On the Asymptotic Behavior of
Electromagnetic Form Factors

Masao YAMADA

Department of Physics, University of Tokyo, Tokyo

(Received March 27, 1968)

Electromagnetic form factors for composite particles are discussed. It is shown that the
form factor of \( \pi \)-meson behaves like \((\log t)^{n}/t^{2}\) (\(n=1\) or \(2\)) for the large positive momentum
squared \(t\) when the \(\pi\)-meson is assumed to be a bound state of the nucleon and anti-nucleon
system. For the spinless nucleon this is verified without any approximation other than the
B-S equation with a general kernel and for this case it is shown that \(n=1\) some approximations
are taken when we include the spin of the nucleon, and we find \(n=2\) for this case,
but it is argued that this conclusion may be true in general. Some discussions of the large \(t\)
behavior of the form factor of nucleon in a bound state are added.

§ 1. Introduction

It seems to us that the so-called "nuclear democracy"\(^{11}\) is one of interesting
hypotheses in the elementary particle physics. It asserts that all hadrons are
bound states and that there are no "aristocratic particles". Many authors have
worked along this hypothesis to predict the values of masses, coupling constants,
Regge parameters, etc., self-consistently by using N/D methods,\(^{2}\) generalized
super-convergence relations,\(^{3}\) and so on.

On the other hand, it has been widely believed that the analysis of the
electromagnetic form factors of hadrons may give informations about the structure
of the particles. Along this line we have shown\(^{4}\) that we can roughly under­
stand the electromagnetic decay phenomena of resonances, if they have a struc­
ture such as the bootstrap hypothesis asserts.

It now seems to be well established experimentally\(^{5}\) that the nucleon elec­
tromagnetic form factors behave like \(t^{-2}\) at the large space-like momentum
transfer \(t\). Moreover, some recent experiments\(^{6}\) indicate that there are no es­
tential differences between the electromagnetic form factor of \(\pi\)-meson and that
of the nucleon.

Very recently, Ball and Zachariasen\(^{7}\) have shown that the form factors of
composite particles behave essentially like \(t^{-1}\) in the simplest approximation.
They take the ladder approximation in the strong interaction form factors. Their
approximation is crude, indeed, but contains at least some contributions from
those states which, in the \(t\)-channel, contain arbitrarily large numbers of particles.

\(^{1}\) Present address: Department of Physics, The University of Texas at Austin, Austin, Texas
78712, U.S.A.
Their conclusion is interesting.

In this article we discuss the electromagnetic form factor of the \( \pi \)-meson, which we assume is a bound state of nucleon and anti-nucleon system. We will show that the form factor behaves at most like \( (\log t)/t^2 \) as \( t \to \infty \), without any approximation such as the ladder approximation, if we neglect the spin of nucleon. Moreover, we show that it behaves like \( (\log t)^2/t^2 \), when we include the effects of the spin of nucleon under some approximation.

In 1955, Mandelstam\(^8\) discussed generally how to determine the matrix element of any dynamical variable between two bound states. According to him, the electromagnetic form factor of a bound state \( \left| a \right> \) can be written as

\[
\left< a \left| j_\mu (z) \right| a \right> = \int dx_3 dx_4 dy_3 dy_4 \chi_a(x_3, x_4) G_\mu(x_3, x_4; z, y_3 y_4) \chi_a(y_3, y_4), \tag{1}
\]

where \( \chi_a \) denotes the B-S amplitude and \( G_\mu \) is a sum of irreducible graphs. The meaning of the above equation is shown in Fig. 1. This is our basic equation.

![Graphical representation of Eq. (1).](image)

We discuss the contributions of all types of graphs belonging to \( G_\mu \) and \( K \), by using the rules widely discussed by Federbush and Grisaru,\(^9\) Polkinghorne,\(^10\) Tiktopoulos\(^11\) and some others\(^12\) to obtain high energy limits of Feynman graphs. In § 2 we treat the large \( t \) behavior of the electromagnetic form factor of a spinless bound state composed of a scalar proton and a scalar antineutron. The form factor of \( \pi \)-meson is discussed in § 3, where \( \pi \)-meson is assumed to be a bound state of the real nucleon anti-nucleon system. In § 4 we discuss briefly the form factor of nucleon when it is considered to be a bound state of a \( \pi \)-meson and a nucleon. Some difficulties involved in the nucleon case is stressed there.

§ 2. Electromagnetic form factor of a spinless bound state

In this section we treat the electromagnetic form factor of a scalar meson \( A \) (mass = \( M \)) which is a bound state of a scalar proton and a scalar antineutron (mass = \( m \)). The form factor can be written as

\[
\int d^4p \int d^4p' f^*(\mathbf{p'}, -\mathbf{k'}) G_\mu(k, -\mathbf{k'}; \mathbf{p}, \mathbf{p'}) f(p, k), \tag{2}
\]

as discussed in the last section, where \( f(p) \) (\( f(p') \)) is the B-S amplitude and
\( p(p') \) is the relative momentum of the constituents, while \( 2k(-2k') \) is the center of mass momentum. (See Fig. 2.)

For a moment, we omit the graphs involving vertex parts. It is easy to generalize the arguments to include them, as can be seen below.

The function \( G_p \) can be represented graphically as shown in Fig. 3.

\[
\begin{align*}
\frac{1}{2} &+ \frac{1}{2} + X + \cdots
\end{align*}
\]

Fig. 3. Series of the irreducible graphs \( G_p \).

