Progress of Theoretical Physics, Vol. 40, No. 5, November 1968

On a Class of Non-Static Fluid Spheres without Energy Flow

Hidekazu NARIAI

Research Institute for Theoretical Physics
Hiroshima University, Takehara, Hiroshima-ken

(Received June 28, 1968)

In order to find some common features of relativistic fluid spheres without energy flow, the velocity field in the fluid spheres is examined from the standpoint of its covariant decomposition. The examination shows that, if the velocity field is shear-free, its 4-divergence depends only on time and the converse is also true. It is also shown that the fluid spheres without energy flow dealt with recently by McVittie, Thompson and Whitrow, Bonnor and Faulkes, Bondi and the author are all shear-free.

§ 1. Introduction and summary

In previous papers, the author found a class of analytical solutions of Einstein’s field equations for a fluid sphere with pressure gradient, but without energy flow, and studied their physical properties. One of them corresponds to a pulsating sphere with finite amplitude and the other two represent bouncing spheres, but all of them are regular at any time. The procedure for finding the solutions was to impose a certain restriction on the metric tensor in a manner analogous to that employed by Buchdahl for the derivation of a static fluid sphere resembling the Emden polytrope of index 5. This type of study was also performed by McVittie, Vaidya, Thompson and Whitrow, Bonnor and Faulkes and Bondi. Among these studies, McVittie’s seems to be the most systematic, but his procedure does not necessarily lead to a class of physically plausible solutions. However, the author’s solutions belong to a special class of McVittie’s classification of the fluid spheres without energy flow.

Another approach, in order to deal with the problem of gravitational collapse for a fluid sphere consisting of matter with a realistic equation of state, of Misner and his coworkers and of Podurets was to reduce Einstein’s field equations for the fluid sphere to the system of partial differential equations of the first order, each one of which has a clear physical meaning.

Numerical analyses of cold neutron stars by May and White and Podurets and that of massive stars \(M=10^4M_\odot\) by Voropino et al. relied substantially on the second formalism. This type of study aims at solving Cauchy’s problem for a realistic fluid sphere and so it is physically superior to the former type of study. At the present stage of our knowledge, however, the amount of information to be obtained numerically is much less than that to be obtained analytically.
In the above circumstances, it is worthwhile to make the former type of study stand on a more solid basis by means of finding some physical condition in terms of which the metric for a fluid sphere takes a particular form such as those adopted, for instance, by McVittie\(^3\) and Thompson and Whitrow.\(^5\) For this purpose, the covariant decomposition of a velocity field in the sense of Ehlers\(^13\) is useful. In § 2 the velocity field in a fluid sphere is decomposed and its surviving components are examined. We shall prove in § 3 that, if there is no energy flow, the vanishing of shear-part of the velocity field is equivalent to the position-independence of its 4-divergence. It is shown further that the metric for a fluid sphere without energy flow and with a shear-free velocity field is reduced to the form adopted by Thompson and Whitrow.\(^5\) In § 4 we shall reduce Einstein’s field equations for a fluid sphere with the velocity field discussed in § 3 in order to show that the reduced equations include as a special case the equations appearing in McVittie’s study.\(^3\) We shall deal in § 5 with the dynamical equations for the fluid sphere from another standpoint and we shall rederive in § 6 our regular solutions by making use of the formalism developed in § 5.

\section{Velocity field in a fluid sphere}

The metric for a spherically symmetric space-time can be written as

\[ ds^2 = -e^{2\phi}dt^2 + e^{2\lambda}dr^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

where \(\phi, \lambda\) and \(R\) are functions of \(t\) and \(r\). If we put \(x^i = (x^0, x^i) (i = 1, 2, 3)\), the metric tensor \(g_{\mu \nu}\) corresponding to the above metric is of the form

\[ g_{\mu \nu} = -e^{2\phi}, \quad g_{11} = e^{\lambda}, \quad g_{22} = g_{33} = R^2, \quad (-g)^{1/2} = e^{\lambda + 1/2}R^2 \sin \theta, \]

where \(g = \text{det}(g_{\mu \nu})\).

