>-0.1089</td>
<td>+0.5663 +0.1468i</td>
<td>+0.1300 +0.4765i</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>30</td>
<td>+1.045</td>
<td>+0.0485</td>
<td>+0.7139 +0.0808i</td>
<td>+0.2736 +0.2966i</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>+1.138</td>
<td>+0.1338</td>
<td>+0.8115 +0.0882i</td>
<td>+0.2095 +0.1081i</td>
<td>+0.5689 -0.3036i</td>
<td>-</td>
</tr>
<tr>
<td>50</td>
<td>+1.186</td>
<td>+0.1861</td>
<td>+0.8226 +0.0041i</td>
<td>+0.1975 +0.0001i</td>
<td>+0.5535 -0.0165i</td>
<td>-</td>
</tr>
<tr>
<td>60</td>
<td>+1.217</td>
<td>+0.2179</td>
<td>+0.8985</td>
<td>+0.0579</td>
<td>+0.2859</td>
<td>-0.3035</td>
</tr>
<tr>
<td>70</td>
<td>+1.237</td>
<td>+0.2367</td>
<td>+0.9221</td>
<td>-0.0971</td>
<td>+0.1798</td>
<td>-0.2803</td>
</tr>
<tr>
<td>80</td>
<td>+1.253</td>
<td>+0.2530</td>
<td>+0.9415</td>
<td>-0.2032</td>
<td>+0.0864</td>
<td>-0.1634</td>
</tr>
<tr>
<td>90</td>
<td>+1.250</td>
<td>+0.2500</td>
<td>+0.9375</td>
<td>-0.2500</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
There are reflected and refracted S waves for all angles of emergence, whereas a reflected P wave exists only for $e > 54^\circ$ and a refracted P wave only for $e > 39^\circ$. For the transmitted (refracted) S wave only one curve has been drawn in Figure 4 (for $e = 50^\circ$), as the curves for all other $e$-values agree closely with the one given here. $\delta$ and $\delta'$ are in the same quadrant for all values of $e$. There are greater variations among the $\delta$-curves for the reflected waves. $\delta$ and $\delta_1$ lie in different quadrants for $e = 17^\circ$ to $25^\circ$ and for $e > 64^\circ$, but are in the same quadrant for other $e$-values. The $\delta$-curve for the reflected S wave for $e = 50^\circ$ agrees very closely with the curve for $e = 0^\circ$ and has therefore been omitted from Figure 4.

![Figure 4](https://academic.oup.com/gsmnras/article-abstract/4/1/106/653734)

**Fig. 4.**—Refraction and reflection at the base of the crust for incidence from below.

### 7. Transmission through continuously varying media (all transverse waves)

A continuously varying medium acts as if it were composed of a large number of thin layers with properties varying by small amounts from layer to layer, provided the wave length is short enough. Therefore, if a property belonging to
the incident side is denoted $X$, the corresponding property of the next layer is denoted $X' = X + dX$, e.g. $\rho' = \rho + d\rho$, $c_s' = c_s + d c_s$, $\beta' = \beta + d\beta$, $B' = B + dB'$, etc.

For an incident $SV$ wave the potentials now become as follows, considering the fact that almost all of the incident energy passes into the transmitted wave of the same type, whereas all other waves will be represented by small quantities:

$$
\begin{align*}
\Psi &= B \exp[i\kappa(x-\beta z - \omega t)] + dB_1 \exp[i\kappa(x+\beta z - \omega t)] \nonumber \\
\Psi' &= (B+dB') \exp[i\kappa(x-\beta' z - \omega t)] \\
\Phi &= dA_1 \exp[i\kappa(x+\alpha z - \omega t)] \\
\Phi' &= dA' \exp[i\kappa(x-\alpha' z - \omega t)].
\end{align*}
$$

$$
\begin{align*}
\frac{dB'}{B} + \frac{dB_1}{B} &+ \frac{dA'}{B} - \frac{dA_1}{B} = -d\beta \nonumber \\
\frac{dB'}{B} &- \frac{dB_1}{B} + \frac{dA'}{B} - \frac{dA_1}{B} = 0 \nonumber \\
\frac{dB'}{B} &+ \frac{dB_1}{B} - \frac{dA'}{B} - \frac{dA_1}{B} = -d(\mu^2) \nonumber \\
\frac{dB'}{B} &- \frac{dB_1}{B} + \frac{dA'}{B} - \frac{dA_1}{B} = -\frac{d(\mu(1-\beta^2))}{\mu(1-\beta^2)}.
\end{align*}
$$

![Diagram](https://academic.oup.com/gsmnras/article-abstract/4/1/106/653734)

**Fig. 5.**—Particle orbits before and after transmission through the base of the crust from below in a particular case ($\phi = 10^\circ$; $\delta = 45^\circ$).