The first term of this series \( G_p^{(1)} \) is equal to

\[
2[m^2 + (k' - p)^2] (p' + k' - p - k) \delta(p - k + p' - k').
\]

(3)

The contribution of this term to the formula (2) is a generalization of the usual triangle graph. This contribution is discussed later, since we need further information about the B-S amplitude \( f(p) \) for this purpose.

The second term contribution of the series, \( G_p^{(2)} \), to the form factor can be represented by Fig. 4. The large \( t = (2k + 2k')^2 \) limit of this graph can be evaluated at once as

\[
\frac{32}{t^2} (k - k')_\rho \text{Re}[F_1^* (M^2) F_1(M^2)],
\]

where

\[
F_1(M^2) = \int f(p, k) d^4 p
\]

and
On the Asymptotic Behavior of Electromagnetic Form Factors

\[ k_\mu F_2(M^2) = \int p^* f(p, k) d^4 p. \]

Here we notice that \( f(p) \) behaves like \( 1/p^3 \) for large \( p^2 \) when we take ladder approximation in the B-S equation, as we can see directly. We assume here that \( f(p) \) damps faster than or as fast as \( 1/p^6 \) in general.

As to the third term, or more complicated graphs of \( G_\mu \), we analyze large \( k \cdot k' \) behaviors of the Feynman graphs, using the results discussed by Federbush and Grisaru, Polkinghorne, Tiktopoulos and others. We should discuss such a type of graphs as shown in Figs. 5a, 5b, 5c, where the dashed line is a scalar meson of mass \( \mu \). It should be noted that the momenta \( p \) and \( p' \) are integration variables. Therefore we can take new variables \( Q \) and \( Q' \) so that only one external line is dependent on \( k(k') \). If the \( k(k') \)-dependent external line is charged, the transformation of the integration variables \( p(p') \) into \( Q(Q') \) is given by \( Q = k + p \) \((Q' = k' + p')\), and otherwise \( Q = k - p \) \((Q' = k' - p')\).

In order to discuss generally the contributions of those graphs to the form factor, we determine the momenta of all lines by the following procedures. First, we take the loops each of which passes through one of the combinations of the terminal points \((A, B, E), (A, D, E), (C, B, E)\) and \((C, D, E)\). We call the loops \( l_{AB}, l_{AD}, l_{CB}, l_{CD} \) respectively. In some cases, all of them are not necessarily distinct, but we treat them differently according to which combination of the terminal points we are noticing. We determine the two paths \( A \rightarrow E \) and \( B \rightarrow E \) for the loop \( l_{AB} \), and similarly for the other loops. The paths must be chosen so that no line should be common to two paths on a loop.

Secondly, we compare the numbers of the lines of the two paths just we have found on each loop, and pick out the smaller numbers from those loops.

Fig. 5. Various types of graphs in the series of \( G_\mu \). The momenta of all lines are denoted explicitly by the three rules. \( A, B, C \) and \( D \) are four terminal points and \( E \) is the point where the photon line is attached.

---

\( * \) Here we take the interaction as \( \tilde{\psi}(x) \phi(x) \phi(x) \).

\( ** \) Sometimes, we have more than one loops for a combination, \((A, B, E)\). In this case we choose the loop \( l_{AB} \) in such a way that the number of the lines on \( l_{AB} \) is smaller than those on the other loops. Moreover, there are some cases when some of those loops are not present. But, at least one such loop exists except for the graph \( G_\mu \).
Then we can determine the largest number among these numbers. We call the loop, to which the largest number belongs, the basic loop. For example, the basic loops of the diagrams of Figs. 5a, 5b and 5c are $l_{AB}$, $l_{CB}$ and $l_{AB}$, respectively.

Thirdly, if the basic loop of a graph is $l_{AB}$, for example, we denote the momentum of the external line which flows in through $A$ ($B$) by $2k - Q (2k' - Q')$, and consequently, the momentum of the other external line which flows in through $C$ ($D$) is determined to be $Q (Q')$. Then we attach $2k - Q (2k' - Q')$ to each line on the path $A \rightarrow E (B \rightarrow E)$, and $Q (Q')$ to each line on the path $C \rightarrow E (D \rightarrow E)$.

Finally, we find all the independent loops from the graphs, including the basic loop. "The loop momenta" (e.g. $I$ and $v$ in Fig. 5c) are attached to all the lines on the loops. We write the loop momentum of the basic loop as $I$.

The above four rules are sufficient to determine the momenta of all lines of any graphs of $G_\rho$ except for $G_\rho^{(0)}$.

By the above rules we can see at once the following four properties for any graphs* of $G_\rho$ except for $G_\rho^{(1)}$, $G_\rho^{(2)}$ and a contribution of a graph as shown in Fig. 5d.

1. No internal lines can depend on both $k$ and $k'$.
2. The number of the lines dependent on $k(k')$ is equal to or larger than 2.
3. One common integration variable $l$ is attached to all $k$- and $k'$-dependent lines.
4. The sign of $l$ relative to $k(k')$ is the same for all $k$-(k'-)dependent lines.

After carrying out the integration over $l$, the contribution of any graph can be represented as

$$
\int_0^1 \prod_{i=1}^n d\alpha_i \frac{h_\rho(\alpha_i)}{[C(\alpha_i)]^{n-1}} \delta(\sum_{i=1}^n \alpha_i - 1),
$$

where we omit the integration symbol with respect to $v$, etc. $h_\rho(\alpha_i)$ is linear with respect to $\alpha_i$, and $\alpha_i's$ are Feynman parameters, which are attached only to the $l$-dependent lines. According to the above properties 1, 3 and 4, the coefficients of $k \cdot k'$ in $C$ in the formula (4) is proportional to $(\sum' \alpha_i)(\sum'' \alpha_i)$, where $(\sum' \alpha_i) \times (\sum'' \alpha_i)$ is the sum of the Feynman parameters attached to $k$-$k'$-dependent lines. For a moment, we discuss those diagrams which do not include self-energy parts. To include the self-energy part, we should express all lines in terms of Lehmann's spectral representation. If the $Z$-factors of all lines are not zero, the following arguments can be easily generalized.