Let us assume that the interior region of a fluid sphere is represented by the metric (2·1) in such a way that the coordinates \(x^\mu\) are comoving with the fluid matter. Then the fluidal 4-velocity field \(u^\mu\) takes the form

\[ u^\mu = e^{-\phi} \delta^\mu_0, \quad u_\mu = -e^{\phi} \delta^\mu_0 \quad (u_\mu u^\mu = -1). \]

According to Ehlers\(^13\) however, the covariant derivative of a velocity field can be decomposed in the following way:

\[ u_{\mu \nu} = \omega_{\mu \nu} + \sigma_{\mu \nu} + (\theta/3) h_{\mu \nu} - a_\mu a_\nu, \]

with

\[ \theta = u^\mu_u = (-g)^{-1/2} \{ (-g)^{1/2} u^\mu \} \quad (\text{the divergence part}), \]

\[ a_\mu = u_\mu u^\mu, \quad a_\mu u^\mu = 0 \quad (\text{the vector part}), \]

\[ \omega_{\mu \nu} = u_{(\mu ; \nu)} h^{\mu \nu} - \omega_{\nu \mu} \quad (\text{the rotation part}), \]

\[ \sigma_{\mu \nu} = u_{(\mu ; \nu)} h^{\mu \nu} - (\theta/3) h_{\mu \nu} \quad (\text{the shear part}), \]
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where $h_{\mu\nu}$ is the projection operator into a hyperplane orthogonal to $u^\nu$, i.e.

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad h_{\mu}^\nu = g^{\mu\nu} h_{\mu\nu},$$

(2.6)

so that

$$h_{\mu\nu} u^\nu = 0, \quad h_{\mu}^\nu h_{\mu}^\nu = h_{\mu}^\mu.$$

(2.7)

It is easily seen from Eqs. (2.5) and (2.7) that both $\omega_{\mu\nu}$ and $\sigma_{\mu\nu}$ are orthogonal to $u^\nu$ and traceless, i.e. $\omega_{\rho}^\rho = \sigma_{\rho}^\rho = 0$.

Now we shall examine each part of the velocity field specified by Eq. (2.3). If we consider Eq. (2.2), the 4-acceleration $a_\mu$ is of the form

$$a_\mu = \delta_\mu^\rho \phi',$$

(2.8)

where a dash denotes differentiation with respect to $r$. Similarly, the scalar $\theta$ is given by

$$\theta = e^{-\phi} (3/2 + 2\dot{F}/F),$$

(2.9)

with

$$F(t, r) = R e^{-\lambda/2},$$

(2.10)

where a dot denotes differentiation with respect to $t$.

On the other hand, it follows from Eqs. (2.2), (2.3) and (2.6) that

$$h_{\mu\nu} = 0, \quad h_{\mu\mu} = g_{\mu\mu},$$

and

$$h_{\mu}^\mu = h_{\mu}^\mu = 0, \quad h_{\mu}^\nu = \delta_{\mu}^\nu.$$

(2.11)

By inserting Eqs. (2.2), (2.3) and (2.11) into the last two expressions in Eqs. (2.5), we obtain

$$\omega_{\mu\nu} = \omega_{\nu\mu} = 0,$$

$$\sigma_{\mu\nu} = \sigma_{\rho}^\nu = 0 \quad \text{and} \quad \sigma_{\mu}^\nu = e^{-\phi} (\dot{F}/F) (\delta_{\mu}^\nu/3 - \delta_{\mu}^\nu \delta_{\nu}^\nu),$$

(2.12)

so that $\sigma_{\mu}^\nu = \sigma_{\rho}^\nu = -\sigma_{\mu}^\nu/2$.

Thus it is shown that the only surviving and independent parts of the velocity field under consideration are $\theta, a_\mu$ and $\sigma_{\mu}^\nu$.

§ 3. Fluidal motion without energy flow

Since we have assumed that the interior region of a fluid sphere is specified by the metric (2.1), Einstein's field equations for the fluid sphere are reduced to

$$8\pi G T_{\mu}^\nu = e^{-\phi} \left[ 2 (R'/R') + 3 (R'/R)^2 - \lambda' R'/R \right] - 1/R^3$$

$$- e^{-\phi} \left[ (\dot{R}/R)^2 + \dot{\lambda}/R \right],$$

(3.1)
\[ 8\pi G T^t_0 = e^{-\lambda} \left\{ (R'/R)^2 + \frac{2\phi'}{R'} + \frac{3(\phi - \lambda' + 1)}{R' + R''} \right\} - \frac{1}{R^2} \]

