§ 1. Introduction.

Recently Tomonaga and Schwinger have independently developed a relativistically covariant formulation of the quantum theory of field and have succeeded to the explanation of the level shift in the hydrogen atom and the anomalous magnetic moment of the electron. Since, however, the present field theory cannot be formulated in a relativistically and gauge covariant way without introduction of the singular delta-function of Jordan and Pauli, the present theory gives the divergent results to such field reaction problems and the remaining finite part can be not free from any ambiguity arising from the singular nature of delta function.

One typical example of the appearance of such an ambiguity is the photon self energy. As first pointed out by Schwinger, the photon self energy should be zero from the gauge covariant point of view, while, Wentzel showed the photon self energy finite and non gauge covariant. Recently Pauli and Villars succeeded to give unique result for the problem of the photon self energy by using "regulators" called by them. As the non gauge covariant and ambiguous terms appear also in the problem of meson decay, we used Pauli's regulator in order to obtain the unique and gauge covariant form. In the first place, in the γ-decay of the scalar neutretto firstly calculated by Fukuda and Miyamoto there appears the convergent but non gauge covariant term, which further requires the necessity of condition \[ \int \sqrt{x} \rho(x) dx = 0, \sum c_i \mu_i = 0 \] in addition to Pauli's conditions on their regulator in order to obtain the gauge covariant formulation. Secondly, there also appears the convergent but non gauge covariant term in the pseudovector coupling matrix element of the pseudoscalar neutretto, which vanishes by using Pauli's regulator. This non gauge covariant term is able to separate into non gauge covariant term not contributing to real transition process and gauge covariant one, but should be dropped off, on which we discussed in the previous paper. Further, in spite of being possible to prove formally the equivalence between pseudoscalar and pseudovector couplings of the pseudoscalar meson field before using the explicit expressions of \( d \)-functions, the identity relation does not hold only for the first non gauge covariant term of the pseudovector coupling and
On the Decay of Heavy Mesons. I.

the first gauge covariant one of the pseudoscalar coupling except the remainings. This fact seems to suggest the ambiguity of the first integral of the pseudoscalar coupling terms. Really this integral gives unique result if we use Schwinger's expressions of $\delta$-functions, but gives another different result when another expressions of $\delta$-functions are used. This situation is clearly seen in the fact that the value of $\tilde{\delta}$ at the origin is infinite in the Fourier representation but becomes zero in the Schwinger one. From this point of view, the first gauge covariant term of the pseudoscalar coupling should be dropped out, which obliges Pauli's regulator to have another condition, i.e., $\int \rho(x)/\sqrt{x} \, dx = 0$, $\sum c_i/m_i = 0$. Further, this condition compels the second term of the scalar coupling of the scalar neutretto to be zero.

But the use of this condition unrestricctedly gives rise the failure of the equivalence theorem in another case of meson decay problem. Namely, in the decay of a pseudoscalar neutral $\tau$-meson into a vector neutral $\pi$ meson with tensor coupling and a photon, the pseudoscalar coupling term is equivalent to the pseudovector coupling one without using $\int \rho(x)/\sqrt{x} \, dx = 0$, except the first diverging and the second finite terms of the pseudoscalar coupling which vanish by the charge renormalization regulator $\int \rho(x) \log |x| \, dx = 0$ and $\int \rho(x) \, dx = 0$ and the first finite but non gauge covariant term of the pseudovector coupling which disappears by the condition $\int \rho(x)/\sqrt{x} \, dx = 0$. On the contrary, the use of $\int \rho(x)/\sqrt{x} \, dx = 0$ destroys the equivalence theorem in this case. This situation tells us the following fact, that the condition $\int \rho(x)/\sqrt{x} \, dx = 0$ should be used only after the relation $f_i = (2x_i/x)f_i^e$ was set up, where $f_i$ and $f_i^e$ denote the pseudoscalar and the pseudovector coupling constants and $x_i$ and $x$ means the reciprocal Compton wave length of proton and meson respectively. But the use of the condition $\int \rho(x)/\sqrt{x} \, dx = 0$ after the relation $f_i = (2x_i/x)f_i^e$ or $f_i = (x/2x_i)f_i^e$ was substituted removes sometimes even the term which satisfies the identity relation and gauge covariancy. As this result seems undesired, we shall adopt the following procedure.