* Inclusion of corrections to all vertices for such graphs does not change the following arguments at all.
Here we consider the asymptotic expansion of the formula (4) as \( k \cdot k' \to \infty \), disregarding \( v \)- and other possible integrations. Then the leading term comes from the neighbourhood of \( \sum' \alpha_r = \sum'' \alpha_s = 0 \). There are two or more than two Feynman parameters in the sum \( \sum' \alpha_r \) and \( \sum'' \alpha_s \) by the property 2 discussed above. Under these circumstances, the order of the leading term of the formula (4) becomes \((\log k \cdot k')^{\mu} / (k \cdot k')^{\nu-a}\), where \([r, s]\) is the minimum between the number of \( k \)-dependent lines and that of \( k' \)-dependent lines, and \( u \leq [r, s] \).

The coefficients of all terms in the expansion contain \( v \)- and other integrations. It can be shown that these integrations always give finite values, so that we can interchange the order to take the limit \( k \cdot k' \to \infty \) and to integrate with respect to \( v \), etc., in the formula (4). For example, if the remaining integration is only one as in the case of Fig. 5c, the integrand of the coefficient of the leading term contains a factor \( 1/[C(\alpha_r = \alpha_s = 0)]^2 \), which behaves like \( 1/(v^2) \) for large \( v \), and, moreover, \( h_r(\alpha_r) \) contains one \( v \)-dependent propagator. This circumstance is similar to all other terms in the expansion.

Actually, the contribution of \( G_\mu^{(3)} \), Fig. 5a, to the electromagnetic form factor becomes

\[
\lim_{k \cdot k' \to \infty} \int d^4Q \int d^4Q' f(Q) G_\mu^{(3)}(k, k'; Q, Q') f(Q) \propto \frac{1}{m^2} \log (k \cdot k') \frac{1}{(k \cdot k')^2},
\]

where

\[
F_1'(M^2) = \int d^4Q f(Q)
\]

and

\[
k_\mu F_1'(M^2) = \int d^4QQ_\mu f(Q).
\]

Finally, we calculate the contribution of the graph, Fig. 5d, to the electromagnetic form factor. This is most easily done when we take the momenta as shown in Fig. 5d, instead of the above rules, because we should integrate with respect to \( Q \) and \( Q' \) without any detailed information on \( f(Q, k) \). The leading term becomes \( O((\log k \cdot k')/(k \cdot k')^2) \).

From the above analysis we can say that the contributions of all terms except for the first one in the series of \( G_\mu \) to the electromagnetic form factor of the bound state \( A \) behave at most like \((\log k \cdot k')/(k \cdot k')^2 \) as \( k \cdot k' \to \infty \). This conclusion does not depend on the details of the B-S amplitude. We need only one assumption that for large \( p^2 \) (the relative momentum squared) the B-S amplitude \( f(p) \) damps faster than or as fast as \( 1/p^\delta \), which can be verified in the ladder approximation.

We notice here that, if there appears elementary higher spin particle ex-
change in $G_\nu$, our conclusion should be altered essentially. But, if the particles are bound states of spinless particles, the contributions are included already in our arguments, as our $G_\nu$ is quite general.

Next we discuss the contribution of $G_\nu^{(1)}$ which is represented by Eq. (3). For this purpose we should know more detailed knowledge of the B-S amplitude $f(\rho)$. $f(\rho)$ satisfies the B-S equation whose kernel is $K(\rho, q; k)$, and $K$ can be expanded graphically as Fig. 6. Substituting Eq. (3) and the kernel $K$ into

\[ \int d^4q' \int d^4q f^*(q') J_\mu(k, k'; q, q') f(q), \]

we obtain the contribution of $G_\nu^{(1)}$ to the electromagnetic form factor as

\[ J_\mu(k, k'; q, q') = \int d^4p \frac{A(k' - p) K^*(k' + k - p, q'; k') K(p, q; k)}{[m^2 + (2k' + k - p)^2] [m^2 + (k + p)^2] [m^2 + (k - p)^2]}. \]

We transform the integration variables $q$, $q'$ and $p$ by

\[ Q = k \pm q, \quad Q' = k' \pm q' \quad \text{and} \quad l = k - p. \]

Then $J_\mu$ becomes

\[ J_\mu(k, k'; Q, Q') = \int d^4l \frac{A(k' - k + l) K^*(k' + l', \pm (Q' - k'); k') K(k - l, \pm (Q - k); k)}{[m^2 + (2k' + k - p)^2] [m^2 + (2k - l)^2] [m^2 + l^2]}, \]

where the double signs in front of $Q' - k' (Q - k)$ depend on which transformation we take in Eq. (6).

It we insert the expansion of $K$ into Eq. (7), $J_\mu$ can be represented as Fig. (7). Ball and Zachariasen's calculation corresponds to the contribution of the first term of this series. This contribution can be calculated by the Mellin transformation technique\textsuperscript{10} to all orders with respect to $k \cdot k'$, and the leading term becomes

\[ \int \ldots \]

Fig. 6. Graphical representation of the kernel $K$.