\[ - e^{-\phi} \left\{ 2(R/R')^2 + 3(\dot{R}/R)^2 - 2\dot{\phi} + \ddot{R}/R \right\}, \quad (3\cdot2) \]

\[ 8\pi G T^t_\alpha = e^{-\lambda} \left\{ \phi'' + (\phi' - \lambda'/2)(\phi' + R'/R) + R''/R \right\} \]

\[ - e^{-\phi} \left\{ \dot{\lambda}/2 + (\dot{\lambda}/2 - \phi)(\dot{\lambda}/2 + \ddot{R}/R) + \ddot{R}/R \right\} \]

and

\[ 8\pi G T^t_\alpha = - e^{-\lambda} \left\{ \phi'' - (\phi' - \lambda'/2)(\phi' + R'/R) - \dot{R}'/2R \right\} \]

\[ \quad - e^{-\phi} \left\{ 2(R'/R)\left( \phi' - \lambda'/2 \right) + \ddot{R}'/2R \right\}, \quad (3\cdot3) \]

where \( T^\alpha_\beta \) is the energy-momentum tensor and \( G \) the Newtonian gravitation constant. Since \( T^\alpha_\beta \) consists of the part specified by \( (\rho + p)u^\mu u^\nu + \rho \delta^\alpha_\nu \) (\( \rho \) and \( p \) denote the total density and pressure, respectively, and \( u^\alpha \) is the velocity field specified by Eq. (2·3)) and the part representing energy flow, we can reduce Eq. (3·4) to

\[ 12\pi G T^t_0 = - e^{-\lambda} \left\{ \phi'' + (\phi' + R'/R) + R''/R - (\phi' - \lambda'/2) \right\} \]

\[ \quad - e^{-\phi} \left\{ \dot{\lambda}/2 + (\dot{\lambda}/2 - \phi)(\dot{\lambda}/2 + \ddot{R}/R) + \ddot{R}/R \right\} \]

by the use of Eqs. (2·9) and (2·10).

In order to construct the physical component of the part of \( T^\alpha_\beta \) which specifies the energy flow, we shall introduce the space-like unit vector \( b^\alpha \) in the \( r \)-direction such as

\[ b^\alpha = e^{-\lambda/2} \delta_i^\alpha \quad (b^\alpha b_\alpha = 1, \ u_\alpha b^\alpha = 0). \quad (3\cdot6) \]

Then the physical component is given by

\[ T^r_\alpha = u^\mu b^\nu T^\mu_\nu = e^{\lambda/2 - \phi} T^\lambda_\lambda. \quad (3\cdot7) \]

By inserting Eq. (3·5) into Eq. (3·7), we obtain

\[ T^r_\alpha = - (1/12\pi G) \left\{ D_r \theta + D_\alpha D_\lambda (\ln F) + 3D_r (\ln F + \lambda/2) D_\lambda (\ln F) \right\}, \quad (3\cdot8) \]

where

\[ D_r = u^\nu \partial_\nu = e^{-\phi} \partial_\nu, \quad D_\nu = b^\nu \partial_\nu = e^{\lambda/2} \partial_\nu. \quad (3\cdot9) \]

Similarly, it follows from Eqs. (2·9) and (2·12) that

\[ \theta = (3/2) D_\lambda \lambda + 2D_\lambda (\ln F), \quad (3\cdot10) \]

\[ \sigma_i^j = (\partial_i^j - \delta_i^j \delta_j^i) D_\lambda (\ln F), \quad \sigma = (\frac{3}{2} \sigma_i^j \sigma^i_j)^{1/2} = |D_\lambda (\ln F) / \sqrt{3}|. \]

Since \( T^r_\alpha \) is a 4-scalar, the invariant representation of the condition that there is no energy flow in the interior region is given by \( T^r_\alpha = 0 \). In this case, Eq. (3·8) is reduced to

\[ D_r \theta + (D_r + 3D_\lambda (\ln F + \lambda/2)) D_\lambda (\ln F) = 0. \quad (3\cdot11) \]

On taking account of Eqs. (3·10) and (3·11), we have

(i) \( D_r \theta = 0 \) or \( \dot{\theta} = \theta (t) \) (if \( \sigma_i^j = 0 \)),

(ii) \( \{ D_r + 3D_\lambda (\ln F + \lambda/2) \} \sigma_i^j = 0 \) (if \( D_r \theta = 0 \)).
The case (i) shows that the scalar part $\theta$ of the velocity field is position-independent if there is no shear. On the contrary, the case (ii) suggests that the reverse is not necessarily true.