The conditions for continuity of $u$, $w$, $p_{zz}$ and $p_{xx}$ lead to the following system of equations, if products, squares and higher powers of all small quantities are neglected:
Polarization of transverse seismic waves

with
\[ p = \frac{\lambda (1 + \varepsilon^2) + 2\mu z^2}{2\mu \beta} \]

and
\[ q = \frac{2\varepsilon}{1 - \beta^2}. \]

The unknowns are \( dB'/B, dB_1/B, dA'/B, dA_1/B \). Solving for \( dB'/B \) we find
\[ \frac{dB'}{B} = -\frac{3\beta^2 + 1}{2\beta(1 + \beta^2)} \frac{dp}{d\mu} - \frac{d\mu}{2\mu}. \] (39)

We do not need to assume that Poisson's relation \( \lambda = \mu \) holds, which is also not the case, especially not in the deeper parts of the mantle. If we put \( \lambda = m\mu \), where \( m \) is arbitrary, the expression for \( dB'/B \) will be unchanged. In a case like this, where we neglect squares and products of small quantities, it is simpler to determine \( dB'/B \) from the energy equation. This gives
\[ \frac{dB'}{B} = -\frac{d(\beta \rho)}{2\beta \rho} \] (39')
which by means of the refraction law can be shown to be identical with equation (39).

For an \( SH \) wave in a continuously varying medium we have similarly that
\[ \begin{align*}
\nu &= C \exp[i\kappa(x - \gamma z - \omega t)] + dC_1 \exp[i\kappa(x + \gamma z - \omega t)] \\
\nu' &= (C + dC') \exp[i\kappa(x - \gamma' z - \omega t)].
\end{align*} \] (40)

The conditions for continuity of \( \nu \) and \( \nu' \) for \( z = 0 \) give the relations
\[ \begin{align*}
\frac{dC'}{C} - \frac{dC_1}{C} &= 0 \\
\mu \gamma' \frac{dC'}{C} + \mu \gamma \frac{dC_1}{C} &= -d(\mu \gamma)
\end{align*} \] (41)
from which we solve \( dC'/C \):
\[ \frac{dC'}{C} = -\frac{\beta}{2\beta} \frac{d\mu}{2\mu} \] (42)
as \( \gamma = \beta = \tan \varepsilon \).

We consider only linearly polarized \( S \) waves, which is approximately correct, considering the probable fact that the variation from an incident linear vibration will be quite small. Moreover, it is easy to demonstrate by examples that assuming elliptically polarized \( S \) waves as being linearly polarized, i.e. neglecting the imaginary term, is permitted as a good approximation in most computations of \( \delta \). Therefore, as in (15),
\[ \tan \delta' = \frac{A_{SH}'}{A_{SH}} = \frac{C'/C}{B'(1 + \beta'^2)^{1/2}} \frac{B(1 + \beta^2)^{1/2}}{E'/E} \] (43)
where \( E = B(1 + \beta^2)^{1/2} \).
For an infinitesimal variation we have $C' = C + dC'$, $E' = E + dE'$ and
\[ \frac{\tan \delta'}{\tan \delta} = 1 + \frac{dC'}{C} - \frac{dE'}{E}. \] (44)

By integration of $[(dC'/C) - (dE'/E)]$ over a finite depth interval we should find the corresponding change in $\delta$. Now
\[ \frac{dE'}{E} = -\frac{d\beta}{2\beta} - \frac{d\mu}{2\mu}, \] (45)
and therefore
\[ \frac{dC'}{C} - \frac{dE'}{E} = 0 \] (46)
and
\[ \delta' = \delta. \] (47)

At the deepest point of the ray path, where total reflexion takes place, equations (37) and (38) are no longer valid. But it may be demonstrated in an analogous way or simpler by means of the energy equation that there is no change of $\delta$ in the deepest point either.

The result is that there is no change of the vibration angle in a continuously varying medium. This is valid under the assumptions we have made, i.e. in addition to those underlying all computations in this paper:

(1) that products, squares and higher powers of all small quantities may be neglected;
(2) that the waves may be considered as linearly polarized.

However, none of these assumptions will severely limit the validity of our result, which may therefore be considered at least as approximately true for the real Earth. This agrees with the empirical results of Monakhov (1950).

8. Reciprocity of the vibration angle

A question of importance, especially in connection with earthquake mechanism studies, concerns the possible reciprocity of the vibration angle. An incident $S$ wave at a discontinuity surface has the vibration angle $\delta$ and the transmitted wave has the vibration angle $\delta'$. If now an $S$ wave with a vibration angle $\delta'$ is sent in exactly opposite direction, giving a transmitted $S$ with vibration angle $\delta''$, the question is if $\delta'' = \delta$ (exact reciprocity) or not.

As before, we consider only linearly polarized incident waves. We consider first transmission through Moho and will use the same system of equations for $SV$ and $SH$ as in Section 6, but now for incidence from above in addition to incidence from below. The details need not be given here. It may be shown that in case of linear polarization of the transmitted wave from below, the reciprocity is exact, i.e. $\delta'' = \delta$. But when the wave transmitted from below is elliptically polarized, the reciprocity breaks down and the wave transmitted back into the lower medium is in general also elliptically polarized. In a special case with $\epsilon = 20^\circ$ - $\phi$ (lower
Polarization of transverse seismic waves

medium), \( e' = 45^\circ \). For the upper medium we found the following corresponding values of \( \delta \) and \( \delta'' \):

\[
\begin{array}{c|c|c|c|}
\theta & \delta & \delta'' \\
0 & 0 & 0 \\
20 & 23.5 & \\
40 & 48.0 & \\
60 & 68.1 & \\
80 & 83.2 & \\
\end{array}
\]

The condition for reciprocity of \( \delta \) upon reflexion at any surface is that

\[
\frac{|B_1|}{B} = \frac{|C_1|}{C}.
\]

There is no reciprocity in \( \delta \) for reflexions at Moho from below, except for \( e = 0^\circ, 90^\circ \), as is evident from Table 3.