Fig. 7. Graphical representation of $J_\mu$. 

\[ \int \ldots \]

\[ \int \ldots \]

\[ \int \ldots \]
Notice that our result Eq. (8) and that of Ball and Zachariasen differ by a factor \(\log k \cdot k'\), which must be present in their calculation, if a more careful calculation is performed in their method.

More generally, quite the same arguments as \(G_\mu\) can be applied to \(J_\mu\). For example, the fourth term of the series \(J_\mu\) in Fig. 6 satisfies the conditions 1, 2, 3 and 4 as is shown in Fig. 8. In general, we have contributions from the series of \(J^\#\) to the form factor at most \((\log k \cdot k')^m / (k \cdot k')^n\) as \(k \cdot k' \to 0\), where \(m \geq 2\) and \(m \geq n\).

![Fig. 8. The fourth graph of Fig. 7. The momenta of all lines are denoted by the previous three rules.](image)

We close this section by saying that the electromagnetic form factor of spinless bound state composed of spinless nucleon and anti-nucleon behaves at most like \((\log t) / t^2\) under the assumptions;

1) B-S amplitude damp faster than or as fast as \(1/p^6\), where \(p\) is the relative momentum,

and

2) All possible higher spin bosons are bound states of spinless particles.

### § 3. Electromagnetic form factor of \(\pi^+\)-meson

In this section we discuss the electromagnetic form factor of \(\pi^+\)-meson when it is considered to be a bound state of spin 1/2 proton and anti-neutron. Inclusion of spin does not affect the structure of the coefficient of \(k \cdot k'\) in \(C\) in the formulae corresponding to the formulae (4) for \(G_\mu\) and \(J_\mu\) of the last section. But the numerator of the propagator of spin 1/2 particle can affect the \(k \cdot k'\) dependence of each graphs, and a more detailed analysis is required for each graphs.

The B-S equation can be written as

\[
m + i\eta^{(1)} \cdot (k + p) [m + i\eta^{(2)} \cdot (k - p)] f(p) = \int d^4q K_{ij}(p, q; k) \rho_i^{(1)} \rho_j^{(2)} f(q),
\]

where \(\rho_i^{(1)}\) and \(\rho_j^{(2)}\) are 16 independent matrices and \(K_{ij}\) is a kernel function.
$f(p)$ is 16-components and can be represented by $4 \times 4$ matrix whose rows (columns) are numbered by $1(2)$ particle index. The operation of $\gamma^{(i)}$ matrix can be defined as

$$\gamma^{(i)}_a f(p) = \gamma_a f(p)$$

and

$$\gamma^{(i)}_a f(p) = f(p) \gamma_a^T.$$ 

Using the charge conjugation matrix $C$, we define the amplitude $g(p)$ by

$$g(p) = f(p) C.$$ 

The equation for $g(p)$ becomes

$$[m + i\gamma \cdot (k + p)] g(p) \left[ m - i\gamma \cdot (k - p) \right] = \sum_{ij} d^4 q (\pm) K_{ij}(p, q; k) \rho_j g(q) \rho_i,$$

(10)

where double sign $\pm$ comes from $e^{-i\rho_j^T c} = \pm \rho_i$.

In terms of $g(p)$, the electromagnetic form factor of $\pi$-meson can be written as

$$\int d^4 p' \int d^4 p \text{Sp}[\gamma_5 g^*(p') \gamma_i g(p; p', k; k')] g(p; k),$$

(11)

where the momenta are taken as shown in Fig. 2. The amplitude $g(p)$ can be represented in general as

$$g(p) = \gamma_5 g_4(p) + i\gamma_5 \cdot k g_4(p) + i\gamma_5 \cdot p g_3(p) + i\gamma_5 \sigma_{\mu} \rho k_{\mu} G_1(p),$$

(12)

where $g_i(p) = g_i(p^2, k \cdot p)$. From Eqs. (11) and (12), we see that existence of the $g_i(p)$'s other than $g_1(p)$ can also make large $k \cdot k'$ behavior worse than that of the spinless case owing to the coefficients $\gamma \cdot k, \gamma \cdot p$ and $\sigma_{\mu} k_{\mu} p_{\nu}$, as well as the numerator of the spin 1/2 propagator of internal lines.

We take here the ladder approximation, which corresponds to the contribution of the first term of Fig. 7, whose dashed lines are also substituted by $\pi$-meson, and hence $M^2 = p^2$. The reasons why we take this approximation are firstly that it will provide us a typical example to see the roles of spin 1/2 propagator and of the amplitudes, $g_i(p)$'s and secondly that it is expected to give us the leading term to the asymptotic electromagnetic form factor, as we have seen in the spinless case.

Then the kernel of Eq. (10) becomes

$$K(p, q; k) = \frac{\lambda}{\pi^2} \frac{1}{\mu^2 + (p - q)^2} h(q^2),$$

(13)

and

$$\rho_i = \rho_j = \gamma_5.$$ 

In Eq. (13) we introduced a cutoff function $h(q^2)$, which cannot be set equal to 1, as we treat $\pi$-meson as a bound state. It must damp for large $q^2$ for fixed.
On the Asymptotic Behavior of Electromagnetic Form Factors

$p$ and $k$. For simplicity, we neglect the possible $p, k$ dependence of $h(q^2)$. The results are independent of the detailed form of $h(q^2)$. Introducing new integration variables $Q$ and $Q'$ by the transformation (6), we can write down at once the contribution of the graph shown in Fig. 9 to the formula (11). The Feynman parameters $\alpha_i$ are taken as shown in Fig. 9.