In order to determine whether the above suggestion is true, let us rewrite Eq. (3·13) as follows:

$$\frac{F}{F} + 3 \frac{F'}{F} - \left( \phi' - 3 \lambda' \right) \frac{\phi}{F} = 0,$$

the integration of which leads us to

$$F = H(t) e^\frac{\phi - 3\lambda}{2},$$

(3·14)

where $H(t)$ is an arbitrary function of $t$. On the other hand,$^{10}$ the expressions for $e^{\phi}$, $e^\lambda$ and $R$ in the neighborhood of $r=0$ must be of the form

$$e^{\phi} = 1 - b(t) r^2 + \cdots,$$
$$e^\lambda = a(t) \left( 1 + 2m(t) r^2 + \cdots \right),$$
$$R = a(t) r(1 + \cdots),$$

(3·15)

in order that both the scalar curvature and the physical components of $T_{\mu\nu}$ may be regular at $r=0$, where $a(t)$, $b(t)$ and $m(t)$ are regular functions of $t$, such as $a(t) > 0$ and $\dot{m} = b \dot{a} / a$ (if $T^r_\tau = 0$). Then, by the use of Eq. (2·10), we have

$$F = r(1 - m r^2 + \cdots),$$

(3·16)

$$e^{\phi - 3\lambda} = a^{-2} \left( 1 - \frac{1}{2}(b + 6m) r^2 + \cdots \right).$$

In order that Eq. (3·14) at $r=0$ may be consistent with the above expressions, we must have $H(t) = 0$, so that $\dot{F} = 0$ or $F = f(r)$. This means that Eq. (3·13) is reduced to $\sigma^r_\tau = 0$.

Accordingly, we arrive at the following theorem: "If there is no energy flow in the interior of a fluid sphere, the requirement that the velocity field is shear-free is equivalent to the requirement that its four-dimensional divergence is position-independent."

From now on we shall assume that the velocity field is shear-free. Then we can reduce Eqs. (2·9) and (2·10) to

$$e^{\theta} = (3\dot{\lambda}/2) \theta^{-1}(t),$$
$$R = e^{\frac{\lambda}{2}} f(r),$$

(3·17)

where

$$f(0) = 0, \quad f'(0) = 1.$$  

(3·18)

Note that the restriction (3·17) imposed on the metric tensor (2·2) is equivalent to that adopted by Thompson and Whitrow.$^5$
§ 4. Reduction of Einstein's field equations

Since we have assumed that \( T^{r}_{t} = D_{t} = 0 \), the surviving components of \( T_{\mu}^{\nu} \) are only those corresponding to the part of a perfect fluid, so that

\[
T_{\theta}^{\theta} = -\rho, \quad T_{t}^{t} = T_{t}^{\theta} = 0.
\] (4·1)

Accordingly, by the use of Eqs. (3·17) and (4·1), Eqs. (3·1) and (3·2) are reduced to

\[
8\pi G\rho = \frac{\theta^{2}}{3} - e^{-\lambda} \left\{ \lambda'' + 2 (f'/f) \lambda' + (\lambda'/2)^{2} + (2f'f'' + f''')/f^{3} \right\},
\] (4·2)

\[
8\pi G\rho = -4\theta^{2}/9 - \theta^{2}/3 + e^{-\lambda} \left\{ (\lambda' + 2f'/f) (\lambda'/2) + (f'/f) \lambda' \right\}
+ (\lambda'/2)^{2} + (f'' - 1)/f^{3}.
\] (4·3)

Moreover, we can replace Eq. (3·3) by \( u^{a} T_{t}^{a} = 0 \), which is reduced to

\[
\dot{\theta} + (3/2) (\rho + \rho) \dot{\lambda} = 0.
\] (4·4)

By inserting Eqs. (4·2) and (4·3) into Eq. (4·4), we obtain

\[
\frac{\lambda'' - (f'/f) \lambda' - (\lambda'/2)^{2}}{\dot{\lambda}} + \frac{\lambda'' - (f'/f) \lambda' - (\lambda'/2)^{2}}{2}
+ (f'f'' - f'' + 1)/f^{3} = 0,
\] (4·5)

which is the differential equation for \( \lambda = \lambda(t, r) \), provided that \( f(r) \) is given in conformity with the boundary conditions (3·18).