Now we shall consider this situation more profoundly. When we prove the equivalence theorem between the pseudoscalar and the pseudovector couplings, the following procedure is used:

$$ H_{pv} = \frac{\partial}{\partial x} \int \cdots dx' \, dx'' + 2x_0 H_{pv} + A, $$

where $H_{pv}$ and $H_{pv}$ denote the Hamiltonian density for the real transition process of the pseudovector and pseudoscalar coupling respectively, and $A$ expresses the term multiplies by $\delta(x)$, which is obtained by transforming the pseudovector coupling term as above using the Dirac equation of $\tilde{\delta}$-function $\gamma_\mu \partial_\mu \tilde{\delta}(x) - x_0 \tilde{\delta}(x) = -\delta(x)$. So that when we will regulate the matrix elements to preserve the equivalence theorem, both sides should be regulated as the same order of magnitude with respect to $x_0$. If such procedure for regulating method is used, the
removal of divergences, gauge covariancy and identity relation are always preserved by using only Pauli's regulators and \( \int \rho(x) \sqrt{x} \, dx = 0 \). But there exists another alternative; namely when we regulate the both sides, the method which regulates the expression divided by \( x_0 \) in the formula cited above is also allowed. But in this case, the another condition \( \int \rho(x) \sqrt{x} \, dx = 0 \) for regulator is necessary besides the above conditions in order to satisfy all requirements mentioned above. It is also noted that those alternatives yield the same results for removing the undesired terms. Further, it is interesting that the conditions \( \int \sqrt{x} \rho(x) \, dx = 0 \) or \( \int \rho(x) / \sqrt{x} \times dx = 0 \), which were necessary to preserve the gauge covariancy and the identity relation for the former and the latter alternative respectively makes the divergent terms vanish, appearing in the decay of a heavy meson into lighter mesons, for example, \( \tau \rightarrow \pi^\pm + \pi^0 \) or \( \tau \rightarrow \pi^\pm + \gamma \). From the considerations mentioned above, it may be concluded that careful treatment for regulating method gives almost satisfactory result.

§ 2. Details of the above considerations.

Fukuda and Miyamoto first pointed out that the non gauge covariant and finite terms appears in the first term of the matrix element for the scalar coupling in the \( \gamma \)-decay of scalar neutrino. But as the gauge covariancy for the matrix element is formally proved, such a non gauge covariant term must be dropped out. Or, such a term should be removed out by the following reason. Generalized Schrödinger equation for such a term is

\[
i \frac{\partial \Psi}{\partial x_\mu(x)} = GA \Psi, \quad G = \frac{\gamma^2}{8\pi^2} \left( -\frac{\partial}{\partial x_\mu} \right)^2 \int_{-\infty}^{\infty} \frac{dx_\mu}{x^2},
\]

where \( A_\mu \) and \( \varphi \) denote electromagnetic potential and wave function of scalar neutrino respectively and \( \Psi \) means state vector and \( x = mc/\hbar, m \) proton mass. This equation is gauge invariant for the following gauge transformation \( A_\mu \rightarrow A_\mu - \frac{\partial A}{\partial x_\mu} \), if we take

\[
\Psi = e^{-iV} \Psi',
\]

\[
I' = G \int_\varphi \left( A(x') \frac{\partial A(x')}{\partial x'_\mu} - 2A_\mu(x')A(x') \right) dx'_\mu,
\]

but \( \overline{\Psi}'F_{\mu\nu}\Psi \) is not gauge covariant, for \( F_{\mu\nu} \) does not commute with \( I' \).

As, however, in this case there appears the odd power of mass \( m \), Pauli's regulator method for even power of mass \( m \) can be not used as it stands. Therefore, if the following integral is utilized, which holds for positive and negative \( x \), where \( x = mc^2/\hbar \),

\[
\sqrt{\frac{2m}{x^2}} = \int_{-\infty}^{\infty} e^{2m^2x^2} dx,
\]
we can regulate as the term of even power of \( m \). The non gauge covariant term for the scalar neutrino with scalar coupling is produced from the following integral:

\[
x \sqrt{x} \int d\xi d\gamma (dk) (dk') (dk'') \int da db dc \left( \frac{a}{|a|} + \frac{b}{|b|} \right) \left( \frac{c}{|c|} + \frac{a + b}{|a + b|} \right) \exp \left( ik_0 \xi + k_0 \eta_0 \right) \\
\times \exp \left( i(a + b + c) (k'' + x) \right) \exp \left[ - \frac{i(ak_0 + bk_0}{a + b + c} \right] \mathcal{S}_p (\gamma \mu \gamma) A_\mu (x + \xi) A_\nu (x - \eta) \varphi (x)
\]