![Fig. 9. Feynman parameters attached to the lines.](https://academic.oup.com/ptp/article-abstract/40/4/848/1926547)

Following Federbush, Grisaru and Tiktopoulos, we introduce new variables $\rho_i$ and $\beta_j$ by

$$\begin{align*}
\alpha_1 &= \rho_1 \beta_1, \\
\alpha_3 &= \rho_4 \beta_3, \\
\beta_1 + \beta_3 &= 1 \\
\alpha_2 &= \rho_2 \beta_2, \\
\alpha_4 &= \rho_2 \beta_4, \\
\beta_2 + \beta_4 &= 1 \\
\alpha_5 &= \rho_5 \\
\text{where } \sum_{i=1}^{3} \rho_i &= 1.
\end{align*}$$

Then the formula (11) can be written after $p, p'$ integrations as

$$\sum_{j=1}^{3} d^4 Q d^4 Q' k((Q' - k')^2) h((Q - k)^2) g_i^*(Q') I_{i,j} g_j(Q),$$

where

$$
\begin{align*}
g_i(Q) &= g_1(Q - k), \\
g_i'(Q) &= g_3(Q - k), \\
g_i'(Q) &= g_4(Q - k)
\end{align*}$$

while

$$g_3(Q) = g_2(Q - k) - g_3(Q - k).$$

Here $I_{i,j}$ is given by

$$I_{i,j} = 2k^2 \frac{1}{1} \prod_{l=1}^{4} \frac{d^4 \beta_l}{2} \delta(\beta_1 + \beta_3 - 1) \delta(\beta_2 + \beta_4 - 1) \{x_{\mu,ij} I_0(\tau) + \sum_{r=1}^{3} x_{\mu,ij} I_r(\tau) + y_{\mu,ij}^{(r)} I_0(\tau) \}$$

where

$$I_0(\tau) = \frac{1}{1} \prod_{n=1}^{3} \frac{d \rho_n}{C_n} \delta(\sum_{n=1}^{3} \rho_n - 1), \\
I_r(\tau) = \frac{1}{1} \prod_{n=1}^{3} \frac{d \rho_n}{C_n} \frac{\partial}{\partial \tau} \delta(\sum_{n=1}^{3} \rho_n - 1),$$

and

$$\frac{d \rho_n}{C_n}.$$
If $S(r) = PIP_2(I)^{1/2}dP^{1/2}P_1(r)\delta(\sum_{n=1}^{\infty} \rho_n - 1)$, and $r$ and $C$ are defined by

$$r = 2\{4k \cdot k' - 2\beta_3k \cdot Q' - 2\beta_3k' \cdot Q + \beta_1\beta_2 Q \cdot Q'\}$$

$$+ \beta_1(2k' - Q')^2 + \beta_2(2k - Q)^2 + (\beta_3 + \beta_4) m^2 + (\beta_1 + \beta_2 - \beta_3 - \beta_4) \mu^2$$

and

$$C = \rho_1\rho_2 + \rho_3\rho_4 \{\beta_1(2k' - Q')^2 + (1 + \beta_5) m^2 + \beta_5 \mu^2\} + \rho_3\rho_2 \{\beta_2(2k - Q)^2(1 + \beta_5) m^2 + \beta_5 \mu^2\}$$

$$+ \rho_1^2 \{\beta_3\beta_5 Q'^2 + \beta_3 m^2 + \beta_1 \mu^2\} + \rho_2^2 \{\beta_4\beta_5 Q + \beta_4 m^2 + \beta_2 \mu^2\} + \rho_3^2 m^2.$$

The $x_{\alpha,i}'s$ and $y_{\alpha,i}'s$ are spurs of the products of $\gamma$-matrices, and they are given in Appendix 1.

As $k \cdot k' \rightarrow \infty$, $r$ behaves like $8k \cdot k'$, so we can use the asymptotic expansions of $I(\tau)$'s and $I'(\tau)$'s, and we will give them in Appendix 2. The leading terms of $I(\tau)$'s and $I'(\tau)$'s are as follows:

$$I_0(\tau), I_0'(\tau) = O(\log \tau / \tau^2),$$

$$I_r(\tau), I_r'(\tau) = O(1/\tau^2)$$

and

$$I'_s(\tau) = O((\log \tau)^3 / \tau^4),$$

while

$$I'_{st}(\tau) < I'_s(\tau) \quad (r, s, t = 1 \text{ or } 2).$$

Moreover, we notice that the leading term of each $I(\tau)$ can depend only on $Q$ or $Q'$, and not on both variables. With these properties of $I(\tau)$'s we can now discuss the large $\tau$ behavior of $I_{st}^{ij}$.

For $i=j=1$, we can immediately see that, except for the term

$$-16i\beta_1(k \cdot Q' k' - k \cdot k' Q') I_1'(\tau) + 16i\beta_2(k' \cdot Q k - k \cdot k' Q) I_2'(\tau)$$

all others damp faster than or as fast as $(\log \tau)^3 / \tau^2$. If we substitute $I_r'(\tau)$ in (15) by their leading terms, it contribute nothing to the integral (14), because such an integral as
On the Asymptotic Behavior of Electromagnetic Form Factors

\[ \int d^4Q (k \cdot Q k, - k \cdot k' Q') f(Q^2, k \cdot Q) \]

is zero owing to Lorentz invariance. Similarly, for \( i=j=2 \), although the coefficients of \( I_{ii}^s(\tau) \) and \( I_{ii}^s(\tau) \) are of order \( \tau^3 \), but such terms do not contribute to the integral for the same reason as the term (15). We can see that all terms except for the following one

\[ 4ik \cdot k' \{ (2k_\mu - \beta_1 Q_\mu) [I_1(\tau) - 2m^2 I'_1(\tau)] - (2k_\mu - \beta_2 Q_\mu) [I_2(\tau) - 2m^2 I'_2(\tau)] \} \]

(18)
damp faster than or as fast as \((\log \tau)^3/\tau^3\), for \( i=j=2 \). As is shown in Appendix 2, \( I_1(\tau) - 2m^2 I'_1(\tau) \) and \( I_2(\tau) - 2m^2 I'_2(\tau) \) are of the order of \((\log \tau)^3/\tau^3\), and so (18) behaves like \((\log \tau)^3/\tau^3\).