In order to show that the fluid sphere under consideration has a close connection with that envisaged by McVittie,\(^{3}\) let us assume for a moment that

\[
e^{\phi} = S^{2}(t) e^{n\eta}, \quad e^{\phi} = Q(r)/S^{n}(t),
\] (4·6)

where \( S(t) \) and \( Q(r) \) are arbitrary functions of their arguments and \( n \) is a nonvanishing parameter. By inserting Eq. (4·6) into Eq. (3·17), we have

\[
e^{\phi} = (3\dot{S}/S) \theta^{-1} (t) (1 - n\eta_{r}/2),
\] (4·7)

where \( \eta_{r} = d\eta /dz \). Since \( e^{\phi} = (-g_{00})^{1/2} \), we can assume without loss of generality that

\[
\theta(t) = 3\dot{S}/S,
\] (4·8)

so that Eq. (4·7) is reduced to

\[
y = e^{\phi} = 1 - n\eta_{r}/2.
\] (4·9)

By inserting Eq. (4·6) into Eq. (4·5) and by making use of Eq. (4·9), we obtain

\[
y_{\eta} + \{(a-1) - (2-\gamma)/n\} y_{\eta} + y \{ (1-\gamma)(a-1)/n
- (1-\gamma)^{2}/n^{2} + b \} = 0,
\] (4·10)

provided that

\[
(Q'' - f'Q'/f)/Q = a (Q'/Q)^{3},
\] (4·11)
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\[(ff'' - f''^2 + 1)/f^3 = b (Q'/Q)^2,\]  
\[(4\cdot12)\]

which are necessary for the consistency of Eq. (4·6) with Eq. (4·5) where \(a\) and \(b\) are arbitrary constants. On comparing Eqs. (4·9), (4·10), (4·11) and (4·12) with their counterparts in McVittie's paper,\(^9\) we easily see that the latter equations correspond to a special case of \(n = 1\) in the former ones.

In order that the fluid sphere to be specified by Eqs. (4·2), (4·3), (4·6), (4·9), (4·10), (4·11) and (4·12) may not be unphysical, we must choose a suitable set of the three parameters \((a, b, n)\) in such a way that the metric is regular in the interior region and both \(\rho\) and \(p\) are at least non-negative. It is interesting to study how many sets are permissible, but we shall remark here only the fact that the author's\(^9\) regular solutions belong to the set: \(a = 3, b = 0\) and \(n = 1\).

§ 5. Integration of Eq. (4·5) and further reduction

In the previous section, we have reduced Eq. (4·5) to the ordinary differential equation (4·10) for \(y = e^\phi\) with the assumptions (4·6) and (4·8). Of the two assumptions, the latter is always permissible and so we shall also adopt it in this section. On the other hand, the former is not compulsory. In fact, without recourse to Eq. (4·6), we can integrate Eq. (4·5) as follows:

\[Y'' - f'Y'/fY - Y((ff'' - f''^2 + 1)/f^3 - 3h(r) Y) = 0 (Y = e^{-\lambda^2}),\]  
\[(5\cdot1)\]

where \(h(r)\) is an arbitrary function of \(r\) such as

\[h(0) = 0.\]  
\[(5\cdot2)\]

This is necessary in order that Eq. (5·1) may be consistent with the boundary conditions

\[Y = a^{-1}(t), f'Y'/fY = Y''/Y = -2m(t) \quad \text{at} \quad r = 0,\]  
\[(5\cdot3)\]

derived from Eqs. (3·15),\(^8\) (3·17) and (3·18).