((4)) is regulated by using the relation (3) and

\[
\int x \sqrt{x} \rho (x) \exp \left( i(a + b + c) x \right) dx = - \frac{1}{\sqrt{\pi i}} \int_{-\infty}^{\infty} \mathcal{K}'' (a + b + c + x^2) dx
\]

((4)) \(-\frac{4}{\sqrt{\pi i}} \int d\xi d\gamma \sum_{-\infty}^{\infty} \frac{du}{u^3} \int_{-\infty}^{\infty} \frac{dv}{v^3} \left( \mathcal{S}_p (\gamma \mu \gamma) R'' (x^2 + x^2) \right) \exp \left[ - \frac{i}{4h^2} \left( y - x \right) \left( v^2 - 1 \right) \right] A_\mu (x + \xi) A_\nu (x - \eta) \varphi (x)
\]

As the first term of the formula ((5)) is non gauge covariant, it must be dropped off, which requires the following condition for regulator:

\[
\int_0^{\infty} R (\gamma) / \gamma'^2 dy = 0, \quad or \quad \int_{-\infty}^{\infty} \sqrt{x} \rho (x) dx = 0, \quad or \quad \sum c_i m_i = 0.
\]

Next, also in the case of the pseudoscalar neutrino with pseudovector coupling there appears the non gauge covariant term in the first of matrix element, which is able to separate into non gauge covariant term not contributing to real transition process and gauge covariant one, but should be dropped off, on which we already discussed in the previous paper 7). This first convergent but non gauge covariant term disappears by Pauli’s regulator \( \int_d x = 0 \) or \( \int (0) = 0 \) or \( \sum c_i = 0 \). Further, the matrix element due to the pseudovector coupling is transformed into the four dimensional divergent part not contributing to real transition process and the pseudoscalar coupling term multiplied by \( 2x \) (reciprocal Compton wave length of proton).
\[ \times A_\mu(x') A_\nu(x'') \varphi(x) + \frac{2gf_\pi}{x_\pi} \left( \frac{e}{\hbar c} \right)^2 \int \int d^4x' d^4x'' \]

\[ \times \text{Sp} \{ S^{(1)}(x''-x') \bar{\gamma}_5 S(x-x') \bar{\gamma}_5 \bar{S}(x'-x'') \gamma_\mu + \cdots \} A_\mu(x') A_\nu(x'') \varphi(x), \]  

(7)

where \( x_\pi \) means reciprocal Compton wave length of \( \pi \)-meson. The first part vanishes by the law of conservation of energy and momentum for real transition process. Therefore, if we put \( f_i = (2x/x_\pi)^f_2 \), Nelson and Dyson's argument is established, which hereafter we call the equivalence theorem, while the first non gauge covariant term of \( H_\mu^\pi \) is not equal to the first gauge covariant one of \( H_\nu^\pi \) and the equivalence theorem exactly holds for remaining terms of \( H_\mu^\pi \) and \( H_\nu^\pi \). Namely the matrix elements for pseudoscalar and pseudovector coupling are the following:

\[ H_\mu^\pi = \frac{f_1}{4\pi^2 x_\pi} \left( \frac{e}{\hbar c} \right)^2 \int \int du \int \int d^4x \left\{ \frac{1}{u^2} + \frac{1-\nu^2}{4x^2} \right\} \]

\[ \times \left( F_{\mu 1} F_{\nu 1} + F_{\mu 2} F_{\nu 2} + F_{\mu 3} F_{\nu 3} \right) \varphi \]

(8)

\[ H_\nu^\pi = \frac{f_2}{8\pi^2 x_\pi} \left( \frac{e}{\hbar c} \right)^2 \int \int du \int \int d^4x \left\{ \frac{1}{u^2} \sum_{\rho} \frac{1}{x^2} \frac{1-\nu^2}{u^2} \right\} \]

\[ \times \sum_{\lambda} \sum_{\rho} F_{\lambda \rho} \partial_\lambda F_{\rho 1} \partial_\varphi \]

(9)

Therefore, it seems appropriate to us that the first gauge covariant term of the pseudoscalar coupling (8) may be removed out. Really this integral gives unique result if we use Schwinger's expressions of \( \delta \)-functions, but gives another different result when another expressions of \( \delta \)-functions are used\(^7\). The removal of such first term requires further the condition \( \int R(x) / \sqrt{\gamma} dy = 0 \), or \( \int \rho(x) / \sqrt{\gamma} dx = 0 \), or \( \sum_{\pi, n} = 0 \) for regulators, which is obtained by the calculation as before. Further, this condition compels the second term of the scalar coupling of the scalar neutral neutrino to be zero (see (5)).