Except for \( i=1, j=2 \) and \( i=2, j=1 \), all other combinations of \( i \) and \( j \) to \( I_{ii}^s \) contribute only the \( O((\log \tau)^3/\tau^3) \) terms to the integral (14) for the same reason as above. For \( i=1, j=2 \) and \( i=2, j=1 \), we can see \( I_{12}^s \) and \( I_{21}^s \) damp faster than or as fast as \((\log \tau)^3/\tau^3\), except for the terms

\[ 8mi(k + k_\mu)(2k \cdot k' - \beta k \cdot Q') I'_1(\tau) \]

(19)

and

\[ -8mi(k + k_\mu)(2k \cdot k' - \beta k \cdot Q') I'_2(\tau). \]

(20)

These terms (19) and (20) should be cancelled, as they are proportional to \( k_\mu + k_\mu' \) which violate gauge invariance. Actually this is true owing to the minus sign in front of the term (20) and the relation \( I_1(\tau) = I_1(k \rightarrow k', Q \rightarrow Q', \beta_1 \rightarrow \beta_2, \beta_2 \rightarrow \beta_1) \).

In conclusion, we can say that the formula (14) behaves like \((\log k \cdot k')^3/(k \cdot k')^3\) as \( k \cdot k' \rightarrow \infty \). The possibility of making the behavior worse by spin 1/2 propagator and by the existence of the amplitudes, \( g_\pi(p) \), is suppressed by the gauge invariance of the theory. Here we remark that the contribution of \( G_\pi^{(e)} \), Fig. 5a, also behaves like \((\log k \cdot k')^3/(k \cdot k')^3\).

\section{Discussion}

In our calculation in the last section, we have made a ladder approximation. There are many graphs in the series of \( G_\rho \) and \( J_\rho \), indeed. But we can say that our conclusion may be true in general. The reasons are as follows. First, if we take any complicated graphs from \( G_\rho \), the really important roles are played only by \( k \)- and \( k' \)-dependent nucleon propagators for each graph. If the number of such nucleon lines is increased, the number of lines to short to obtain a \( k \cdot k' \)-large behavior is also increased. This means that the net effects of the order of \( k \cdot k' \) will be the same as the case which we have calculated. Secondly, the reason why the possibility that the spin 1/2 propagator makes the behavior worse is suppressed is a very general principle, gauge invariance, and such a
principle works for all graphs of $G_\mu$ and $J_\mu$. One remark is added here that the conclusion of the last section does not alter, if we take a scalar meson exchange instead of the $\pi$-meson exchange in the ladder approximation, provided that the strong interaction vertex function damps appropriately.

Next, we discuss the electromagnetic form factors of nucleon. Ball and Zachariasen$^7$ argued that the form factors also behave like $(\log t)/t^2$ as $t \to 0$, under the simplest model. Their essential point was that the strong interaction vertex function behaves like $1/Q^2$ for a large off-shell mass $Q^2$. But their conclusion is not decisive, since the nucleon propagator should be different from the usual one and nucleon itself is assumed to be a bound state.

If we use the usual nucleon propagator, and if we consider the ladder approximation in which the nucleon exchange is used instead of scalar meson exchange, the circumstance is very bad, and we cannot find such a cancellation of $O(1/\tau^2)$ terms as in our $\pi$-meson case. This is due to the fact that we have two $k$-dependent and two $k'$-dependent nucleon lines instead of one and one in the $\pi$-meson case, respectively, while the number of lines to short to obtain the large $k \cdot k'$ behavior is the same for the two cases.

Strictly speaking, we cannot formulate even the B-S equation for such a problem since the bound state itself is a constituent particle. Therefore the calculation may be meaningless for the nucleon form factors. But we agree completely with Ball and Zachariasen's conclusions that the form factor of a composite particle dies off at large $t$ much faster than that of an elementary one, and that the fact that the nucleon form factor vanishes at least as $t^{-1}$ is the evidence of the compositeness of the nucleon. Moreover, the $\pi$-meson can be considered to be a bound state of a nucleon and anti-nucleon system, consistently at least with respect to the large $t$-behavior of the form factor, as we shown in this article. This may be a support to the hypothesis, "nuclear democracy".

Acknowledgements

It is a pleasure to thank Professor H. Miyazawa for his kind interest in this work. The author also thanks J. Arafuné for helpful discussions and a reading of the manuscript, and Y. Shimizu and A. Sato for helpful discussions.