By inserting Eq. (4·8) and \(Y = e^{-\lambda^2}\) into Eq. (3·17), we obtain

\[e^\phi = - (\dot{Y}/Y)/(\dot{S}/S),\]  
\[(5\cdot4)\]

so that \((e^\phi)_{r=0} = d(\ln a)/d(\ln S)\) is not necessarily unity. Similarly, we can reduce Eqs. (4·2) and (4·3) to

\[8\pi G \rho = 3(\dot{S}/S)^2 - 3Y^2 (Y'/Y) - 2(f'/fY)' + (f'^2 - 1)/f^3 + 2hY,\]  
\[(5\cdot5)\]

\[8\pi G \rho = -2e^{-\phi}(\dot{S}/S)^2 - 3(\dot{S}/S)^2 - 2(f'/fY)' + (f'^2 - 1)/f^3 - 2(f'/fY)'(Y/f)' \phi'.\]  
\[(5\cdot6)\]

On the other hand, it follows from Eqs. (3·17), (5·1) and (5·5) that

\(^8\) In this equation, we have settled the \(t\)-coordinate in such a way that \(e^\phi = 1\) at \(r = 0\), but Eq. (5·3) can hold irrespective of such a settlement of \(t\).
which is equivalent to Eq. (19) in Thompson and Whitrow's paper. The above equation shows that \( \rho \) is \( r \)-independent if \( h(r) = 0 \) and its converse is also true by virtue of the boundary conditions (3.18) and (5.2). Moreover, Thompson and Whitrow considered solely uniform-density models collapsing to a point-singularity of infinite density, while Bondnor and Faulkes and Bondi considered uniform-density models with an oscillating motion and a bouncing one, respectively.

For simplicity's sake, let us assume here that

\[
ff'' - f' + 1 = 0,
\]

which corresponds to the case \( b = 0 \) in Eq. (4.12). Then it follows from Eqs. (5.8) and (3.18) that

\[
f(r) = \begin{cases} A \sin(r/A) & (k=1) \\ r & (k=0) \\ A \sinh(r/A) & (k=-1) \end{cases},
\]

so that

\[
x = \int_0^r f(r) dr = \left(\frac{1}{2}\right) \begin{cases} (2A)^2 \sin^2(r/2A) & (k=1) \\ r^2 & (k=0) \\ (2A)^2 \sinh^2(r/2A) & (k=-1) \end{cases},
\]

where \( A \) is a positive constant such that \( k/A^2 \) is the curvature of a 3-space specified by the metric \( d\sigma^2 = dr^2 + f^2(r) (d\varphi^2 + \sin^2\varphi d\varphi^2) \), and \( \bar{f}(x) = f\{r(x)\} \) = \( \{2x(1-kx/2A^2)\}^{1/2} \).

By inserting Eqs. (5.9) and (5.10) into Eq. (6.1), we obtain

\[
Y_{xx} + 2q_{xx} Y = 0,
\]

where \( Y_{xx} = \partial^2 Y/\partial x^2 \) and \( q_{xx} = d^2 q/dx^2 = 3h\{r(x)\}/4x(1-kx/2A^2) \) which is as yet arbitrary, but must be non-negative, if we require that \( \rho' \leq 0 \) (cf. Eq. (5.7)). Similarly, Eqs. (5.5) and (5.6) are reduced to

\[
8\pi G\rho = 3(\dot{S}/S)^2 + 3Y^2[k/A^4 + 2(1-kx/A^2)] (Y_x/Y) - 2x(1-kx/2A^2)
\]

\[
\times \{ (Y_x/Y)^2 + 4q_{xx}Y^2/3 \},
\]

\[
8\pi G\phi = -2e^{-\phi}(\dot{S}/S)^2 - 3(\dot{S}/S)^2 - Y^2[k/A^4 + 2(1-kx/A^2)] (Y_x/Y - \phi_x)
\]

\[
-2x(1-kx/2A^2) \{ (Y_x/Y)^2 - 2\phi_x Y_x/Y \},
\]

where \( e^\phi \) is given by Eq. (5.4).

On the other hand, it follows from Eqs. (5.3) and (5.10) that

\[
Y = a^{-1}(t), \quad Y_x/Y = -2m(t) \quad \text{at} \quad x = 0,
\]

which are the boundary conditions for \( Y(t, x) \). Moreover, at the outer boundary
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$x = x_0$, we must have

$$p_b = p(t, x_0) = 0,$$

which, together with Eq. (5.13), enables us to obtain

$$\{Y_b^{-3}(\dot{S}/S)^3\} = (\dot{Y}/Y)_b = 0,$$

$$2x_0(1 - kx_0/2A^2) \{3(Y_a/Y)_b^3 - 2(2Y_a/(YY)_b^3 - 2(2Y_x/(YY)_b^3)\} = 0,$$

since $(e^b)_b = - (\dot{Y}/Y)_b/(\dot{S}/S)$ and $(q_b)_b = (\dot{Y}_a/Y - Y_a/Y)_b$. We can regard Eq. (5.16) as the differential equation for the scale factor $S(t)$.