But the use of this condition unrestrictedly yields the failure of the equivalence theorem in another case of meson decay problem. Namely, in the decay of a pseudoscalar neutral \( \tau \)-meson into a vector neutral \( \pi \)-meson with tensor coupling (interaction constant \( g_2 \)) and a photon, the pseudoscalar coupling term (interaction constant \( G_i \)) is equivalent to the pseudovector coupling one (interaction constant \( G_i \)), except the terms mentioned below. The matrix element for pseudoscalar coupling of \( \tau^0 \) becomes as follows:

\[ H_\mu^\tau = \frac{1}{4\pi^2} \frac{G_i g_2}{(2\pi)^2} \int \int d^4x' d^4x'' \]

\[ \times \text{Sp} \{ \gamma_5 \gamma_\mu \gamma_\alpha \gamma_\beta \} \varphi_\alpha \varphi_\beta \]

\[ \times \left\{ \int_0^\infty dw / \frac{\sin w}{w} \cos w + \frac{1}{6} \right\} \varphi_\alpha \varphi_\beta \]

\[ \times \left\{ \int_0^\infty dw / \frac{\sin w}{w} \cos w + \frac{1}{6} \right\} \varphi_\alpha \varphi_\beta \]
where $\varphi, \chi_{\mu}$ and $F_{\mu\nu}$ denote the wave functions of scalar $\pi_{s}$, vector $\pi_{v}$ mesons and electromagnetic field respectively. The first diverging and the second finite terms are removed out by Pauli's regulators $\int \rho(x) \log |x| \, dx = 0$ and $\int \rho(x) \, dx = 0$. The matrix element for pseudovector coupling becomes

$$H_{\mu v} = -\frac{x}{8(2\pi)^{2}} \frac{G_{\pi} G_{\mu}}{n_{\pi}^{2}} \text{Sp} (i\gamma_{\mu} \gamma_{\nu} \gamma_{\alpha} \gamma_{\beta}) \left\{ 2A_{\mu} \chi_{\alpha} \partial_{\nu} \varphi + \frac{1}{2} \varphi \chi_{\nu} F_{\mu\nu} \right\} \frac{\int_{-\infty}^{\infty} \frac{dn}{n^{2}}} {\int_{-\infty}^{\infty} \frac{dn}{n^{2}}} \times \left[ \frac{(1-\tau^{2}+u-2\tau^{2}) \frac{x_{-}^{2}}{x^{2}} - (1-\tau^{2}+u) \frac{x_{+}^{2}}{x^{2}}} {2} \right] \times \left[ \frac{(1+\tau) (1-u) \frac{x_{-}^{2}}{x^{2}} - \frac{1-\tau^{2}}{2} \frac{x_{+}^{2}}{x^{2}}} {2} \right] \times \left[ \frac{(1-\tau^{2}+u-2\tau^{2}) \frac{x_{-}^{2}}{x^{2}} - (1-\tau^{2}+u) \frac{x_{+}^{2}}{x^{2}}} {2} \right] \times \left[ \frac{(1+\tau) (1-u) \frac{x_{-}^{2}}{x^{2}} - \frac{1-\tau^{2}}{4n^{2}} \frac{x_{+}^{2}}{x^{2}}} {2} \right],$$

(10)

the first non gauge covariant term of which is dropped by the condition $\int \sqrt{x} \times \rho(x) \, dx = 0$. And if we put $G_{v} = (2x/x_{v})G_{s}$, the equivalence theorem holds exactly between the remaining terms of (10) and (11) except the terms dropped out by regulator. If we, further, use the condition $\int \rho(x) \log |x| \, dx = 0$ and $\int \rho(x) \, dx = 0$, the equivalence theorem is established by regulating after the relation $f_{1} = (2x/x_{v})f_{2}$ was sub-
stituted. In this case it is sufficient for establishment of the theorem to use only
Pauli’s condition \( \int \rho(x) \, dx = 0 \). In the case of (10) and (11), if we put \( G = (2x/x_0)G \), the condition \( \int \sqrt{x} \rho(x) \, dx = 0 \) guarantees the theorem. Therefore, when
we regulate by putting \( H_{\mu} = 2xH_{\mu} \), the requirement of the removal of divergency, gauge covariancy and equivalency theorem are accomplished sufficiently by Pauli’s
regulator and the condition \( \int \sqrt{x} \rho(x) \, dx = 0 \).