Appendix I

In this appendix we write down $x_{\mu\nu}$ and $y_{\mu\nu}$:

$$x^{(1)}_{\mu\nu} = -\frac{1}{2} \text{Sp}[m (\Gamma_i \gamma_\mu \Gamma_j - \gamma_\mu \Gamma_i \Gamma_j) + i\gamma_\alpha \Gamma_i \gamma_\mu (\gamma \cdot k' \gamma_\rho \gamma_\tau - \gamma_\rho \gamma_{\tau} \gamma \cdot k) \Gamma_j],$$

$$x^{(0)}_{\mu\nu} = \frac{i}{4} (2k' - \beta_2 Q') \alpha \times \text{Sp}[\gamma_\alpha \Gamma_i \gamma_\mu \gamma_\nu \Gamma_j + \gamma_\beta \Gamma_i \gamma_\alpha \gamma_\nu \gamma_\beta \Gamma_j + \gamma_\beta \Gamma_i \gamma_\alpha \gamma_\nu \gamma_\beta \Gamma_j],$$

$$y^{(0)}_{\mu\nu} = -m \text{Sp}[m \Gamma_i \gamma_\mu \gamma_\nu \Gamma_j - 2im \Gamma_i (\gamma \cdot k' \gamma_\rho \gamma_\tau - \gamma_\rho \gamma_{\tau} \gamma \cdot k) \Gamma_j + 4\Gamma_i \gamma_\tau \gamma_\rho \gamma_\mu \gamma_\nu \Gamma_j].$$
On the Asymptotic Behavior of Electromagnetic Form Factors

\[ y_{\mu ij}^{(1)} = -(2k' - \beta_i Q')_a S_0^{(m)} + 2m^2 \delta_{\rho a} \Gamma_j \]

\[ + 4i \tau \Gamma_j (\tilde{\gamma}_a \tilde{\gamma}_a - \tilde{\gamma}_a \tilde{\gamma}_a - k) \Gamma_j + 4i \tau \Gamma_j (\tilde{\gamma}_a \tilde{\gamma}_a - \tilde{\gamma}_a \tilde{\gamma}_a - k) \Gamma_j \]

\[ y_{\mu ij}^{(1,2)} = (2k' - \beta_i Q')_a (2k' - \beta_i Q')_b S_0^{(m)} + 2m^2 \delta_{\rho a} \Gamma_j \]

\[ + 4i \tau \Gamma_j (\tilde{\gamma}_a \tilde{\gamma}_a - \tilde{\gamma}_a \tilde{\gamma}_a - k) \Gamma_j \]

\[ y_{\mu ij}^{(2)} = (2k' - \beta_i Q')_a (2k' - \beta_i Q')_b S_0^{(m)} + 2m^2 \delta_{\rho a} \Gamma_j \]

\[ + 4i \tau \Gamma_j (\tilde{\gamma}_a \tilde{\gamma}_a - \tilde{\gamma}_a \tilde{\gamma}_a - k) \Gamma_j \]

where

\[ \Gamma_j = \tilde{\gamma}_a \Gamma_j \tilde{\gamma}_a \Gamma_j \]

In this appendix we prove that the function \( I(\tau) \)'s satisfy the relations of Eq. (16) as well as \( I_2(\tau) = 2m^2 I^2(\tau) = O((\log \tau)^{n-1}) \) for \( r = 1, 2 \), for large \( \tau \). For the purpose we show first that we need only to consider the case when the momentum \( Q \) is space-like in the following integral.

\[ \int d^4Q \int d^4Q' \prod_{n=1}^{3} d\rho_n d\rho_n \prod_{i=1}^{4} d\beta_i \frac{g_i^{*}(Q') g_i(Q) f(\rho, \beta)}{[C]^{m}} \]

\[ \times \delta \left( \sum_{n=1}^{3} \rho_n - 1 \right) \delta (\beta_1 + \beta_2 - 1) \delta (\beta_3 + \beta_4 - 1) \]

\[ (A.1) \]

where \( f \) is an entire function and \( C \) and \( g_i^* \) are defined in \( \S \) 3. Here, we write \( C = \rho_1 \rho_2 A + \rho_2 \rho_3 B + \rho_3 \rho_4 A + \rho_4 \rho_1 B \). For large positive value of \( k' \), \( \tau \) approaches to \( 8k' \), and \( C \) is positive definite, if \( \rho_1 \) and \( \rho_2 \) are not zero. Even for extremely small values \( \rho_1 \) and \( \rho_2 \), \( C \) is also positive, as \( c = m^2 \). B-S amplitudes \( g_i(Q') \) and \( g_j(Q) \) have only cuts and poles on the real axes of \( Q_0 \) and \( Q_0' \) planes. So that we can transform the integration contours to the imaginary axes of \( Q_0 \) and \( Q_0' \) planes. Then, changing the integral variables \( Q_0 \) and \( Q_0' \) to \( iQ_0 \) and \( iQ_0' \) we find that only the contribution of space-like \( Q \) and \( Q' \) are needed to consider in the above integral.

As \( Q \) and \( Q' \) are space-like, \( a \), \( b \) and \( c \) are positive definite, as can be seen directly in the definition of \( C \) in \( \S \) 3. Owing to the positive definitness of \( a \), \( b \)
and $c$, we can use Mellin transformation technique.\textsuperscript{10} The Mellin transformation $I'_\lambda (\lambda)$ of $I'_\tau (\tau)$ can be written as

$$I'_\lambda (\lambda) = \frac{1}{2} \Gamma (-\lambda) \prod_{i=1}^{\infty} d\tilde{\rho}_i (\tilde{\rho}_i)^{-1+2} (\tilde{\rho}_i)^{-1+1} (\tilde{\rho}_i)^{-2} \exp \left[ - \frac{D(\tilde{\rho})}{\tilde{\rho}_i} \right]. \quad (A\cdot 2)$$

where $D = C - \rho_0 \rho_3$. $I'_\lambda (\lambda)$ has a simple pole at $\lambda = -2$ and double and simple poles at $\lambda = -3, -4, \ldots$. By the inverse transformation we obtain

$$I'_\tau (\tau) = \sum_{n=2}^{\infty} \frac{1}{\tau^n} a_n^{(r)} (k, k'; Q, Q') + \sum_{n=2}^{\infty} \frac{\log \tau}{\tau^n} b_n^{(r)} (k, k'; Q, Q') \quad (r = 1, 2), \quad (A\cdot 3)$$

where $a_n^{(r)}$ and $b_n^{(r)}$ do not contain $k \cdot k'$. We notice here that $a_2^{(r)}$, the coefficient of the leading term, can be given by

$$a_2^{(r)} = \frac{1}{2} \int_{0}^{1} d\rho_1 d\rho_3 \frac{\delta (\rho_1 + \rho_3 - 1)}{[D(\rho_1 = 0)]^2}.$$ 