Thus our task is reduced to solving the two differential equations (5.11) and (5.16) in such a way that the metric components $e^b$ and $R = f/Y$ are regular and the total density, $\rho$, and the total pressure $p$ given by Eqs. (5.12) and (5.13), respectively, satisfy the inequalities such as $\rho/3 \geq \rho \geq 0$, $\rho_a \leq 0$ and $p_a \leq 0$. For this purpose, we must select a suitable set of $q(x)$, $a(t)$ and $m(t)$, which are adjustable functions of their arguments.

§ 6. Rederivation of our regular solutions

It seems rather difficult to integrate Eqs. (5.11) and (5.16) so as to assure all the requirements mentioned in the last paragraph of § 5 and, so, it may be useful to show how our regular solutions are derivable from the formalism developed in § 5.

Let us assume that

$$q(x) = e(1 + 2\alpha x)^{-1/2},$$

$$a(t) = (S + e)^3/S, m(t) = - e\alpha/(S + e),$$

where $e(> 1)$ and $\alpha(> 0)$ are arbitrary constants permitting the condition $q_b = q(x_b) = 1$ or $\alpha = (e^3 - 1)/2x_b$. Then it follows from Eqs. (6.1) that

$$qq_x = 3a^2,$$

which is reduced to Eq. (4.11) with $a = 3$ and $Q(r) = q \{x(r)\}$. By inserting Eq. (6.1) into Eq. (5.11), we obtain

$$Y_{qq} + (3/q)(Y_q + 2Y^2) = 0,$$

where $Y_q = \partial Y/\partial q$. Similarly, by the use of Eq. (6.2), we can reduce Eq. (5.14) to

$$Y = S(S + e)^{-2}, Y_q = -2S(S + e)^{-3} \text{ at } q = e.$$

It is easily seen that both equations (6.4) and (6.5) are satisfied by

$$Y = S(S + q)^{-1},$$

which is probably a unique solution.

By inserting Eq. (6.6) into Eq. (5.16) and performing the integration, we obtain
\[ S^2 = S^4 (S + 1)^{-6} \left[ \gamma S - \frac{(4\alpha + k/A^2)}{\epsilon^4} - k(S/A)^2 \right], \tag{6.7} \]

where \( \gamma \) is a positive constant. The above equation shows that the case \( k = 1 \) corresponds to a pulsating model such as \( S_1 \geq S(t) \geq S_1 > 0 \) \( (S_1 + S_2 = \gamma A^2 \) and \( S_1 S_2 = (1 + 4\alpha A^4)/\epsilon^4) \), and the other two cases, \( k = 0 \) and \( k = -1 \) lead us to bouncing models such as \( S(t) \geq S_1 (= 4\alpha/\gamma \epsilon^4 \) or \( \gamma A^2/2) \{\sqrt{1 + 4(4\alpha A^2 - 1)/(\gamma A^2 \epsilon^4)^2} - 1 \} \) according as \( k = 0 \) or \( -1 \).

Now it is an easy matter to check that, in order to assure the conditions \( \rho > 0, \ p > 0, \ \rho_s \leq 0 \) and \( \rho_s \leq 0, \) we must have

\[ (S_2 >) S_1 > e(> 1), \tag{6.8} \]

where \( S_2 = \gamma A^2 - S_1, \infty \) or \( \gamma A^2 + S_1 \) according as \( k = 1, 0 \) or \( -1 \). In addition, we can assure the condition \( \rho/3 \geq \rho \) as well, without destroying the inequalities \( (6.8) \). The 'regular solutions thus obtained are nothing but the ones derived in the previous papers. 

**Remarks:** We have also studied the uniform-density models in connection with the studies due to Thompson and Whitrow,\(^5\) Bonnor and Faulkes\(^6\) and Bondi.\(^7\) The contents will be published in a separate paper.

**References**