For example, in the decay problem of \( \pi^\pm \to \pi^0 \pi^0 \), there appear diverging
integrals proportional to even and odd power of \( x \), the former of which vanishes
by Pauli’s condition \( \int \rho(x) \log |x| \, dx = 0 \). Although in the latter diverging integral
there appears apparently the term proportional to \( \int \rho(x)/\sqrt{x} \, dx \) or \( \int \rho(x)/y \, dy \),
this term exactly cancels out, which guarantees the removal of divergency by
using only the condition \( \int \sqrt{x} \rho(x) \, dx = 0 \) or \( \int \rho(y)/y \, dy = 0 \). For example, this
integral is written as follows:

\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{\omega}} \sigma (\nu^2 \omega) R'(\nu^2 \omega + x^2) \exp (i\nu^2 \omega) A =
\]

\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y-x^2} \sigma (y-x^2) R'(y) \exp i(y-x^2) A
\]

\[
= -2iA \int_{-\infty}^{\infty} dx R(x^2) + \cdots + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y-x^2} \sigma (y-x^2) R'(y) \exp i(y-x^2) A
\]

\[
- iA \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y-x^2} \sigma (y-x^2) R(y) \exp i(y-x^2) A
\]

\[
= -2iA \int_{-\infty}^{\infty} R(x^2) \, dx + \cdots
\]

\[
+ \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y-x^2} \sigma (y-x^2) R(y) + iA \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y-x^2} \sigma (y-x^2) R(y) + \cdots
\]

\[
- iA \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y-x^2} \sigma (y-x^2) R(y) + A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma (y-x^2) R(y) \, dy \, dx \cdots
\]

(12)

In this formula the third and the fourth term cancels out. The second term is
transformed into \( \int \rho(y)/\sqrt{y} \, dy \), the vanishing of which is necessary to obtain the
convergency. The first term is transformed to \( \int \rho(y)/\sqrt{y} \, dy \), the removal of
which is unnecessary in this case. Although the condition \( \int \rho(y)/\sqrt{y} \, dy = 0 \) was
settled to preserve the gauge covariancy, it is interesting that it is also useful to
remove the diverging integral.

Now there exists another alternative for regulating method, which regulates
the term after the relation \( H_{\mu} = 1/2xH_{\mu} \) was set up. But in this case, the
another condition \( \int \rho(x)/\sqrt{x} \, dx = 0 \) is necessary besides the above conditions in
order to satisfy all requirement mentioned above. However those alternatives
yield the same results for removing the undesired terms. For example, in the
\( \gamma \)-decay problem of pseudoscalar neutretto if we adopt the latter alternative, the
first terms of (8) and (9) are dropped by the condition $\int \mu(x) \sqrt{x} \, dx = 0$. If the former alternative is adopted, the same first terms of (8) and (9) are removed by the condition $\int \mu(x) \sqrt{x} \, dx = 0$. And in the decay problem of $\pi_0$ meson into $\pi_0$ meson and a photon, according to the former alternative, the first diverging and the second finite terms of (10) and the first non gauge covariant one of (11) are dropped out by the condition $\int \sqrt{x} \rho(x) \, dx = 0$. Regulating according to the latter alternative, the first diverging term of (10) is dropped out by the condition $\int \rho(x) \log |x| \, dx = 0$ and the second term of (10) and the first one of (11) are removed by $\int \rho(x) \, dx = 0$. It is evident that those alternatives yield the same result. In the decay of $\tau^\pm \rightarrow \pi^\pm + \pi^0$ ($\tau$ scalar, $\pi^\pm$, $\pi^0$ pseudoscalar) the diverging integral appears, which according to the latter alternative vanishes by the condition $\int \rho(x) \sqrt{x} \, dx = 0$.

According to such an idea, as the same situation exists in the scalar and vector coupling of scalar meson, i.e.,

$$H' = 2xH_r - 2xH_x,$$

it seems appropriate to regulate the terms of scalar coupling multiplied by $x$, which removes the first and second terms by the conditions $\int x\mu(x) \, dx = 0$ and $\int \mu(x) \, dx = 0$ in the case of the former alternative. Even according to the latter alternative i.e., $H_r = (1/2x)H_x$ the first and second terms of scalar coupling are dropped by the conditions $\int \sqrt{x} \rho(x) \, dx = 0$ and $\int \rho(x) \sqrt{x} \, dx = 0$. Thus, the both alternatives yield the same result. It may be concluded that all requirements mentioned above are satisfied whenever we use the regulator by considering such a relation.

References.

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