Similarly, $I'_\tau (\tau)$ can be given by

$$I'_\tau (\tau) = \sum_{n=2}^{\infty} \frac{1}{\tau^n} a_n^{(s)} + \sum_{n=2}^{\infty} \frac{\log \tau}{\tau^n} b_n^{(s)}, \quad (A\cdot 4)$$

and

$$b_2^{(s)} = \frac{1}{2m^3}.$$ 

Before going to $I_{n\lambda} (\tau)$ ($r, s = 1, 2$), we should discuss $I_{n\lambda} (\tau)$. $I_{n\lambda} (\tau)$ can be rewritten as

$$I_{n\lambda} (\tau) = \prod_{i=1}^{\infty} d\rho_i \frac{\rho_i \rho_3 \rho_i}{[C(\rho)]^s} \delta (\sum_{i=1}^{\infty} \rho_i - 1) = \frac{1}{2} \prod_{i=1}^{\infty} d\rho_i \frac{\rho_i \rho_3 \rho_i}{[C(\rho)]^s} \exp \left[ - \frac{C(\rho)}{\rho_i} \right]. \quad (A\cdot 5)$$

By this expression, we can obtain Mellin transform $I_{n\lambda} (\lambda)$, which has a simple pole at $\lambda = -2$ and a triple, a double and a simple pole at $\lambda = -3$, and so on. We notice here that the residue of the simple pole at $\lambda = -2$ is equal to the one in $I'_\lambda (\lambda)$. So we obtain the relation

$$I'_\tau (\tau) - I_{n\lambda} (\tau) = O ((\log \tau)^{r}/\tau^s). \quad (A\cdot 6)$$

Next, we discuss $I_{n\lambda} (\tau)$, which is a function of the form

$$I_{n\lambda} (\tau) = \prod_{i=1}^{\infty} d\rho_i \frac{\rho_i \rho_3 \rho_i}{[C(\rho)]^s} \delta (\sum_{i=1}^{\infty} \rho_i - 1). \quad (A\cdot 7)$$

Tiktopoulos\textsuperscript{13} discussed generally such a type function, and using his result with
a slight modification, we obtain

\[ I_{11}'(\tau) = O((\log \tau)^3/\tau^7). \]  

(A·8)

\( I_{11}'(\tau) \) can be rewritten as

\[ I_{11}'(\tau) = \int_0^1 \prod_{i=1}^3 d\rho_i \frac{\rho_i \rho_3 \delta(\sum_{i=1}^3 \rho_i - 1)}{[C(\rho)]^3} = \int_0^1 \prod_{i=1}^3 d\rho_i \frac{\rho_1 \rho_2 \rho_3 (1 - \rho_1 - \rho_2) \delta(\sum_{i=1}^3 \rho_i - 1)}{[C(\rho)]^3}. \]

(A·9)

Substituting Eqs. (A·6) and (A·8) into Eq. (A·9), we obtain

\[ I_{11}'(\tau) = O((\log \tau)^3/\tau^7). \]  

(A·10)

For \( I_0(\tau) \) and \( I_0'(\tau) \), we can use the result by Tiktopoulos\(^{11} \) as for the case of \( I_{11}'(\tau) \), we obtain

\[ I_0(\tau) = O(\log \tau/\tau^2) \]  

(A·11)

and

\[ I_0'(\tau) = O(\log \tau/\tau^2). \]  

(A·12)

It is trivial by definition that \( I_{10}(\tau) \) is smaller than \( I_{10}'(\tau) \) for \( r, s, t = 1, 2 \). Finally, we discuss \( I_1(\tau) \), which can be written as

\[ I_1(\tau) = \int_0^1 \prod_{i=1}^3 d\rho_i \frac{\rho_1 \rho_2 \rho_3 \delta(\sum_{i=1}^3 \rho_i - 1)}{[C(\rho)]^3} = 2 \int_0^1 da \int_0^1 \rho_1 \rho_2 \rho_3 \prod_{i=1}^3 d\rho_i \frac{d\delta(\sum_{i=1}^3 \rho_i - 1)}{[\rho_1 \rho_2 \rho_3 + ad]^3}, \]  

(A·13)

where

\[ d = \frac{A}{m^2} \rho_1 \rho_3 + \frac{B}{m^2} \rho_2 \rho_3 + \frac{a}{m^2} \rho_1^3 + \frac{b}{m^2} \rho_2^3 + \frac{c}{m^2} \rho_3^3. \]

Then, we can see that the leading term comes from \( \rho_3^3 \) of \( d \), and we obtain

\[ I_1(\tau) - 2m^2 I_1'(\tau) = O((\log \tau)^3/\tau^7). \]  

(A·14)

References

1) G. F. Chew, Rev. Mod. Phys. 34 (1962), 394.
2) Too many references to cite here.
3) S. Mandelstam, University of California preprint, (1967).