The vibration angle is also generally not reciprocal for reflexions at the Earth's core or at the Earth's surface, not even when the reflected wave is linearly polarized. The \( SV \) component is generally decreased at every reflexion, whereas \( SH \) is unchanged, i.e. \( \delta \) increases for every reflexion. This corresponds to the observation that \( S \) waves reflected several times at the Earth's surface (\( SSS, \ldots \)) contain proportionally more of \( SH \) motion than of \( SV \). Reciprocity occurs only for \( e = 0^\circ, 90^\circ \) for reflexions at the Earth's core (Table 1) and for \( e = 0^\circ, 45^\circ, 90^\circ \) for reflexions at the Earth's surface (Table 2).

One consequence of these results is the following. If from observations of \( S \) waves at a seismograph station one wants to deduce the motion at the source, it is generally not permitted to follow the rays backwards to the source and correct for the changes on the way. Instead, it is necessary in making such corrections to follow the rays in the same direction as they propagate. Such a computation will require knowledge of reflexion and transmission coefficients at each interface.

9. Consequences for earthquake mechanism studies

The present study was begun already in 1946, when I had arrived at the idea that in earthquake mechanism studies use could be made of all seismic waves and not only of the first motion of \( P \) waves, which was practically the only procedure used at that time. I collected quite a number of readings of all possible phases both at Kew and at Uppsala, but these observations were not good enough to permit definite conclusions. I found it necessary to perform much theoretical computation, before the observations could possibly be interpreted. In 1946, I also visited Sir Harold Jeffreys in Cambridge for the first time and had the fortunate opportunity to discuss some of these problems with him. He drew my attention to his paper of 1926, which in fact has stimulated both the computations presented in this paper as well as a number of similar computations related to other waves.

Some investigations of the polarization of \( S \) waves were made already in the early days of instrumental seismology, as e.g. by Galitzin (1911) and by Geiger & Gutenberg (1912). Later Neumann (1930) published a paper on observations of \( S \) waves and their use in focal mechanism studies. In recent years many seismologists have used \( S \) waves for earthquake mechanism studies, e.g. Heinrich & Haill (1952), Dehlinger (1952), Ingram (1953), Gutenberg (1955), Keylis-Borok (1957),
Ritsema (1957), Adams (1958), Nuttli (1958), Byerly & Stauder (1958), Stauder & Byerly (1958), and Stauder (1959). Honda (1957), who used S waves already many years ago, has published a comprehensive survey of the earthquake mechanism problem, including an extensive bibliography.

There are three main problems involved in such investigations:

1. The relation between focal mechanism and vibration of S waves, as they leave the source. This problem has been dealt with by Monakhov (1950), Dehlinger (1952), Gutenberg (1955), Honda (1957), Keylis-Brook (1957), Nuttli (1958), Stauder & Byerly (1958), and others.

2. The changes in the vibration properties during the propagation from source to station. This problem is studied in the present paper.

3. The relation between vibration of an incident S wave and seismograph records. This problem has been treated by Gutenberg (1952), Ingram (1953) and others.

Returning to point (2) above, we can summarize our results as follows. Observations of direct S waves are most trustworthy, as there is practically no change in the vibration angle during the propagation through the mantle (a continuously varying medium). Changes are expected to arise as the waves pass the base of the crust towards the station, but as the Moho is only one or two wavelengths from the station, it is likely that the effect is much smaller than it would be at a greater distance from a discontinuity surface. Moreover, the change of the vibration angle on transmission through Moho is quite small, as is evident from Figure 4. On the other hand, in using other transverse waves (ScS, sS, SS, ...) we must necessarily take the changes of the vibration angle on reflexions at the core boundary or at the Earth’s surface into account.

Different methods have been used in observations of transverse waves for deducing focal mechanisms, namely:


2. Observation of the direction of vibration by combination of simultaneous amplitudes on two or preferably three matched components (Neumann 1930, Dehlinger 1952, Ritsema 1957, Stauder 1959). This is equivalent to the method using the amplitude ratio $SH/SV$ (Keylis-Borok 1957).

3. Observation of amplitude ratios between P and S waves (Keylis-Borok 1957, Honda 1957).

First-motion observations of S waves are generally less reliable than observations of polarization, owing to the already existing motion in a seismogram. Moreover, it must be observed that the direction of initial motion may deviate considerably from the direction defined by the vibration angle ($\delta$), which is related to the fully developed motion after the initial stage.

Seismological Laboratory,
University of Uppsala,
Uppsala, Sweden:

1960 September.
References


